Here are some references for the course:

1. Neukirch, “Algebraic Number Theory”
2. Neukirch, Schmidt and Wingberg, “Cohomology of Number Fields”
3. Artin and Tate, “Class Field Theory”
5. Fesenko and Vostokov, “Local Fields and their Extensions”
6. Serre, “Local Fields” ("Corps Locaux")
7. Serre, “Galois Cohomology” ("Cohomologie Galoisiennne")
8. Serre, “Algebraic Groups and Class Fields”
9. Milne, “Arithmetic Duality Theorems” (might contain mistakes)
31.2. Here is some kind of a table of contents for the course.

(1) **Local CFT.** Let us recall the main result of this theory, before continuing with the table of contents. The main goal is to understand the absolute Galois group of a local field in terms of the field itself (not in terms of its extensions). This is difficult, but the maximal abelian quotient of the absolute Galois group turns out to be more manageable.

**Main Result.** Let $K$ be a local field. There exists a natural homomorphism with dense image, $\mathbb{K}^\times \to \text{Gal}(\overline{K}/K)^{\text{ab}}$.

Moreover, if we replace $\mathbb{K}^\times$ with its completion with respect to the topology defined by the open subgroups of finite index, the induced map becomes an isomorphism $\hat{\mathbb{K}}^\times \cong \text{Gal}(\overline{K}/K)^{\text{ab}}$.

Do not confuse with the group of characters of $\mathbb{K}^\times$.

How can we visualize this isomorphism?

Here is a slightly more precise statement:
We have a commutative diagram:

\[
\begin{array}{ccc}
\hat{\mathbb{Z}} & \longrightarrow & \text{Gal}(\overline{K}/K) \\
\downarrow & & \downarrow \\
\widehat{(K^*)^\times} & \longrightarrow & \text{Gal}(K^{nr}/K)
\end{array}
\]

Here, $K^{nr}$ is the maximal unramified extension of $K$, and $\overline{K}$ is the residue field of $K$.

31.3. Example. Take $K = \mathbb{Q}_p$. We will show later in the course that $\mathbb{Q}_p^{ab} = \mathbb{Q}_p(\mu_{\infty})$ (where $K^{ab}$ denotes the maximal abelian extension of $K$). Let us now explain how to construct the isomorphism

\[\hat{\mathbb{Q}}_p^\times \overset{\sim}{\longrightarrow} \text{Gal}(\mathbb{Q}_p(\mu_{\infty})/\mathbb{Q}_p)\]

"by hand". First recall that $\mathbb{Q}_p^\times = \mathbb{Z} \times \mathbb{Z}_p^\times$.

On the other hand,

\[\mathbb{Q}_p(\mu_{\infty}) = \mathbb{Q}_p(\mu_{p^\infty}) \cdot (\mathbb{Q}_p(\mu_{p^\infty}))^\prime\]

where $\mu_{p^\infty}$ denotes the group of roots of unity of order prime to $p$. 
Moreover, \( Q_p(\mathbb{M}_p^\circ) = Q_p^{nr} \).

Now \( \mathbb{Z}_p^x \) acts on the group \( \mathbb{M}_p^\circ \) in the obvious way (if \( a \in \mathbb{Z}_p^x \), it acts on \( \mathbb{M}_p^\circ \) via \( u \mapsto u\alpha^{-1} \), which has an obvious definition when \( \alpha^{-1} \in \mathbb{Z} \), and can be extended to all of \( \mathbb{Z}_p \) by continuity), which gives an action of \( \mathbb{Z}_p^x \) on \( Q_p(\mathbb{M}_p^\circ) \) by \( Q_p \)-automorphisms.

§1.4. Now we continue with the table of contents.

1. The main isomorphism of local CFT (in fact, today we will write a more or less explicit formula for this isomorphism, due to Neukirch).

2. Hatzelwinkel's construction of the isomorphism \( \hat{\mathbb{Z}}^x \cong \text{Gal}(\overline{K}/K)_{ab} \) (the opposite direction to the one considered by Neukirch).

3. The Nakayama-Tate approach to local CFT (based on the notion of a "class formation"). This approach is less explicit than the first two.

4. Lubin-Tate theory. (It is about explicitly constructing \( K_{ab} \) together with an action of the group \( \hat{\mathbb{Z}}^x \).)
Remark: All the four approaches to local CFT mentioned above are very complementary to each other.

(5) Poincaré duality (an approach to class field theory through Galois cohomology)

We think of a local field (from the point of view of cohomology) as a 2-dimensional object, more or less fibered over a circle with the fibers also being circles (but the fibration is nontrivial, so we should not think of a local field as a torus).

This approach is due to Tate (originally); it was later developed (to take p-torsion coefficients into account) by Fontaine-Herr. In fact, this is a part of Fontaine's theory of p-adic Galois representations.

(6) Geometric CFT (Lang & Serre) and its de Rham version (which is a local theory of PDE with analytic coefficients).

(7) Ramification theory (the theory of conductors and \( \varepsilon \)-factors).

Note that all of the topics listed above have to do with the local theory.
§1.5. Here are the topics which we hope to cover that have to do with global class field theory.
(6) Neukirch's explicit approach
(9) Global CFT via the approach of Nakayama and Tate (main ingredient: computation of the Brauer group of a global field, which can be done quite easily using zeta functions of central division algebras)
(10) Poincaré duality (an approach due to Poitou and Tate); computations of some important Euler characteristics (Tate)

§1.6. An aside. Let us explain the seemingly strange appearance of inverses in our description of the action of $\mathbb{Z}^\times_p$ on $\mathbb{Q}_p(M_p)$. The reason comes from the requirement of compatibility of local CFT with global CFT. Let us fix a global field $K$. We have

\[ \frac{11}{\mathfrak{m}_1} \mathbb{Gal}(K_0/K_0)^{ab} \rightarrow \mathbb{Gal}(K/K)^{ab} \]

something in between $\mathbb{Q}$ and $\mathbb{Q}

\( \frac{11}{\mathfrak{m}_2} \mathbb{R}_v \) local CFT

the kernel of this map is $K^\times$
From a certain viewpoint, global CFT is a little bit easier than local CFT, due to the fact that every finite extension of $K$ is unramified at all but finitely many places, which determines the map

$$\prod \hat{K}_v^\times \to \text{Gal}(\overline{K}/K)^{ab}$$

essentially uniquely. This, in turn, determines the normalization of the main isomorphism of local CFT, by compatibility with the global theory.

§1.7 Table of contents, continued

(11) Geometric global CFT (Lang, Serre)
(12) The global version of the Lubin-Tate theory exists only for certain classes of global fields. (This has to do with complex multiplication and Drinfeld's theory of elliptic modules.)
(13) Analytic theory: zeta-functions and $L$-functions (Hasse, Iwasawa-Tate)
(14) Values of $L$-functions at negative integers (Shintani; and a much more beautiful and canonical approach due to Nori)
(15) Iwasawa theory (Kolyvagin's theory of Euler systems)
§1.8. Neukirch's approach to local CFT.

The first thing we need to understand is:

how to make the picture

\[ \text{Gal} \left( \mathbb{K}/\mathbb{K} \right)^{ab} \xrightarrow{\cong} \hat{\mathbb{K}}^\times \]

\[ \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\nu} \]

more rigid? Note first that if we have a finite extension $L \supset K$, then we have the inclusion map $K^\times \hookrightarrow L^\times$, and a canonical map $\text{Gal}(L^{ab}/L) \rightarrow \text{Gal}(K^{ab}/K)$.

At first glance this seems inconsistent. Fortunately, we also have the norm homomorphism $L^\times \longrightarrow K^\times$.

Let us formulate the main result.

**Theorem.** There exists a unique system of isomorphisms

\[ \hat{r}_K : \text{Gal}(K^{ab}/K) \xrightarrow{\cong} \hat{K}^\times \]

one for each local field $K$ (where "$r$" stands for "reciprocity"), subject to two compatibility conditions:

(i) $\hat{r}_K$ is compatible with projections to $\hat{\mathbb{Z}}$

(ii) for every finite extension $L \supset K$, the following diagram commutes:
§1.9. First we would like to prove the uniqueness part of the theorem above.

Let us consider a finite extension \( L \supseteq K \), and define
\[
\text{Gal}_{ab} (L/K) := \text{Coker} (\text{Gal}_{ab} (L) \to \text{Gal}_{ab} (K)),
\]
where \( \text{Gal}_{ab} (K) \) def \( = \text{Gal}(K_{ab}/K) \).

This is the Galois group of the maximal abelian extension of \( K \) contained in \( L \).

The diagram in Theorem 1.8 (ii) must then induce an isomorphism
\[
\text{Gal}_{ab} (L/K) \cong \hat{K}^\times / N_{L/K}(L^\times).
\]

**Theorem.** Fix a local field \( K \).

(a) For every finite extension \( L \supseteq K \), \( N_{L/K}(L^\times) \) is an open subgroup of finite index in \( K^\times \), (This is fairly easy, and is left as an exercise.)

(b) Every open subgroup of finite index in \( K^\times \) contains a subgroup of the form \( N_{L/K}(L^\times) \) for some finite extension \( L \supseteq K \). (This part is much harder, and will be proved later.)

For now, we assume the statement of this theorem.
Thus we see that if the reciprocity maps \( r_K \) as in Theorem 1.8 exist, they give rise to identifications

\[
\rho_{L/K} : \text{Gal}(L/K) \cong K^\times / N_{L/K}(L^\times).
\]

(Conversely, given a system of isomorphisms like this, for all finite extensions \( L \supseteq K \) of local fields, and if they satisfy suitable compatibility conditions, then we can define the maps \( r_K \) by a limiting procedure.)

\[\begin{array}{c}
\text{§1.10.} \\
\text{We keep working with a finite extension of local fields } L \supseteq K.
\end{array}\]

Let us assume that it is Galois, since for the uniqueness part of Theorem 1.8, we only need to consider Galois extensions.

We have the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\text{nr}} & L^{\text{nr}} \\
\downarrow & & \downarrow \\
L \cap K^{\text{nr}} & \xrightarrow{\text{Galois group } \mathbb{Z}/m\mathbb{Z}} & K \\
\end{array}
\]

For some \( m \).
We have:
\[ \text{Gal}(L^{nr}/K) \xrightarrow{\alpha} \text{Gal}(L/K) \times \text{Gal}(K^{nr}/K) \]

\[ \downarrow \]
\[ \text{Gal}(L/K) \times \frac{\mathbb{Z}}{\mathbb{Z}/m\mathbb{Z}} \]

Now consider the following set:
\[ \mathcal{Q} := \{ L^{nr} \supset F \supset K \mid F/K \text{ is finite, } L^{nr}/F \text{ is unramified} \} \]

Claim: \( \mathcal{Q} \) can be identified with the set \( \{ \psi \in \text{Gal}(L^{nr}/K) \mid \text{the image of } \psi \text{ in } \text{Gal}(K^{nr}/K) \cong \mathbb{Z} \text{ is a natural number} \} \).

Proof: We have a map from \( \mathcal{Q} \) to this set, given by sending \( F \) to the corresponding Frobenius \( \psi = F^{L^{nr}/F} \in \text{Gal}(L^{nr}/F) \subset \text{Gal}(L^{nr}/K) \), because \( L^{nr} = F^{nr} \).

This makes sense, because \( L^{nr} = F^{nr} \).

The map in the opposite direction is given by \( \psi \mapsto (L^{nr})^\psi \). Let us check that \( (L^{nr})^\psi \) is finite over \( K \).
Clearly, it is enough to do this after replacing $q$ with one of its powers. Hence, we may assume that $q$ fixes $L$, in which case it is evident that $(L^{nr})^P$ is finite over $L$, hence also over $K$.

Next we need to check that $L^{nr}$ is unramified over $(L^{nr})^P$. This is left as an easy exercise (using the same idea as in the previous paragraph). This proves the claim.

\section{Observations}

1. The composition
   $$\Phi \hookrightarrow \text{Gal}(L^{nr}/K) \rightarrow \text{Gal}(L/K)$$
   is surjective.

   (We will call the elements of $\Phi$ the "Frobenius elements" in $\text{Gal}(L^{nr}/K)$.)

   This is an easy consequence of the fact that the composition
   $$\mathbb{N} \hookrightarrow \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$$
   is surjective.

2. Now consider an element of $\Phi$ as an intermediate extension $K \subseteq F \subseteq L^{nr}$.

   Then $N_{F/K}(\mathcal{O}_F^x) \subseteq N_{L/K}(L^x)$. 


How do we prove (2)? First note that the claim is obvious when $F \subset L$, simply by the transitivity of norms.

In general, we proceed as follows. Consider $F' = F \subset L$. By the remark above, we have $N_{F'/K}(F' \times) \subset N_{L/K}(L' \times)$.

$$N_{F'/K} \left( \mathcal{O}_{F'}^\times \right) \cup N_{F'/K} \left( \mathcal{O}_{F}^\times \right) = N_{F'/K}(O_{F'}^\times).$$

Claim: $N_{F/K}(O_{F}^\times) = N_{F'/K}(O_{F'}^\times)$. We have an inclusion.

**Proof:** A priori, we have an inclusion.

To show equality, it suffices to check that $N_{F'/F}(O_{F'}^\times) = O_{F}^\times$. Again, by the transitivity of norms.

But $F'/F$ is a finite unramified extension, so we are reduced to the following lemma:

**Lemma:** For every finite unramified extension of local fields $F' \subset F$, we have $N_{F'/F}(O_{F'}^\times) = O_{F}^\times$.

**Proof of the lemma.**

Consider the standard filtration on $O_{F}^\times$:

$$O_{F}^\times > 1 + m_{F} > 1 + m_{F}^2 > \cdots$$

and similarly for $O_{F'}^\times$. 

Filtration on $O_{F}^\times$.
The norm map is compatible with these filtrations, and by completeness, it suffices to show that the induced map on the associated graded pieces is surjective.

However, on the first piece we get the norm map for the residue field extension, and on all the other pieces we get the trace map for the residue field extension, both of which are surjective (an easy exercise).

\[ \text{§1.12. An "explicit formula".} \quad \text{We claim that if we have a compatible system of maps } r_K \text{ as in Theorem 1.8 (we are not even assuming that they are isomorphisms), then for every finite extension } L \supset K \text{ of local fields, the induced map} \]

\[ r_{L/K} : \text{Gal}(L/K) \to K^*/N_{L/K}(L^*) \]

must be given by the following formula. Let } \sigma \in \text{Gal}(L/K), \text{ let } F \in \mathfrak{P} \text{ be any lift of } \sigma, \text{ and let } \omega_F \in F \text{ be a uniformizer, then} \]

\[ r_{L/K}(\sigma) = N_{F/K}(\omega_F). \]
§1.13. Why does this formula make sense? Note at least that the map \( F \to \text{Gal}_{ab}(L/K) \) is surjective by Observation \( 1.11(1) \), and the image of \( NF/K(\sigma_F) \) in \( K^x/N_{L/K}(L^x) \) is independent of the choice of \( \sigma_F \) in \( F \) by Obs. \( 1.11(2) \). However, if we want to use the formula above in the proof of existence, then we also need to prove independence of the choice of \( F \).

§1.14. First we will finish the proof of the fact that if \( K \)'s exist, then the formula of §1.12 must hold.

Consider \( 1 \in \mathbb{Z}_F^{\text{val}} \) and \( F \to F^\times \to K^x \to N_{F/K} \). Let \( r_F \) be the assignment in \( \text{Gal}_{ab}(F) \) to \( F \), and let \( r_K \) be the assignment in \( \text{Gal}_{ab}(K) \) to \( K \). Then \( r_{L/K} \) is the assignment in \( \text{Gal}_{ab}(L/K) \) to \( L/K \). Thus we might as well take \( \tau_F = r_F(\tilde{\sigma}) \), where \( \tilde{\sigma} \) corresponds to \( F \), and this proves the claim of §1.12.
Thus we completed the uniqueness part in the proof of Theorem 1.8. So far what we had were some simple observations, rather than an actual theorem.

The difficult part is proving that the formula given in §1.12 is well defined (i.e., gives an answer which is independent of the choice of $F$), and that it gives a group isomorphism. This proof will be explained in one of the later lectures.

In the rest of today's lecture we will explain a general dictionary (which has nothing to do specifically with local CFT).

Dictionary. Recall the picture:

\[ \begin{array}{c}
\text{Gal}_{ab}(L) \\
\downarrow \\
\text{Gal}_{ab}(K)
\end{array} \xrightarrow{\text{will be constructed}}
\begin{array}{c}
L \\
\uparrow \\
K
\end{array} \xrightarrow{\text{constructed}}
\begin{array}{c}
N_{L/K}
\end{array} \xrightarrow{\text{constructed}}
\begin{array}{c}
L \\
\uparrow \\
K
\end{array} \xrightarrow{\text{constructed}}
\begin{array}{c}
N_{L/K}
\end{array}
\]

The idea of the construction comes from topology, where for a finite covering $X \rightarrow Y$ of topological spaces, we have a transfer map $H_1(Y) \rightarrow H_1(X)$ given by taking full preimages of 1-cycles.
In fact, it turns out that transfer is a purely group-theoretical construction, which can be formulated in a much more general setting.

§1.17. $G$-modulations. Let us fix a profinite group $G$. All $G$-modules we consider will be continuous with respect to the discrete topology.

We would like to consider objects that are more general than $G$-modules.

Let us define an additive category $\mathcal{C}_G$:

- objects = finite $G$-sets (where the stabilizer of every point is open in $G$)

- morphisms: $\text{Hom}_{\mathcal{C}_G}((X,Y), (Z[X], Z[Y])) = \{G$-equivariant correspondences between $X$ and $Y$\}

Think of this equality as the definition of the RHS. In the second interpretation, composition of morphisms is the standard composition of correspondences.
Why is the second interpretation nice? In our applications, we will take $G = \text{Gal}(K^{\text{sep}}/K)$ for some field $K$, so that finite $G$-sets are the same as finite étale schemes over $K$, and then we recover correspondences in the usual sense of algebraic geometry.

§1.18. The category $E_G$ has an obvious symmetric monoidal structure coming from products of $G$-sets.

Note also that $E_G$ is self-dual, i.e., we have a natural equivalence $E_G \cong E_G^{\text{op}}$. Actually, it is just the duality functor in the sense of the general theory of monoidal categories (check that $E_G$ is rigid!).

Explicitly, the functor $E_G^{\text{op}} \to E_G$ is the identity on objects, and the transposition at the level of objects.

§1.19. We can also describe $E_G$ by "generators and relations."

We have an obvious functor

$$
\mathbb{Z} \{ \text{finite } G\text{-sets} \} \longrightarrow E_G \\
\downarrow \text{duality} \\
E_G^{\text{op}} \longrightarrow E_G
$$
Exercise. $E$ is freely generated by the subcategories (finite $G$-sets), (finite $G$-sets)\textsuperscript{op}, subject to two relations:

(1) Given a cartesian square of $G$-sets

\[
\begin{array}{cccc}
X & x & Y & \to X \\
\downarrow & & \downarrow & \\
Y & \to & Z
\end{array}
\]

the two induced compositions $\xrightarrow{X} Y$ are equal.

(2) Given any morphism $\xrightarrow{X} Y$ of finite $G$-sets, the induced composition $\xrightarrow{Y} Y$ is the multiplication by $\deg (X/Y) = \text{the degree function}$ which is the $G$-invariant function on $Y$ defined by $y \mapsto |\pi^{-1}(y)|$.

§1.20. Definition. A $G$-modulation in an additive category $A$ is an additive functor $E \xrightarrow{G} A$. (If $A$ is not specified, we take it to be the category of abelian groups.)
So if $M$ is a $G$-modulation in $A$, then for every finite $G$-set $X$, we get an object $M_X \in A$. In particular, for any open subgroup $U \subset G$ of finite index, we get an object $M_{G/U} \in A$.

The exercise in §1.19 allows one to reformulate the definition of a modulation in a more concrete way (it is a rule which assigns an object to every finite $G$-set $X$, assigns a morphism $X \to Y$ of $M_X \in A$, and to every morphism $X \to Y$ of finite $G$-sets assigns a morphism $M_X \to M_Y$ satisfying the two relations described above).

§1.21. The notion of a $G$-modulation will allow us to reformulate the main theorem of local CFT in a more precise way (as a statement about an isomorphism of modulations).

§1.22. Remark. The notion of a $G$-modulation is "self-dual" in the sense that both a covariant, and a contravariant, additive functor between additive categories transforms $G$-modulations into $G$-modulations. (This is just using the duality $\mathcal{E}_G \cong \mathcal{E}_G^{\text{op}}$.)

If we take $A = \{\text{finite abelian groups}\}$, then we get a notion of duality for modulations by using $\text{Hom} (-, \mathbb{Q}/\mathbb{Z})$. 
§1.23. Relation to $G$-modules.

Note that for any profinite group $G$, we have a canonical $G$-module action with values in the category of $G$-modules, namely,

$$X \longrightarrow \mathbb{Z}[X].$$

There is a functor

$$(G\text{-modules}) \longrightarrow (G\text{-modulations})$$

$$N \longrightarrow (X \longrightarrow \text{Hom}_{G\text{-mod}}(\mathbb{Z}[X], N)).$$

We claim that this functor has a right adjoint. Namely, given a $G$-module $M$, we form

$$M_G := \lim_{U \subset G} M_{G/U}$$

where $U \subset G$ is an open normal subgroup.

This has a natural $G$-module structure, and the functor $M \mapsto M_G$ is the desired right adjoint.

§1.24. Example. What is a $\mathbb{Z}/p\mathbb{Z}$-module?

We only have two subgroups, so a $\mathbb{Z}/p\mathbb{Z}$-module is a diagram

$$M_0 \longrightarrow M_1,$$

of abelian groups satisfying some relations.

Exercise. Describe these relations.

§1.25. Final remark. Compute the right derived functor of

$$(G\text{-modules}) \longrightarrow (G\text{-modulations})$$

You get the data which to a $G$-module $N$ assigns the collection of the cohomology groups $H^*(G/U, N_U)$.
§1.26. An application.

Take \( N = \mathbb{Q}/\mathbb{Z} \), with the trivial \( G \)-action. The corresponding first derived functor is the \( G \)-modulation which assigns

\[
G/U \mapsto H^1(U, \mathbb{Q}/\mathbb{Z}) = \text{Hom} (U, \mathbb{Q}/\mathbb{Z}) = \text{Hom} (U_{ab}, \mathbb{Q}/\mathbb{Z}).
\]

Applying the duality \( \text{Hom} (-, \mathbb{Q}/\mathbb{Z}) \) again, we get the modulation \( G/U \mapsto U_{ab} \). This defines the transfer maps that were mentioned previously.

Explicitly, the transfer maps can be defined as follows. Consider open normal subgroups \( U \subset V \subset G \). We will define \( \text{tr} : V_{ab} \rightarrow U_{ab} \) by choosing a system of representatives \( R \subset V \) modulo \( U \).

Given \( v \in V \) and \( r \in R \), write \( v \cdot r = r' \cdot u_r \) with \( r' \in R \) and \( u_r \in U \).

**Exercise:** Prove the formula

\[
\text{tr} (v) = \prod_{r \in R} u_r
\]