First part: Vadim Vologodsky
Another approach to Tate duality

§7.1. We wish to explain another approach to the Tate duality theorem discussed last time, which is more abstract, but does not rely on any difficult results.

Let $G$ be an abstract (resp., profinite) group, and let $\mathcal{M}$ be the category of $G$-modules (resp., continuous discrete $G$-modules). We have

$$
\xymatrix{ D^+(\mathcal{M}) \ar[r]^{Rf_*} & D^+(\text{Ab}) \\
\ar@/^/[u]^f \ar@/_/[u]_{f^*} & \ }
$$

where $f_* : \mathcal{M} \to \text{Ab} = \{\text{abelian groups}\}$

$$(M) \mapsto \text{MG}$$

The functor $f^*$ takes a complex in $D^+(\text{Ab})$ to the same complex with the trivial $G$-action. Moreover, the functor $f^*$ is left adjoint to $Rf_*$. 

§7.2. One can ask if $Rf_*$ also has a right adjoint. Suppose we do have such a functor, call it $f^!$.

(It is only defined at the derived category level, and does not have to be the derived functor of anything.)
Then we have
\[ R\text{Hom}_{D(M)}(M, f^! N) \cong R\text{Hom}_{\text{aff}}(Rf_* M, N). \]

Let us define \( D := f^!(\mathbb{Q}/\mathbb{Z}) \). Then we obtain the following duality isomorphism
\[ Rf_* (\text{RHom}_\mathbb{Z}(M, D)) \cong \text{Hom}_\mathbb{Z}(Rf_* M, \mathbb{Q}/\mathbb{Z}). \]

In the case where \( D \) is concentrated in a single degree, it will give a dualizing module for \( G \).

\[ \text{§7.3.} \quad \text{Our goal is to prove the following theorem.} \]

Let \( G \) be a profinite group of finite cohomological dimension. Then the functor \( Rf_* : D^+(M) \to D^+(\text{aff}) \) has a right adjoint, \( f^! : D^+(\text{aff}) \to D^+(M) \).

To prove the result, we will pass to the unbounded derived categories, and then apply a general theorem on triangulated categories due to Neeman.

\[ \text{§7.4.} \quad \text{Let } \mathcal{T} \text{ and } \mathcal{T}' \text{ be triangulated categories, and let } F : \mathcal{T} \to \mathcal{T}' \text{ be a triangulated functor.} \]

\[ \text{Theorem (Neeman). Assume that the following two conditions are satisfied:} \]
(i) \( \mathcal{T} \) and \( \mathcal{T}' \) have infinite direct sums, and \( F \) commutes with infinite direct sums.

(ii) The category \( \mathcal{T} \) is compactly generated.

Then \( F \) has a right adjoint.

**Terminology.** An object \( X \in \mathcal{T} \) is said to be compact if the canonical morphism

\[
\bigoplus_{i \in I} \text{Hom}(X, Y_i) \to \text{Hom}_{\mathcal{T}}(X, \bigoplus_{i \in I} Y_i)
\]

is an isomorphism for any collection \( \{ Y_i \}_{i \in I} \) of objects of \( \mathcal{T} \). We say that \( \mathcal{T} \) is generated by a collection \( S \) of objects if

\[
(Y \in \mathcal{T}, \text{Hom}(X, Y) = 0 \ \forall X \in S) \Rightarrow Y = 0.
\]

Finally, we say that \( \mathcal{T} \) is compactly generated if it is generated by a set of compact objects.

**Reference** (for Theorem 7.4). Very nice paper:

Neeman, "Grothendieck duality theorem via Brown representability"

### §7.5. Exercise.

Prove Theorem 7.3 using Thm. 7.4.

**Hints.** The category \( D^+(\mathcal{M}) \) does not have infinite direct sums, so first we need to extend \( R^f \) to a triangulated functor \( D(\mathcal{A}) \). For this, represent every object of \( D(\mathcal{A}) \) as a colimit of its truncations that live in \( D^+(\mathcal{M}) \), and
use the assumption $\text{cd}(G) < \infty$.

Check that this extension is still right adjoint to $f^*$. Then apply Theorem 7.4 to prove that $Rf_* : D(M) \to D(\text{Ab})$ has a right adjoint, and finally show that this right adjoint takes $D^+(\text{Ab})$ to $D^+(M)$.

§ 7.6. Remarks. (1) For the proof above to work, we need to know that $H^*(G, \cdot)$ commutes with infinite direct sums, which is why we must assume that $G$ is profinite.

(2) Let us describe a set of compact generators of the category $D(M)$. It consists of $G$-modules of the form $\mathbb{Z}[[G/U]][i]$, where $i \in \mathbb{Z}$ and $U$ runs over all open subgroups of $G$. Note the following formula: if $M^*$ is a complex in $M$ and $H^c(M^*)$ is its $c$-th homology object in $M$, we have

$$H^c(M^*) = \lim_{\text{open}} R^c f_* (M^*),$$

where $f_* : M \to \text{Ab}$ is $N \to N^u$.

This formula implies that the objects $\mathbb{Z}[[G/U]][i]$ generate $D(M)$. 

\[\text{\}4\]
Finally, we explain how to recover the Tate duality theorem discussed last time from Theorem 7.3, and, in particular, how the Tate condition appears in this picture.

From Theorem 7.3 we obtain the existence of a dualizing complex $\mathcal{D}$, which is not necessarily a single module.

We have $\mathcal{D} = f! (\mathbb{Q}/\mathbb{Z})$, so

$$\text{RHom}_{\mathcal{D}} (N, \mathcal{D}) \cong \text{Hom}_{\mathbb{Z}} (R^f* N, \mathbb{Q}/\mathbb{Z})$$

$$\Rightarrow H^i(\mathcal{D}) \cong \lim_{\substack{\to \mathcal{U} \subset G}} \text{Hom}(H^i(G, \mathbb{Z}[G/\mathcal{U}]), \mathbb{Q}/\mathbb{Z}).$$

The Tate condition, stated last time, means precisely that this vanishes for all $i \neq cd(G)$, so that $\mathcal{D}$ is concentrated in a single degree. Thus it defines a dualizing module for $G$.

Beilinson resumes his lectures

Lubin-Tate approach to CFT

8.7.8. Let $K$ be a local field, $q$ the order of its residue field, and $\pi \in K$ a uniformizer.

Definition. A formal power series $f(t) \in \mathcal{O}_K[[t]]$ is called a Lubin-Tate (LT) $\pi$-series if it
satisfies the following two conditions:

(i) \( f(t) \equiv t \mod (t^2) \)

(ii) \( f(t) \equiv t^q \mod (t^q) \)

Example. \( \pi t + t^q \) is a LT \( \pi \)-series.

Remark. These conditions are invariant with respect to the adjoint action.

Important point. One should think of an element of \( \mathcal{O}_K[[t]] \) not as a function on the formal disc \( \text{Spf}(\mathcal{O}_K[[t]]) \), but rather as an endomorphism of this disc.

Let us write \( V = \text{Spf}(\mathcal{O}_K[[t]]) \), viewed as a formal scheme over \( \text{Spec}(\mathcal{O}_K) \).

Notation. \( LT_{\pi} = \) the collection of all \( LT_{\pi} \)-series.

the action is by substitution (need to assume that the constant term is 0 to get a well-defined endomorphism)

§ 7.9. Let us take \( f, g \in LT_{\pi} \) and consider the induced action of \( f \) and \( g \) on \( V \) (this is well defined, since \( f \) and \( g \) have zero constant terms, by definition). We also let \( g \) act on the product \( V^n \) diagonally.
Lemma. The map
\[
\text{Intertwiners} \left( \frac{V^n}{\text{diag}(g)} \right) \rightarrow V
\]
\[
\text{Linear maps} \left( \frac{V^n}{\text{diag}(g)} \right) \rightarrow V
\]
is bijective. In particular, automorphisms of $V$ act transitively on $LT^V$ (and those automorphisms that have differential = id act simply transitively on $LT^V$).

Remark. In fact, we could also replace $V$ with $V^m$ for any $m \in \mathbb{N}$, with the induced diagonal action of $g$.

**87.10.** For simplicity, we only prove Lemma 7.9 in the case where $n=1$ ($=m$). Starting with a linear $\varphi^{(1)} : V \rightarrow V$, we build an intertwiner $\varphi : V \rightarrow V$ by induction. Assume that $\varphi^{(l)}(t)$ is defined modulo $I^{l+1}$, where $I = O_K[[t]]$.

Then $\varphi^{(l+1)}$ should have the form
\[
\varphi^{(l+1)}(t) = \varphi^{(l)}(t) + \varphi^{(l+1)}(t)
\]
where $\varphi^{(l+1)}(t)$ is a correction term.

We have an identity of the form
\[
f \cdot \varphi^{(l)} - \varphi^{(l)} \cdot g = E_{l+1}
\]
where $E_{l+1}$ is divisible by $\frac{1}{\mathfrak{a}}$ (because
modulo \( \pi \), \( f \) and \( g \) both act by the Frobenius, which commutes with everything.

Next, we have
\[
\begin{align*}
    f \circ \varphi^{l+1} &\equiv f \circ \varphi^l + \pi \cdot \varphi^{l+1} \quad \text{and} \\
    \varphi^{l+1} \circ g &\equiv \varphi^l \circ g + \pi^{l+1} \cdot \varphi^{l+1} \pmod{\mathbb{I}_{l+2}}
\end{align*}
\]

Thus
\[
    f \circ \varphi^{l+1} - \varphi^{l+1} \circ g \equiv E_{l+1} + (\pi - \pi^{l+1}) \varphi^{l+1},
\]
and since \( E_{l+1} \) is divisible by \( \pi \), we can choose \( \varphi^{l+1} \) so that the RHS is 0.

Moreover, the choice is unique. \( \Box \)

§7.11. Corollary. (1) The group \( \text{Aut}_{\mathcal{O}_K}(V) \) acts transitively on \( \text{LT}_{\pi} \).

(2) The centralizer in \( \text{Aut}_{\mathcal{O}_K}(V) \) of any \( f \in \text{LT}_{\pi} \) naturally identifies with \( \mathcal{O}_K \) (via the derivative map).

(3) \( \sqrt{V} \) carries a canonical structure of a formal group over \( \mathcal{O}_K \) such that the \( \mathcal{O}_K \)-action from (2) is "linear" with respect to this formal group structure.

In fact, this formal group structure is already uniquely determined by the condition that \( f \) is an endomorphism of this formal group structure.
87.12. Proof. (1) & (2) follow immediately from the case \( n=1 \) of Lemma 7.9.

(3) A formal group law on \( V \) is the same as a formal power series \( F(x,y) \in \mathcal{O}_K[[x,y]] \) such that \( F(x,y) \equiv x + y \mod (x,y)^2 \) and \( F \) is associative & commutative.

Now (3) follows by applying Lemma 7.9 with \( n=2 \) and \( n=3 \) (the latter is needed to prove associativity).

87.13. Definition. Let \( R \) be any commutative \( \mathcal{O}_K \)-module on \( \mathcal{O}_K \)-algebra. A formal \( \mathcal{O}_K \)-module on \( \text{Spec } R \) is a commutative 1-parameter formal group \( \mathcal{U} \) over \( R \) (i.e., \( \mathcal{U} \cong \text{Spf } \mathcal{R}[[t]] \)) equipped with an action of \( \mathcal{O}_K \) by endomorphisms such that the derivative of this \( \mathcal{O}_K \)-action is given by the homomorphism \( \mathcal{O}_K \rightarrow R \) defining the \( \mathcal{O}_K \)-algebra structure on \( R \).


Every Lubin–Tate \( \pi \)-series \( f \) defines a unique formal \( \mathcal{O}_K \)-module over \( \text{Spec } \mathcal{O}_K \).
Example. Consider $K = \mathbb{Q}_p$ and $f(t) = (1+t)^p - 1$. We have $f(t) \in \mathbb{L} T_p$. The corresponding formal group is the formal completion of the multiplicative group.

Exercise. Describe the induced action of $\mathbb{Z}_p$ on this formal group (in particular, $p \in \mathbb{Z}_p$ acts by the element $f$).

§7.16. Let $K$ be any local field as above, fix a uniformizer $\pi \in K$ and $f \in \mathbb{L} T_\pi$. Write $E(\pi)$ for the corresponding Lubin-Tate formal group.

For any finite extension $L \supseteq K$, we can consider $E(\pi)(M_L)$. By definition, this is the set $M_L$ equipped with the abelian group structure coming from the formal group structure on $E(\pi)$. Moreover, since $E(\pi)$ is a formal $O_K$-module, we see that $M_L$ becomes, in fact, an $O_K$-module.

Lemma. $E(\pi)(M^K_\Lambda)_{\text{tors}} \cong K/O_K$ as an $O_K$-module.

The torsion is taken with respect to the $O_K$-module structure.
87.17. Exercise. In the situation of 87.15, this becomes an isomorphism between the group of $p$-power roots of 1 in $\overline{\mathbb{Q}}_p$ and the group $\mathbb{Q}_p/\mathbb{Z}_p$. (Note that the isomorphism is non-canonical!)

87.18. Proof of Lemma 7.16. We may assume that $f(t) = t^q + \pi t$. Let $\mathcal{L} \in \mathcal{M}_K$. Since $f(t)$ is a separable polynomial, the preimage $f^{-1}(\mathcal{L})$ has $q^n$ elements.

The points in $E^{(\pi)}(\mathcal{M}_K)$ that are killed by $\pi^n$ are the same as the elements $x \in \mathcal{M}_K$ satisfying $(f \circ f \circ \cdots \circ f)(x) = 0$, because $\pi \in \mathcal{O}_K$ acts on $E^{(\pi)}(\mathcal{M}_K)$ via $f$. In particular, by the previous remark,

$$\left| \ker \left( \pi^n : E^{(\pi)}(\mathcal{M}_K) \to E^{(\pi)}(\mathcal{M}_K) \right) \right| = q^n$$

for every $n \in \mathbb{N}$. This immediately implies the assertion of the lemma.

87.19. Consider the action of $G_K = \text{Gal}(\overline{K}/K)$ on $E^{(\pi)}(\mathcal{M}_K)_{\text{tors}}$. It commutes with all the structures considered above, so it acts by $\mathcal{O}_K$-module endomorphisms.
Now \( \text{End}_{\mathcal{O}_K \text{-mod}} (K/\mathcal{O}_K) = \mathcal{O}_K \); in particular, we get a canonical homomorphism \( G_K \rightarrow \mathcal{O}_K^\times \). (The reason it is canonical, even though the isomorphism \( E(\pi)(M_K^\text{tors}) \cong K/\mathcal{O}_K \) is non-canonical, is that \( \mathcal{O}_K \) is commutative.)

**§7.20. Proposition.** The homomorphism

\[
G_K \rightarrow \mathcal{O}_K^\times
\]

we constructed is surjective.

**Remark.** Taking the invariants of the kernel of this homomorphism on \( \overline{K} \), we obtain an abelian extension of \( K \) inside \( \overline{K} \). From §7.18, we see that this extension can be obtained by adjoining all elements of \( M_{\overline{K}}^\text{tors} \) that are annihilated by a certain power of \( \pi \).

**Remark.** This whole story does not depend on the choice of \( \pi \in \mathcal{L}_{\pi} \), but it depends very seriously on the choice of \( \tau \in K \).

**§7.21.** Let us write \( \chi_{\pi} : G_K \rightarrow \mathcal{O}_K^\times \) for the character we constructed, and

\[
K^\tau = \{ \overline{K} \cap \text{Ker}(\chi_{\pi}) \}.
\]
Thus $K^{(n)}$ is the field generated over $K$ by $E^{(n)}(m_K)$ tors. Let us also write $K^{(n)}$ for the subfield of $\overline{K}$ generated by the points of $E^{(n)}(m_K)$ killed by $\pi^n$.

§7.22. Proof of Proposition 7.20. Again, we may assume that $f(t) = t^q + \pi t$. Let us first check that the composition

$$
G_K \xrightarrow{\chi_{\pi}} \mathbb{Q}_K^\times \rightarrow k^\times
$$

is surjective. By definition, $K^{(1)}$ is obtained from $K$ by adjoining the roots of $f(t)$; equivalently, the roots of $t^{q-1} + \pi$. This gives a cyclic extension of $K$ of degree $q-1$. On the other hand, it is easy to see that the induced map

$$
\text{Gal}(K^{(1)}/K) \rightarrow k^\times
$$

is injective (more or less by the def. of $K^{(1)}$). Since both groups have order $q-1$, this map is also surjective.

More generally, to see that

$$
G_K \xrightarrow{\chi_{\pi}} \mathbb{Q}_K^\times \rightarrow (\mathcal{O}_K/M^n_K)^\times
$$

is surjective for every $n \in \mathbb{N}$, we use the same argument, noting that $f_0 \ldots \pi_0$ is an Eisenstein polynomial, whence irreducible.
§7.23. Remark. The proof given above shows that $\pi$ lies in the image of

$$N_{K^{\pi}/K} : (K_{n}^{\pi})^{\times} \rightarrow K^{\times}$$

for every $n \in \mathbb{N}$.

§7.24. The moral of this story is that the Lubin-Tate approach gives another construction of the main isomorphism of local CFT (in order to prove that this is the correct answer, one needs to use cohomological techniques).

Recall from the classical local CFT:

$$K_{\text{ab}}$$

where $\pi$ acts via the identification $\text{Gal}(K_{\text{ab}}/K) \cong \mathbb{R}^{\times}$

Theorem. (1) We have $(K_{\text{ab}})^{\pi} = K^{\pi}$

(2) The two induced actions of $O_{K}^{\times}$ on $(K_{\text{ab}})^{\pi}$, one coming from classical local CFT and the other coming from the Lubin-Tate theory, are inverse to each other.
§7.25. Idea of the proof of Theorem 7.24: to see what happens to the Lubin-Tate picture when we change it. It is difficult to analyze the situation over $\mathcal{O}_K$; however, once we base change to $\mathcal{O}_K\text{nr}$, we see that all the Lubin-Tate formal groups become isomorphic to each other.

The details of the proof will be explained next time.