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Lectures on the Weil conjectures

Lecture 3: 05/28/2007

Today: • Idea of Weil's original proof of the Riemann hypothesis for curves over finite fields
• some general comments

Recall: $X =$ algebraic variety / \mathbb{F}_q

\Rightarrow we can introduce $Z(X, t)$

It is a rational function of t :

$$Z(X, t) = \frac{\prod \det(1 - t \cdot \text{Fr}^*, H_c^{2i+1}(X))}{\prod \det(1 - t \cdot \text{Fr}^*, H_c^{2i}(X))}$$

If X is smooth, projective and connected, then we get a functional equation for $Z(X, t)$ from the Poincaré duality for X .

Def: A Weil number of weight i is an algebraic integer α such that each complex conjugate of α has absolute value $q^{i/2}$.

If X is smooth and complete, all eigenvalues of Fr^* on $H^i(X)$ are Weil numbers of weight i . If X is either smooth and non-complete, or singular

and complete, then we only have an inequality for the weights of eigenvalues of Fr^* acting on $H_c^j(X)$.

Similar result: $X =$ smoothly ^{connected} projective curve over \mathbb{C}

$F: X \rightarrow X$ algebraic map of degree $q > 0$

Lemma: Then $F^*: H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ is a semisimple endomorphism all of whose eigenvalues have absolute value $q^{1/2}$.

Proof: Let $\Omega^1(X)$ denote the space of holomorphic 1-forms on X ; thus $\dim \Omega^1(X) = g =$ genus of X .

We have the Hodge decomposition:

$$H^1(X, \mathbb{C}) = \Omega^1(X) \oplus \overline{\Omega^1(X)}$$

Moreover, we have a symplectic form on $H^1(X, \mathbb{C})$ coming from the intersection pairing, and $\Omega^1(X), \overline{\Omega^1(X)}$ are Lagrangian subspaces with respect to this form.

However, let us consider instead the Hermitian

namely, $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$ form:

$$(\alpha, \beta) := \int_X \alpha \wedge \overline{\beta}$$

For any $\alpha \in \Omega^1(X)$, one can easily check that if $\alpha \neq 0$, then

$$-i(\alpha, \alpha) = -i \int_X \alpha \wedge \bar{\alpha} > 0$$

[In local coordinates: $dz \wedge d\bar{z} = 2i dx \wedge dy$
 $\Rightarrow -i \cdot dz \wedge d\bar{z}$ is a positive 2-form.]

Now: F is holomorphic \Rightarrow it preserves the Hodge decomposition;
it is also unitary with respect to the Hermitian form (\cdot, \cdot) up to multiplication by q . This proves the lemma. //

Higher-dimensional generalization:

$X =$ connected smooth projective variety of dimension $n \in \mathbb{N}$ over \mathbb{C}

$F: X \rightarrow X$ an algebraic map

Again, we can ask what are the eigenvalues of F acting on $H^i(X, \mathbb{C})$. In general, there is not much that can be said.

However, suppose there exists an ample line bundle \mathcal{L} on X such that $F^*(c_1(\mathcal{L})) = q \cdot c_1(\mathcal{L})$ for some $q > 0$, where $c_1(\mathcal{L}) \in H^2(X, \mathbb{C})$ is the first Chern class of X .

- Remarks:
- 1) This requirement holds, in particular, if $F^*(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$
 - 2) Over \mathbb{F}_q , if F is the Frobenius, it is obvious that $Fr^*(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for any line bundle \mathcal{L} on X .
 - 3) For curves, our condition is clearly equivalent to $\deg(F) = q$.
 - 4) Since F^* also acts on $H^2(X, \mathbb{Z})$, it is clear that $F^*(c_1(\mathcal{L})) = q \cdot c_1(\mathcal{L}) \Rightarrow q \in \mathbb{Z}$, provided $q \neq 0$. (Double-check!)

Theorem: Under these assumptions, the eigenvalues of F^* on $H^i(X, \mathbb{C})$ are Weil numbers of weight i (with respect to q).

The idea of the argument is the same as in the 1-dimensional case. However, one needs to use more serious technical tools: hard Lefschetz theorem, Hodge index theorem, and so on...

Upshot: If one can invent a suitable cohomology theory for varieties over finite fields so that one has a Lefschetz fixed point theorem, a hard Lefschetz theorem, and so on, then one can prove the Riemann hypothesis for varieties over finite fields.

However: There CANNOT exist a good cohomology theory for varieties over finite fields with coefficients in \mathbb{Q} .

For example, one has the supersingular elliptic curves whose endomorphism algebras are too large to act on 2-dimensional vector spaces over \mathbb{Q} (too large: they are rings of integers in quaternionic algebras over \mathbb{Q}).

A topological comment

$X =$ smooth algebraic variety over \mathbb{C}
 $x \in X$ a (closed) point

$\Rightarrow \exists$ Zariski neighborhood U of x which is a $K(\pi, 1)$

In fact: U is an iterated fibration by smooth affine curves

$$U \rightarrow U_1 \rightarrow U_2 \rightarrow \dots$$

Now use the fact that smooth affine curves are $K(\pi, 1)$'s (in fact, for free groups π).

\rightarrow Insert a discussion of the Riemann existence theorem (the general version proved by Artin (?), Grauert & Remmert)

We have good cohomology theories with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, where ℓ is a prime different from $p = \text{char}(\mathbb{F}_q)$.

Using them we can define cohomology with coefficients in \mathbb{Z}_ℓ , and then in \mathbb{Q}_ℓ .

However: This is not very helpful for controlling absolute values of eigenvalues of Fr^* acting on $H^i(X)$, because: if we have an algebraic integer λ and we have good control over λ modulo ℓ^n for each $n \in \mathbb{N}$, we do not know ANYTHING about the archimedean absolute values of λ .

Nevertheless: Deligne managed to prove the ^(difficult part of the) Weil conjectures using ℓ -adic cohomology. This proof is beautiful, but it does not really explain why the statement is true. In fact, it is believed that ℓ -adic cohomology is NOT the correct framework for questions of this sort.

[Remark: Deligne proved a much more general result about cohomology of ℓ -adic sheaves on algebraic varieties over finite fields. This result has numerous applications to the topology of algebraic varieties.]

In ≈ 1965 , Grothendieck invented another approach based on certain conjectures about algebraic cycles which nobody knows how to prove (these are the so-called "standard conjectures"). These conjectures imply all the Weil conjectures in a much more natural way.

Recommended reading: The Grothendieck-Serre correspondence (a bilingual edition has been published by the AMS).

Next: We will explain Weil's proof of the Riemann hypothesis for curves over finite fields, and then say some words about motives.

$X =$ smooth (connected) projective curve over \mathbb{F}_q

Look at $J(X) = \text{Pic}^0(X)$, the Jacobian of X .
 (It is the "motivic" replacement of the first cohomology group of X .)

We have $\text{Fr}_q \curvearrowright X$ and $\text{Fr}_q \curvearrowright J(X)$.

We want to have control on the endomorphisms of $J(X)$. However, usually there are not enough endomorphisms of X itself to produce all endomorphisms of $J(X)$.

Instead, we need to look at correspondences. (8)

Def: A (self-)correspondence on X is a divisor on $X \times X$.

Motivation: In usual topology, one can use Poincaré duality and Künneth formula to identify $H^0(X \times X) \cong \text{End}(H^0(X))$, where X is a smooth compact manifold.

Explicitly:

$c \in H^0(X \times X) \xrightarrow{\alpha} p_{2*}(c \cup p_1^*(\alpha))$.

integration along the fibers of p_2

pullback

Now every divisor on $X \times X$ gives a class in $H^2(X \times X)$, so we can use it to give an endomorphism of $H^1(X)$.

Remark: Over \mathbb{C} , algebraic equivalence for divisors is the same as cohomological equivalence.

Some comments about numerical equivalence...

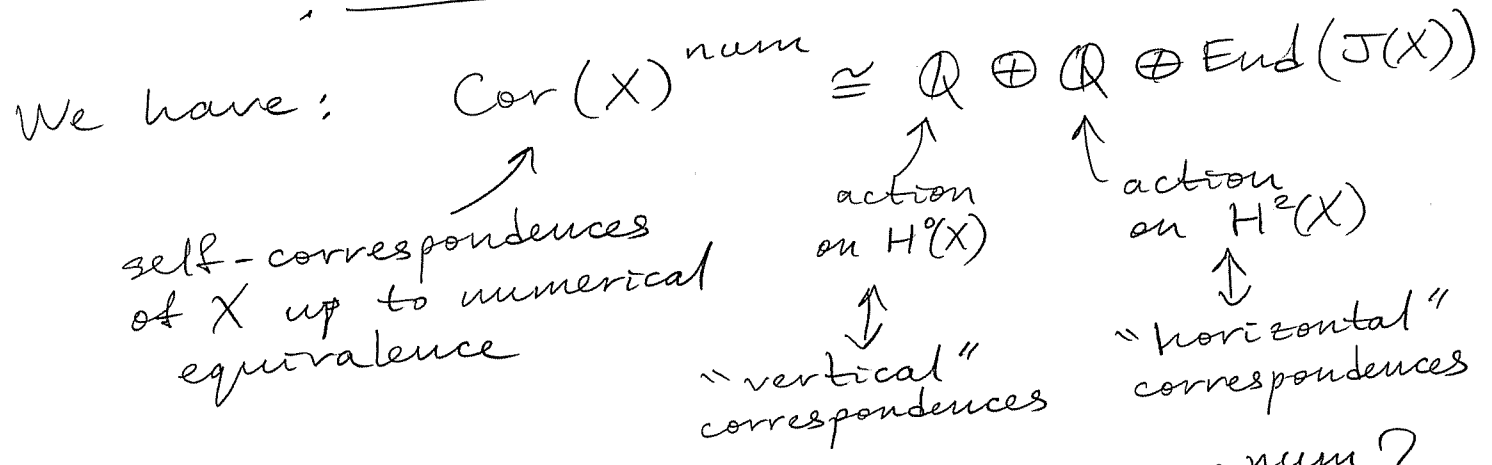
Fact: Endomorphisms of $J(X)$ are essentially the self-correspondences of X .

(Possible reference: Serre's book, "Algebraic Groups and Class Fields")

Now we have the Frobenius acting on $J(X)$, and we want to study its "eigenvalues".

Remark: $\text{End}(J(X))$ is a finitely generated abelian group, and hence it is an integral extension of \mathbb{Z} .

By definition, the roots of the minimal polynomial for Fr over \mathbb{Z} are the "eigenvalues" of Fr .



What are the structures on $\text{Cor}(X)^{\text{num}}$?

- 1) Involution (it comes from switching the two copies of X)
- 2) The trace functional, given by the intersection index with the class of the diagonal.

We will denote these structures by $D \mapsto {}^t D$ and $D \mapsto \text{tr}(D)$, respectively.

Fact: $\langle D, D' \rangle = \text{tr}(D \cdot {}^t D')$
(intersection index) (on the usual topology, this is an immediate consequence of the trace formula)

Basic property: The trace pairing is nondegenerate and has index 1 (i.e., one of its eigenvalues is positive, and all other are negative).
Over \mathbb{C} , one can prove this using Hodge theory.

In general, one can give an algebraic proof using the Riemann-Roch theorem for algebraic surfaces.

This result is called the Index Theorem. Modulo this result, we can prove the Riemann hypothesis for $X =$ smooth connected projective curve over \mathbb{F}_q .

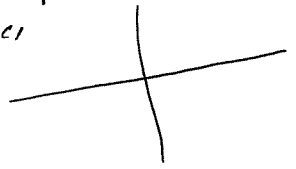
Exercise (easy!): $Fr \cdot {}^t Fr = q$

This is not quite the analogue of unitarity. We need to use some sort of positivity. How do we use the Index Theorem?

Lemma: The intersection product is negative definite on the $\text{End}(H^1(X))$ part of $\text{Cor}(X)_{\text{num}}$.

Proof: How do we find the one positive eigenvalue of the intersection pairing guaranteed by the Index Theorem?

If D is any ample divisor on $X \times X$, then, of course, $\langle D, D \rangle > 0$. So if we take a complement to $\mathbb{Q} \cdot D$, the intersection form will be negative definite on this complement.

In particular, let D be a "cross"  on $X \times X$. It corresponds to the element $(1, 1, 0)$ on $\mathbb{Q} \oplus \mathbb{Q} \oplus \text{End}(H^1(X))$. Hence on $\text{End}(H^1(X))$ the intersection form is negative definite.

But now $\mathbb{Z}[Fr_q^*] \subset \text{End}(H^1(X))$. Hence, by negative definiteness, we get the desired statement about the absolute values of the eigenvalues of Fr_q^* .

To relate this to the original zeta function:

Use the fact that

$$|X(\mathbb{F}_q^n)| = \langle \Gamma_{Fr_q^n}, \Delta \rangle = \text{tr}(\Gamma_{Fr_q^n}),$$

where $\Gamma_{Fr_q^n} \subset X \times X$ is the graph of Fr_q^n and $\Delta \subset X \times X$ is the diagonal.

Grothendieck generalized this idea to higher dimensional varieties. However, his "proof" is based on several "unproved" conjectures. For example, the Index Theorem is not known (it is one of the "standard conjectures").

In general, we know very little about algebraic cycles on varieties of large dimension.

References for the standard conjectures (and the derivation of the Weil conjectures):

- the two volumes in Proceedings of Symposia on Pure Mathematics (published by the AMS), called "Motives"
- the Grothendieck-Serre correspondence.

Remark: Weil's original proof did not use the index theorem (it used something else).

Something about this may be contained in Mumford's book on abelian varieties. There is a section about the index theorem on Mumford's "Lectures on curves on an algebraic surface".