\[8.1.\] Correction to last time. In the description of the category \( \mathcal{C}_G \) by generators and relations given last time, we forgot the simplest possible relation:

\[
\begin{array}{ccc}
\mathcal{S}_G & \cong & \mathcal{E}_G \\
\mathcal{S}_G^\text{op} & \longrightarrow & \mathcal{E}_G \\
\end{array}
\]

Here, \( \mathcal{S}_G \) is the category of \( G \)-sets, \( \mathcal{S}_G^\text{op} \) is the category obtained from \( \mathcal{S}_G \) by keeping only isomorphisms (and discarding all the non-invertible arrows). The relation we forgot to add is that the diagram above should commute.

Example. Let \( G = \mathbb{Z}/p\mathbb{Z} \). A \( G \)-modulation is a diagram \( V_1 \xleftarrow{\beta} V_G \), where \( V_1 \) is an abelian group, \( V_G \) is a \( G \)-module, \( \beta \circ x = \beta \), \( 0 \circ x = 0 \), \( \forall g \in G \), \( \beta \circ g = \beta \), \( \forall g \in G \).
Morally, $\mathbf{a}$ should be thought of as multiplication by $p$, and $\mathbf{g}$ should be thought of as the averaging operator $\frac{1}{p} \sum_{g \in G} g$. Note, however, that the result of the averaging lives in a different space.

\[ \mathbf{2.2.} \] Example of a "double functor" on the category $\mathcal{S}_G : X \mapsto K_0(\mathcal{S}_G(X))$, where $\mathcal{S}_G(X)$ is the category of $G$-equivariant sheaves on $X$. This has a chance of being a modulation, because we have both the pullback and the pushforward functors for equivariant sheaves.

However, the last axiom in the definition of a modulation is not satisfied. In fact, the assignment $X \mapsto K_0(\mathcal{S}_G(X))$ gives what is known as a "Mackey functor" (the notion of a Mackey functor is obtained from our description of a $G$-modulation by discarding the last relation). The notion of a $G$-modulation itself was introduced by Neukirch.
Let us return to local CFT. We have two natural modulations:

\[ K \mapsto G^a_K = \text{Gal}(K^{ab}/K) \]

and \[ K \mapsto \hat{K}^x \]

(To really think of this as a modulation, we should fix a local field \( E \), and consider only finite separable extensions \( K \) of \( E \). Then we get two \( G_E \)-modulations.)

Recall that we have natural quotient maps

\[ G^a_K \longrightarrow \text{Gal}(K^{nr}/K) \cong \hat{\mathbb{Z}} \leftarrow \text{val} \hat{K}^x \]

In fact, we can think of the assignment

\[ K \mapsto \hat{\mathbb{Z}} \]

as a modulation as well, where for a finite extension \( L \supset K \) of local fields, the corresponding maps are:

\[ L \leftarrow \hat{\mathbb{Z}} \quad \text{mult. by } e(L/K) \]

\[ \text{mult. by } f(L/K) \downarrow \]

\[ K \leftarrow \hat{\mathbb{Z}} \]

Here, \( e(L/K) \) = ramification degree for \( L \) over \( K \)

\( f(L/K) \) = degree of the residue field extension for \( L \) over \( K \).
§2.4. Now we can state a more precise version of Theorem 1.8 from the last lecture.

**Theorem.** There exists a unique isomorphism of modulations between $K \mapsto G_K^\text{ab}$ and $K \mapsto \hat{K}^\times$ which is compatible with the two natural homomorphisms $G_K^\text{ab} \to \hat{Z} \leftarrow \hat{K}^\times$.

Recall that the uniqueness statement was proved in the previous lecture. One way to construct the desired reciprocity maps is using a method due to Tate and Nakayama.

§2.5. Before describing the Tate–Nakayama story, we need a general digression.

**Tate cohomology of finite groups.**

Let us fix a finite group $G$ and define

$$\mathcal{H}(G) = \text{the additive category formed by the } \mathbb{Z}[G]\text{-modules that are free as } \mathbb{Z}\text{-modules (i.e., free abelian groups equipped with a } G\text{-action).}$$

The morphisms in this category are defined as follows:

$$\text{Hom}_{\mathcal{H}(G)}(M, N) = \text{Coker} \left( \text{Hom}_{\mathbb{Z}[G]}(M, N) \xrightarrow{\sum_{g \in G} g} \text{Hom}_{\mathbb{Z}[G]}(M, N) \right).$$
The composition is defined unambiguously, because homomorphisms obtained by applying $\Sigma g_i$ form an ideal in the collection of all $g_i \in G$-equivariant homomorphisms.

**Exercise.** A $G$-module homomorphism $M \overset{f}{\to} N$ becomes trivial in $T(G)$ if and only if it factors through a free $\mathbb{Z}[G]$-module.

**Remark.** So $T(G)$ is what some people call the “stable category of $\mathbb{Z}[G]$-modules” (some kind of a quotient of the category of $\mathbb{Z}$-free $\mathbb{Z}[G]$-modules by the subcategory of all projective $\mathbb{Z}[G]$-modules; of course, this is not a Serre quotient!).

The category $T(G)$ has an obvious monoidal structure.

---

\[ \text{2.6.} \] The category $T(G)$ also has a natural triangulated structure. First, given $M \in T(G)$, let us describe $M[1]$. There exists an embedding $M \hookrightarrow F$ of $\mathbb{Z}[G]$-modules, where $F$ is a free $\mathbb{Z}[G]$-module and $F/M$ is free as an abelian group. For instance, we could take $F = \mathbb{Z}[G] \otimes \mathbb{Z}^\infty M$. We define $M[1] = F/M$, and it is easy to check that this is independent of the choice of $F$. 
Similarly, to define $M[-1]$, we consider a surjection of $\mathbb{Z}[G]$-modules $F' \to M$, where $F'$ is a free $\mathbb{Z}[G]$-module, and define $M[-1] = \text{Ker}(F' \to M)$.

**Definition of distinguished triangles.**

First we will describe a special collection of triangles by defining the cone of any morphism in $\mathcal{T}(G)$, and then a distinguished triangle will be defined to be a triangle isomorphic to one of the cone diagrams.

Consider a morphism $f : M \to N$. Choose an embedding $M \to F$ as in the definition of $M[1]$. Then we obtain an embedding $M \to F \oplus N$, and we define $C(f) = (F \oplus N)/M$. Moreover, we obviously obtain a diagram

$$M \to N \to C(f) \to M[1] = F/M.$$ 

This is what we call a "cone diagram", or a "standard distinguished triangle". As explained earlier, we define a distinguished triangle in the category $\mathcal{T}(G)$ to be any triangle which is isomorphic to one of these standard ones.

We omit the verification of the axioms defining a triangulated category.
3.2.7. Another construction of \( T(G) \).

In \( \text{Db}(\mathbb{Z}[G]-\text{mod}) \), we have a full subcategory formed by the perfect complexes, i.e., bounded complexes of arbitrary projective \( \mathbb{Z}[G] \)-modules. One can show that \( T(G) \) is equivalent to the corresponding Verdier quotient.

3.2.8. A third construction of \( T(G) \).

It can also be described as the homotopy category \( K(\text{proj. } \mathbb{Z}[G]-\text{mod}) \) acyclic of arbitrary acyclic complexes of projective \( \mathbb{Z}[G] \)-modules.

3.2.9. Let us describe various functors

\[
\begin{align*}
\mathcal{T}(G) & \quad \xrightarrow{1} \quad \text{Db}(\mathbb{Z}[G]-\text{mod}) \\
& \quad \xrightarrow{2} \quad K^b(\text{proj. } \mathbb{Z}[G]-\text{mod}) \\
& \quad \xrightarrow{3} \quad K(\text{proj. } \mathbb{Z}[G]-\text{mod}) \text{ acyclic}
\end{align*}
\]

The functor 1 is stupid: take any \( \mathbb{Z}[G] \)-module and put it in degree 0.

Let us describe 2. Pick \( M \in \text{Db}(\mathbb{Z}[G]-\text{mod}) \).

We may assume that the terms of \( M \) are free as abelian groups. Now let \( M \xrightarrow{f_\ell} M \) be a quasi-isomorphism, where \( M_\ell \) is a
bounded above complex of free \( \mathbb{Z}[G] \)-modules (there is no problem in constructing it because \( M \) is bounded). In fact, we can also construct a quasi-isomorphism \( M \to M_r \), where \( M_r \) is a bounded below complex of free \( \mathbb{Z}[G] \)-modules.

The composition \( M \to M_r \) is also a quasi-isomorphism, so its cone is an object of \( K(\text{proj. } \mathbb{Z}[G]-\text{mod}) \) acyclic. This is the value of the functor \( \mathbb{2} \) at \( M \).

Finally, the functor \( \mathbb{3} \) is defined by

\[
C \to \ker(d^0 : C^0 \to C^1).
\]

**Remark.** All of this really only depends on the fact that \( \mathbb{Z}[G] \) is a Frobenius algebra in the symmetric monoidal category of free abelian groups.

**82.10. Definition of Tate cohomology.**

If \( M \) is a \( \mathbb{Z}[G] \)-module, or a bounded complex of \( \mathbb{Z}[G] \)-modules, we define the Tate cohomology groups of \( M \) by

\[
\hat{H}^i(G, M) := \text{Hom}_{ \mathcal{T}(G) } (\hat{M}, \hat{M}[i]),
\]

where \( \hat{M} \in \mathcal{T}(G) \) is the object corresponding to \( M \).
§2.11. Explicit description of Tate cohomology groups. Let $M$ be a single $\mathbb{Z}[G]$-module, which is not necessarily free as an abelian group.

**Lemma.** There exists a $\mathbb{Z}[G]$-module $\hat{M}$ and a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to K \to \hat{M} \to M \to 0$$

such that:
- $\hat{M}$ is free as an abelian group
- $K$ is free as a $\mathbb{Z}[G]$-module.

The proof is left as a simple exercise.

Note that, by definition, $\hat{M}$ is the object of $\mathcal{F}(G)$ corresponding to $M$.

**Proposition.** If $M$ is any $\mathbb{Z}[G]$-module, we have:

- $\widehat{H}^i(G, M) \cong H^i(G, M)$ if $i \geq 1$;
- $\widehat{H}^i(G, M) \cong H_{-i-1}(G, M)$ if $i \leq -2$;
- $\widehat{H}^0(G, M) \cong \ker (M_G \xrightarrow{\Sigma g} M_G)$;
- $\widehat{H}^{-1}(G, M) \cong \ker (M_G \xrightarrow{\Sigma g} M_G)$.

§2.12. **Proof.** The left hand side is obviously a cohomological functor on the category of $\mathbb{Z}[G]$-modules. Let us show that so is the right hand side.
To this end, we will interpret the RHS as the cohomology of some naturally constructed complex. Choose:

\[
\mathbb{Z}_l \xrightarrow{q_{is}} \mathbb{Z} \xrightarrow{q_{is}} \mathbb{Z}_r
\]

bounded above complex of free \( \mathbb{Z}[G] \)-modules

Then \( H^c(G, M) \cong H^{-c}(M \otimes_{\mathbb{Z}[G]} \mathbb{Z}_l) \) and \( H^c(G, M) \cong H^i(M \otimes_{\mathbb{Z}[G]} \mathbb{Z}_r) \).

If \( C = \text{cone}(\mathbb{Z}_l \to \mathbb{Z}_r) \), then the RHS in the formula of Proposition 2.11 amounts to the cohomology of \( M \otimes_{\mathbb{Z}[G]} C \).

This easily implies that the RHS is a cohomological functor. Finally, one checks that if \( M \) is free as an abelian group, then both sides give the same answer for \( c = 0 \).

This is enough.

\[\text{Exercise.} \]

Note that the complex \( C \) above was constructed by applying the functor (2) described in §2.8 to the object \( \mathbb{Z} \) of \( \text{D}^b(\mathbb{Z}[G]-\text{mod}) \).
More generally, let \( M \) and \( N \) be arbitrary \( \mathbb{Z}[G] \)-modules that are free as abelian groups, and apply the same construction:

\[
M \to M' \to M'', \quad N \to N' \to N''.
\]

Is it true that

\[
\text{Hom}_{K(\mathbb{Z}[G]-\text{mod})} \left( \text{Cone}(M \to M''), \text{Cone}(N \to N'') \right)
\]

\[
\cong (\text{Cone}(M \to M''), N) = \text{Hom}_{K(\mathbb{Z}[G]-\text{mod})} (M, \text{Cone}(N \to N'')) ?
\]

If so, explain this.

\[\text{§2.14.}\] Next we will prove a theorem due to Tate which explains why Tate cohomology groups are useful.

Theorem (Tate). Let \( M \in \mathcal{T}(G) \). TFAE:

(i) \( M = 0 \) in \( \mathcal{T}(G) \) (equivalently, \( M \) can be represented by a projective \( \mathbb{Z}[G] \)-module)

(ii) \( M \) is cohomologically trivial, which, by definition, means that

\[
\hat{H}^i(H, M) = 0 \quad \text{for every } i \in \mathbb{Z} \text{ and every subgroup } H \triangleleft G.
\]
(iii) For any prime \( p \), there exists \( i(p) \in \mathbb{Z} \) such that if \( H \leq G \) is a \( p \)-Sylow subgroup, then

\[
\hat{\chi}(p)^{(H, M)} = 0 = \hat{\chi}(p+1)^{(H, M)}.
\]

(Since all \( p \)-Sylow subgroups of \( G \) are conjugate, this condition does not depend on the choice of \( H \).)

§2.15. Proof of Theorem 2.14. Clearly, it suffices to show that \((iii) \Rightarrow (i)\). We think of \( M \) as a \( \mathbb{Z}[G] \)-module which is free as an abelian group and we assume that \( M \) satisfies \((iii)\). We need to show that \( M \) is projective as a \( \mathbb{Z}[G] \)-module.

Step 1. Observe that it suffices to verify that for any \( \mathbb{Z}[G] \)-module \( K \) which is free as an abelian group, the \( G \)-module \( \text{Hom}_\mathbb{Z}(M, K) \) has the property

\[
\hat{H}^1(G, \text{Hom}_\mathbb{Z}(M, K)) = 0.
\]

Indeed:

Exercise. Under our assumptions,

\[
\hat{H}^1(G, \text{Hom}_\mathbb{Z}(M, K)) \cong \text{Ext}^1_{\mathbb{Z}[G]}(M, K).
\]

Of course, the vanishing of \( \text{Ext}^1_{\mathbb{Z}[G]}(M, K) \) for every \( \mathbb{Z}[G] \)-module \( K \) which is free as a \( \mathbb{Z} \)-module is equivalent to the projectivity of \( M \).

Step 2. In fact, we will show that under the above assumptions, the \( G \)-module \( \text{Hom}_\mathbb{Z}(M, K) \) is cohomologically trivial. (maybe skip this?)
In fact, it suffices to show that
\[ \hat{H}^i(G, \text{Hom}_\mathbb{Z}(M, K)) = 0 \]
for every \(p\)-Sylow subgroup \(G_p \subset G\) and all \(i\).

[This is a general fact: if \(N\) is any \(G\)-module and \(\hat{H}^i(G, N) = 0\) for every \(p\)-Sylow subgroup \(G_p \subset G\) for all \(p\), then \(\hat{H}^i(G, M) = 0\).]

Thus we are reduced to the case where \(G\) is a finite \(p\)-group.

\[ \textbf{Lemma.}\]
Let \(p\) be a fixed prime, and let \(M\) be a \(\mathbb{Z}[G]\)-module which is free as a \(\mathbb{Z}\)-module.
Let \(G\) be a finite \(p\)-group, and let \(M\) be a \(\mathbb{Z}[G]\)-module which is free as a \(\mathbb{Z}\)-module. If \(\hat{H}^{-2}(G, M) = 0 = \hat{H}^{-1}(G, M)\), then \(M/pM\) is free as an \(\mathbb{F}_p[G]\)-module.

\[ \text{Proof.}\]
The exact sequence
\[ 0 \rightarrow M \rightarrow M \rightarrow M/pM \rightarrow 0 \]
implies that \(H_1(G, M/pM) = \hat{H}^{-2}(G, M/pM) = 0\).

We will show that this implies that \(M/pM\) is free as an \(\mathbb{F}_p[G]\)-module.

Choose a surjection \(F \rightarrow M/pM\) of \(\mathbb{F}_p[G]\)-modules, which is an isomorphism on the coinvariants, and is such that \(F\) is free as an \(\mathbb{F}_p[G]\)-module.
Exercise. Using the fact that
\[ H_1(G, \mathbb{M}/p\mathbb{M}) = 0, \]
show that the map \( F \rightarrow \mathbb{M}/p\mathbb{M} \) must be an isomorphism. This uses the fact that \( G \) is a \( p \)-group, which implies that the augmentation ideal \( \text{Ker}(\mathbb{F}_p[G] \rightarrow \mathbb{F}_p) \) in the group algebra \( \mathbb{F}_p[G] \) is nilpotent.

\[ \text{completion of the proof of Theorem 2.14} \]

We are in the following situation: \( G \) is a finite \( p \)-group and \( \mathbb{M} \) is a \( \mathbb{Z}[G] \)-module, which is free as a \( \mathbb{Z} \)-module and satisfies
\[ \hat{H}^i(G, \mathbb{M}) = 0 = \hat{H}^{i+1}(G, \mathbb{M}) \] for some \( i \in \mathbb{Z} \).

This still implies that \( \mathbb{M}/p\mathbb{M} \) is free as an \( \mathbb{F}_p[G] \)-module. Indeed, replacing \( \mathbb{M} \) with \( \mathbb{M}[+i+2] \in \mathbb{F}_p[G] \), we see that
\[ \mathbb{M}[+i+2]/p\mathbb{M}[+i+2] \]
\( \Rightarrow \) all of the Tate cohomology groups of \( \mathbb{M}[i+2] \) vanish \( \Rightarrow \) the same is true for \( \mathbb{M} \), and now we can apply Lemma 2.16.

Finally, we take any \( \mathbb{Z}[G] \)-module \( K \) which is free as a \( \mathbb{Z} \)-module. Then
\[ \text{Hom}_\mathbb{Z}(\mathbb{M}, K)/p \cong \text{Hom}_\mathbb{Z}(\mathbb{M}/p\mathbb{M}, K/pK) \]
is free as an \( \mathbb{F}_p[G] \)-module, by the argument above.
This implies that all the Tate cohomology groups of \( \text{Hom}_\mathbb{Z}(M, K)/\mathfrak{p} \) vanish. However, in view of the short exact sequence

\[
0 \rightarrow \text{Hom}_\mathbb{Z}(M, K) \rightarrow \text{Hom}_\mathbb{Z}(M, K) / \mathfrak{p} \rightarrow 0,
\]

and the fact that all the Tate cohomology groups of any \( G \)-module are annihilated by a power of \( \mathfrak{p} \), this implies that all the Tate cohomology groups of \( \text{Hom}_\mathbb{Z}(M, K) \) vanish as well. This completes the proof.

§2.18. We now return to our study of local fields.

**Key fact about Galois cohomology.** Let \( K \) be a local field, let \( K^{nr} \) be a maximal unramified extension of \( K \), and consider finite extensions \( K^{nr} \subset S \subset T \), such that \( T \) is Galois over \( S \). Put \( G = \text{Gal}(T/S) \).

**Proposition.** We have \( \hat{H}^0(G, T^x) = 0 \), or, equivalently, \( T^x \) vanishes in the category \( T(G) \).

**Proof.** We have \( \hat{H}^1(G, T^x) = 0 \) for free from Hilbert's Theorem 90. By Tate's
theorem, it is enough to do either of the following things:

(i) check that \( H^2(G, T^x) = 0 \), i.e., that \( S \) has trivial Brauer group;

(ii) to check that the norm homomorphism

\[
N_{T/S} : T^x \longrightarrow S^x
\]

is surjective.

Neither of these is very difficult. For instance, for (i), one can prove that every central division algebra over \( S \) splits over some finite unramified extension of \( S \). However, since \( S \) is a finite extension of \( K_{nr} \), it has no nontrivial unramified extensions (its residue field is separably closed).

I will skip this part (the fact I just stated was proved in my Summer 2007 lectures). In a later lecture we will explain a purely cohomological proof of this statement.

**§2.19.** Now we will apply the key fact proved above. We just remarked that

\[
H^2(K_{nr}/K, (K_{nr})^x) = Br(K)
\]

**iff**

\[
H^2(Gal(K_{nr}/K), (K_{nr})^x).
\]
This allows us to compute the Brauer group, $\text{Br}(K)$, of $K$. Namely, we have a split exact sequence of $\text{Gal}(K^{nr}/K)$-modules

$$0 \rightarrow \mathcal{O}^{x}_{K^{nr}} \rightarrow (K^{nr})^{x} \xrightarrow{\text{val}} \mathbb{Z} \rightarrow 0$$

Moreover, $H^i(\text{Gal}(K^{nr}/K), \mathcal{O}^{x}_{K^{nr}}) = 0$ for all $i \geq 1$, because it is already true "at the finite level", i.e., for finite unramified extensions of $K$ (again, see my summer 2007 lectures).

Thus we only need to compute $H^i(\hat{\mathbb{Z}}, \mathbb{Z})$ (recall that $\text{Gal}(K^{nr}/K) \cong \hat{\mathbb{Z}}$).

The short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

gives $H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \cong H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

So $\text{Br}^{-}(K) \rightarrow \mathbb{Q}/\mathbb{Z}$ canonically.

§2.20. Now the assignment $K \mapsto \text{Br}(K)$ is a modulation. Let us compute the corresponding maps

$$\begin{align*}
\text{Br}(L) & \xrightarrow{\text{inv}_L} \mathbb{Q}/\mathbb{Z} \\
\text{Br}(K) & \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z}
\end{align*}$$
Note that the composition of the up and down arrows is multiplication by \( n = [L:K] \). Moreover, it is easy to check that the up arrow is multiplication by \( n \). Therefore, the down arrow has to be the identity map \( \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \), because
\[
\mathbb{Z} \xrightarrow{\cong} \text{End}_{\mathbb{Z}-\text{mod}}(\mathbb{Q}/\mathbb{Z}).
\]

In this situation, if \( L/K \) is Galois, we obtain
\[
H^2(\text{Gal}(L/K), L^\times) = \ker(\text{Br}(K) \to \text{Br}(L)) \cong \left( \frac{1}{n} \mathbb{Z} \right)/\mathbb{Z}
\]

Let \( \delta_{L/K} \in H^2(\text{Gal}(L/K), L^\times) \) be the element corresponding to \( \frac{1}{n} \) under this identification. Now \( H^2(G, L^\times) = \hat{H}^2(G, L^\times) \) (\( G = \text{Gal}(L/K) \)), so we can think of \( \delta_{L/K} \) as a morphism \( \mathbb{Z} \to L^\times[2] \) in the category \( T(G) \).

\[\text{Lemma.} \quad \text{Let } L \supseteq K \text{ be a finite Galois extension of local fields, with } n = [L:K] \text{ and } G = \text{Gal}(L/K). \text{ The morphism } \mathbb{Z} \to L^\times[2] \text{ constructed above is an isomorphism in } T(G).\]

\[\text{Proof.} \quad \text{We need to show that the cone of } \delta_{L/K} : \mathbb{Z} \to L^\times[2] \text{ is cohomologically trivial (by Tate's theorem). This means that we need to find } \iota \in \mathbb{Z} \text{ with the}\]

Following properties:

1. \( \hat{H}^{i+1}(\mathbb{Z}) \to \hat{H}^{i+3}(L^x) \)

2. \( \hat{H}^{i}(\mathbb{Z}) \to \hat{H}^{i+2}(L^x) \)

3. \( \hat{H}^{i-1}(\mathbb{Z}) \to \hat{H}^{i+1}(L^x) \)

We will take \( i = 0 \). This is an easy exercise.

Namely, (1) holds because \( \hat{H}^{2}(\mathbb{Z}) = 0 \),

(3) holds because \( \hat{H}^{1}(L^x) = 0 \) (by Hilbert's Theorem 90), and (2) holds because we can identify both sides with \( \mathbb{Z}/n\mathbb{Z} \) and compute the map in question explicitly.

32.22. It follows that \( \delta_{L/K} \) induces an isomorphism on every Tate cohomology group:

\[ \hat{H}^i(G, \mathbb{Z}) \to \hat{H}^{i+2}(G, L^x) \quad \forall i \in \mathbb{Z}. \]

Let us now take \( i = -2 \). This produces a canonical isomorphism

\[ \hat{H}^{-2}(G, \mathbb{Z}) \to \hat{H}^0(G, L^x) \]

\[ \ker \to \mathbb{K}^x/N_{L/K}(L^x) \]

This is the desired reciprocity map.

We have described the Tate-Nakayama construction of the local reciprocity maps.