§ 2.1  \[ A = \text{finitely generated } \mathbb{F}_q \text{-algebra} \]
\[ X = \text{Spec } A \]
\[ \Rightarrow \quad Z(X, t) = \prod_{m = \text{maximal ideal of } A} (1 - t^{n(m)})^{-1} \quad (\star) \]

where  \[ n(m) = [A/m : \mathbb{F}_q] \].

§ 2.2  Now suppose \( A \) is an algebra of finite type over \( \mathbb{Z} \) (i.e., a finitely generated ring). What is the correct analogue for \( X = \text{Spec } A \) of the function \( Z(X, t) \)?

Exercise: If \( m \) is a maximal ideal of \( A \), then \( A/m \) is still a finite field.

However, the characteristic of \( A/m \) may depend on the choice of \( m \), so it would be wrong to try to imitate the formula above.

Instead, if we make the change of variables \( t = q^{-s} \) in \( (\star) \), then we obtain the product
\[ Z(X, s) = \prod_{m = \text{max. ideal of } A} (1 - |A/m|^{-s})^{-1} \quad (\star\star) \]
We take \((\ast \ast)\) as the definition of \(\zeta(X, s)\) for \(X = \text{Spec} A\), where \(A\) is any finitely generated ring.

Note that \(\zeta(X, s)\) is no longer a formal power series, so one has to approach it through analytical methods.

**Example:** For \(X = \text{Spec} \mathbb{Z}\), we get the usual (Riemann) zeta function.

### §2.3.

Now we return to the case where \(X\) is an algebraic variety over \(\mathbb{F}_q\). Our goal today is to prove the rationality of \(Z(X, t)\), at least when \(X\) is a curve.

[The Riemann hypothesis is a certain statement about the zeroes of \(Z(X, t)\).]

### §2.4.

Recall from last time the parallel picture in differential topology:

- \(X\) = smooth, compact, oriented manifold
- \(Fr: X \rightarrow X\) = smooth map with isolated fixed points satisfying certain conditions

**Pontryagin duality:** perfect pairing

\[
H^i(X, \mathbb{Q}) \times H^{-i}(X, \mathbb{Q}) \xrightarrow{\cup} H^n(X, \mathbb{Q}) \xrightarrow{\int_X} \mathbb{Q}
\]
K"unneth formula:

\[ H^\alpha (X \times Y, \mathbb{Q}) \cong \bigoplus_{i} (H^i(X, \mathbb{Q}) \otimes H^{\alpha-i}(X, \mathbb{Q})) \]

via the maps

\[ H^i(X, \mathbb{Q}) \otimes H^{\alpha-i}(X, \mathbb{Q}) \to H^\alpha (X \times Y, \mathbb{Q}) \]

\[ \otimes \xrightarrow{\beta} (p_1^* \delta) \cup (p_2^* \delta) \]

where \( X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y \) are the two projections.

Proof of the Lefschetz fixed point formula.

We will deduce the LFPF from the two results quoted above.

\[ X \xrightarrow{\Delta} X \times X \text{ diagonal embedding} \]

Since the normal bundle to \( X \) in \( X \times X \) is oriented, \( \Delta \) determines a cohomology class \( \text{cl}(\Delta) \in H^n(X \times X, \mathbb{Q}) \).

Using Poincaré duality, it can be computed explicitly. Namely, let us choose biorthogonal bases

\( \{ e_{a} \} \) of \( H^i(X, \mathbb{Q}) \)

and \( \{ g_{i} \} \) of \( H^{n-i}(X, \mathbb{Q}) \)

(biorthogonal w.r.t. the cup product pairing).
Exercise:

\[ \text{cl}(\Delta) = \sum_{i,a} (-1)^i (e_a \otimes g_{n-i}) \]

Using this formula, we can prove the LFP formula. Namely, writing \( \Gamma_{Fr} = \text{graph of } Fr \),

\[ |X^{Fr}| = \int_{X \times X} \text{cl}(\Delta) \cup \text{cl}(\Gamma_{Fr}) = \int_X \Delta^* \text{cl}(\Gamma_{Fr}) \]

\[ = \int_X \Delta^* (Fr \times \text{id}_X)^* (\text{cl}(\Delta)) = \sum_{i} (-1)^i \text{tr}(Fr^*; H^i(X,\mathbb{Q})) \]

using the exercise above.

\[ \text{Section 2.5:} \]

This picture also suggests a functional equation for \( \mathbb{Z}((X, Fr), t) \). Namely, \( Fr^* \) acts on \( H^n(X, \mathbb{Q}) \) by some fixed scalar (its degree), and the cup product is compatible with the action of \( Fr^* \), which allows us to relate the eigenvalues of \( Fr^* \) on \( H^i(X, \mathbb{Q}) \) to its eigenvalues on \( H^{n-i}(X, \mathbb{Q}) \). This leads to a formula of the type

\[ \mathbb{Z}((X, Fr), t^{-1}) = (\text{some monomial}) \cdot \mathbb{Z}((X, Fr), t) \]

not sure about this one.
Exercise: Find the precise formula and prove it using the simple identity
\[
\frac{1}{1 - at} = \frac{-a^{-t}t^{-1}}{1 - a^{-1}t^{-1}}
\]

32.6. Fix a smooth projective curve $X$ over $\mathbb{F}_q$. Let $g$ be the genus of $X$. For simplicity, we will write $Z(t) = Z(X, t)$.

32.7. Aside: How do we define the genus? If $X$ is a smooth projective curve, then the genus of $X$ can be defined as
\[
g = \text{"number of handles" of } X\}
\]
\[
g = \frac{1}{2} \dim \mathbb{Q} H^1(X, \mathbb{Q})
\]
\[
= \dim \mathbb{C} \Omega^1(X),
\]
where $\Omega^1(X)$ is the space of holomorphic 1-forms on $X$.

The last definition of the genus makes sense over any base field.

32.8. Riemann–Roch. Recall that we have the notions of line bundles and divisors on the curve $X$. 
Recall that there are two ways of thinking about line bundles (or, more generally, vector bundles):

(1) In terms of total spaces
(a geometric object mapping to $X$ with a linear structure on the fibers satisfying some conditions...)

(2) In terms of sheaves of sections.
In particular, a line bundle on $X$ is the same thing as a locally free sheaf of $O_X$-modules of rank 1.

Recall also the correspondence between line bundles and divisors. In fact, there is a bijection between isomorphism classes of pairs $(L, f)$, where $L$ is a line bundle on $X$ and $f$ is a nonzero meromorphic section of $L$, and divisors on $X$, given by

$(L, f) \mapsto \text{div}(f)$.

§2.9. Complex-analytic picture

$0 \to \text{Pic}^0(X) \to \text{Pic}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0$

↑ group of isomorphism classes of line bundles on $X$

Now choose a divisor $D$ of degree 0 on $X$. We will define an invariant of $D$. 
Since \( \deg(D) = 0 \), it is possible to choose a collection of (say, piecewise smooth) paths in \( X \)

\[ \gamma_i \]

with the property that \( \sum (z_i - x_i) = D \).

In other words, if we put

\[ \gamma = \sum \gamma_i, \]

then \( \int \gamma = D \). Then \( \gamma \) gives rise to a linear functional

\[ \int : \Omega^1(X) \to \mathbb{C} \]

\[ \omega \mapsto \sum \int_{\gamma_i} \omega \]

**Question:** What happens if we choose a different \( \gamma \)?

This amounts to adding a closed 1-chain to \( \gamma \) (i.e., a homology 1-cycle).

**Fact:** Integration defines an embedding

\[ H_1(X, \mathbb{Z}) \hookrightarrow \Omega^1(X)^* \]

as a lattice (i.e., a discrete subgroup).

Note that \( \Omega^1(X)^* \) is a complex vector space of dimension \( g \) and \( H_1(X, \mathbb{Z}) \) is a free abelian group of rank \( 2g \).
§2.10. Thus we obtain a well defined map

\[ \text{Div}^0(X) \to \Omega^1(X)*/\text{H}_1(X, \mathbb{Z}) \]

\( \text{(compact) torus of real dimension } 2g \)

\textbf{Lemma:} The image of \( \text{Div}^0(X) \) under this map depends only on the linear equivalence class of \( D \).

\textbf{Proof:} It is enough to check that the divisor \((f)\) of a meromorphic function \( f \) on \( X \) maps to 0 under this map. Let us view \( f \) as a holomorphic map \( X \to \mathbb{P}^1 \). Pick any path \( \gamma \) in \( \mathbb{P}^1 \) between 0 and \( \infty \), and let \( \gamma' \) be the full preimage of \( \gamma \) under \( f \) (check that \( \gamma' \) is OK as a chain even if \( \gamma \) passes through the ramification points of \( f \)).

If \( \omega \in \Omega^1(X) \), then one easily checks that

\[ \int \omega = \int_{\gamma'} \text{tr}_f(\omega) = 0 \text{ because } \Omega^1(\mathbb{P}^1) = 0. \]

Since \( \partial \gamma' = \text{div}(f) \), this proves the lemma. \( \square \)

§2.11. Need to explain the definition of \( \text{tr}_f(\omega) \) in more detail! (E topological definition; alg. def. via Galois theory; alg. definition using local duality ...)
§2.12. Philosophical remark:

Pic^0(X) is the correct abstract analogue of the first cohomology group of a curve X (E.g., it is used in Weil's proof of the Weil conjectures for curves over finite fields ...)

For varieties of dimension > 1, it is not at all clear what the correct algebro-geometric analogues of all cohomology groups of X are (this is what the theory of motives is about).

§2.13. Reminder on the Riemann-Roch theorem.

X = smooth projective curve of genus g over any field \( k \)

\( L \) = any line bundle on \( X \)

We write \( h^0(L) = \dim_k \Gamma(X,L) \)

Let \( \omega_X \) denote the line bundle of regular 1-forms on \( X \); so \( g = h^0(\omega_X) \).

The Riemann-Roch theorem (actually, combined with Serre duality) is:

\[
    h^0(L) - h^0(\omega_X^\otimes L^{-1}) = \deg(L) + 1 - g
\]

Remark: If we know the approximate form of the result, \( h^0(X) - h^0(\omega_X^\otimes L^{-1}) = \deg(L) + c \), where \( c \) is a constant, then it is easy to compute \( c \) by plugging in \( L = 0_X \).
32.14. Rationality of $\mathbb{Z}(X, t)$.

Let $X$ be a smooth projective curve over $\mathbb{F}_q$.

Recall the definition:

$$\mathbb{Z}(X, t) = \prod_{x \in |X|} \left(1 - t^{\deg x}\right)^{-1} = \sum_{D \geq 0} t^{\deg D}$$

($|X|$ = set of all closed points of the underlying scheme of $X$)

$$= \sum t^{\deg (\text{Div}(y))}$$

pairs $(L, y)$ up to isomorphism

$L = $ line bundle on $X$

$y = $ nonzero regular section of $L$

the existence of such a $L$ forces $\deg (\text{Div}(y)) > 0$ automatically

because we want to get an effective divisor on $X$

$$= \sum_{L \in \text{Pic}(X)} |\text{Pic}(X, L)| \cdot t^{\deg L}$$

$$= \sum_{L \in \text{Pic}(X), \deg L \geq 0} \frac{h^0(L)}{q - 1} \cdot t^{\deg L}$$

Now recall that $\deg (\omega X) = 2g - 2$ (again, this follows easily from the Riemann-Roch formula by taking $L = \omega X$).
Let us analyze RR in more detail:
\[ h^0(L) \neq 0 \implies \deg(L) \geq 0 \]
\[ h^0(\omega_X \otimes L^{-1}) \neq 0 \implies \deg(L) \leq 2g - 2 \]

In particular, if \( \deg(L) > 2g - 2 \), we see that the RR formula gives an exact formula for \( h^0(L) \).

Thus we obtain
\[
Z(X, t) = |\text{Pic}^0(X)| \cdot \sum_{n > 2g - 2} \frac{q^{n+1-g-1}}{q-1} \cdot t^n + \sum_{\substack{L \in \text{Pic}(X) \ 0 \leq \deg(L) \leq 2g - 2}} \frac{h^0(L)-1}{q-1} \cdot t^{\deg(L)}.
\]

The second sum is a polynomial. The first sum is obviously a rational function of \( t \) (easy to compute explicitly).

§2.15. 
Now we would like to rewrite the formula above in a better way, and, in particular, to derive the functional equation for \( Z(X, t) \).

It is easy to check that:
\[ Z(X,t) = |\text{Pic}^0(X)| \cdot \sum_{g \geq 1} \left( \frac{q^{n+g-1}}{q-1} \cdot t^n \right) + \]

\[ + \sum_{g \leq \deg(x) \leq 2g-2} \frac{q^{h^0(x)} \deg(x)+1-g}{q-1} \cdot t^{\deg(x)} + \]

\[ + \sum_{0 \leq \deg(x) \leq g-1} \frac{q^{h^0(x)}}{q-1} \cdot t^{\deg(x)-1} = \]

\[ = |\text{Pic}^0(X)| \cdot t^g \cdot (1-t)^{-1} \cdot (1-q \cdot t)^{-1} + \]

\[ + \text{(other two terms)} \]

\[ = \text{the zeta function of } \mathbb{P}^1 \]

as computed in the first lecture.

one obtains this formula by comparing

the case \( g = 0 \), i.e., \( X = \mathbb{P}^1 \).

\[ \text{(2.16)} \quad \text{Functional equation:} \]

\[ Z(X,q^{-1}t^{-1}) = q^{1-g} t^{2-2g} \cdot Z(X,t). \]

To prove this result, we use the formula for \( Z(X,t) \) obtained in §2.15 and the RR formula.
To that end, consider the Serre duality involution on all isomorphism classes of line bundles on $X$: $L \mapsto \omega_X \otimes L^{-1}$.

Clearly, it interchanges the intervals $g \leq \deg L \leq 2g - 2$ and $0 \leq \deg L \leq g - 1$.

The rest is left as a simple exercise.

### §2.17. What is left?

We need to prove the analogue of the Riemann hypothesis in this situation, namely, the nontrivial zeroes of $Z(X, t)$ have absolute value $\frac{1}{2}$.

This statement is much less trivial than the rationality of $Z(X, t)$ and the functional equation. There are two ways of proving it:

1. **Conceptual** — essentially due to A. Weil (uses the Jacobian $\text{Pic}^0(X)$ and a bit of intersection theory on $X \times X$)

2. **Elementary** — invented by Stepanov; it uses only RR on $X$ itself

Next time we will also say a few words about the Weil conjectures for varieties over $\mathbb{F}_q$ of dimension $> 1$. 