



Quantization of minimal resolutions of Kleinian singularities

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Abstract

In this paper we prove an analogue of a recent result of Gordon and Stafford that relates the representation theory of certain noncommutative deformations of the coordinate ring of the n th symmetric power of \mathbb{C}^2 with the geometry of the Hilbert scheme of n points in \mathbb{C}^2 through the formalism of \mathbb{Z} -algebras. Our work produces, for every *regular* noncommutative deformation \mathcal{O}^λ of a Kleinian singularity $X = \mathbb{C}^2/\Gamma$, as defined by Crawley-Boevey and Holland, a filtered \mathbb{Z} -algebra which is Morita equivalent to \mathcal{O}^λ , such that the associated graded \mathbb{Z} -algebra is Morita equivalent to the minimal resolution of X . The construction uses the description of the algebras \mathcal{O}^λ as quantum Hamiltonian reductions, due to Holland, and a GIT construction of minimal resolutions of X , due to Cassens and Slodowy.

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1. Introduction

Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite nontrivial subgroup, and $\{\mathcal{O}^\lambda\}$ the family of noncommutative deformations of the singularity $X = \mathbb{C}^2/\Gamma$ constructed by Crawley-Boevey and Holland [6]. If (Q, I) denotes the McKay quiver associated to Γ , where Q is the set of edges and I is the set of vertices, then the parameter space for these deformations is naturally identified with \mathbb{C}^I . If $\delta = (\delta_i)_{i \in I} \in \mathbb{N}^I$ is the minimal positive imaginary root for the quiver Q and $\lambda \in \mathbb{C}^I$, the algebra

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\mathcal{O}^λ is commutative when $\lambda \cdot \delta = 0$, and noncommutative otherwise [6, Theorem 0.4(1)]. After a result of Holland [12], for $\lambda \cdot \delta \neq 0$, the algebra \mathcal{O}^λ can be thought of as a *quantization* of X . In the same paper Holland has also constructed a quantization of a certain partial resolution of the singularity X , and he asked if there exists a quantization of the minimal resolution $\tilde{X} \rightarrow X$. The difficulty lies in the fact that \tilde{X} is a non-affine variety, so it is not even clear what one should mean by a noncommutative deformation of \tilde{X} .

On the other hand, in [9], Gordon and Stafford prove a conjecture of Ginzburg that certain rational Cherednik algebras of type A , introduced in [8], are Morita equivalent to certain noncommutative deformations of the Hilbert scheme of n points in \mathbb{C}^2 , which is a crepant resolution of the singularity \mathbb{C}^{2n}/S_n . Their approach is based on the formalism of \mathbb{Z} -algebras, which we review in Section 5. In this paper we use this formalism to prove an analogue of the result of Gordon and Stafford for Kleinian singularities (which was also conjectured by Ginzburg), at the same time answering Holland’s question:

Theorem 1. *Let $\lambda \in \mathbb{C}^I$ be such that $\lambda \cdot \delta = 1$ and the algebra \mathcal{O}^λ has finite global dimension. For each dominant regular weight $\chi \in \mathbb{Z}^I$, there exists a filtered \mathbb{Z} -algebra $B^\lambda(\chi)$ which is Morita equivalent to \mathcal{O}^λ , such that the associated graded \mathbb{Z} -algebra $\text{gr}^\bullet B^\lambda(\chi)$ corresponds to a commutative graded ring $S(\chi)$ with $\text{Proj } S(\chi)$ being equal to the minimal resolution of X corresponding to χ as constructed by Cassens and Slodowy in [4].*

Here, for two vectors $v, w \in \mathbb{C}^I$, we denote by $v \cdot w$ the usual (\mathbb{C} -bilinear) scalar product of v and w . An element $\chi \in \mathbb{Z}^I$ is called a dominant regular weight if $\chi \cdot \delta = 0$ and $\chi \cdot \alpha > 0$ for every positive Dynkin root α of the affine root system associated to the quiver Q . The notion of a \mathbb{Z} -algebra is recalled in Definition 9, and the concept of Morita equivalence used in the statement above is explained in Definition 11.

Remark 2. The algebra \mathcal{O}^λ has finite global dimension if and only if $\lambda \cdot \alpha \neq 0$ for all Dynkin roots α [6, Theorem 9.5].

The second statement of the theorem and its proof imply that $B^\lambda(\chi)$ is a quantization of \tilde{X} (by which we simply mean a noncommutative deformation constructed using differential operators), which is why our result answers Holland’s question. On the other hand, Theorem 1 naturally completes the following diagram (which does not commute!):

$$\begin{array}{ccc}
 ? & \xleftarrow{\sim} & (\mathcal{O}^\lambda\text{-mod})^{filt} \\
 \text{gr} \downarrow & & \downarrow \text{gr} \\
 \text{Coh}(\tilde{X}) & \xleftarrow{\text{pullback}} & \text{Coh}(X)
 \end{array}$$

Here $(\mathcal{O}^\lambda\text{-mod})^{filt}$ stands for the category of filtered finitely generated \mathcal{O}^λ -modules. The question mark can be replaced by a suitable quotient of the category of filtered finitely generated $B^\lambda(\chi)$ -modules (see Section 6 for details), and the top arrow is an equivalence of categories. The significance of this result is that it yields a new way of taking the associated graded of a filtered \mathcal{O}^λ -module as a coherent sheaf on \tilde{X} , by composing the top horizontal arrow with the left vertical arrow.

The corresponding diagram for rational Cherednik algebras of type A has been obtained in [9], and then applications of this result to the representation theory of these algebras have been explored in [10]. A similar study for the algebras \mathcal{O}^λ will appear in [3]. In particular, one can gain a better understanding of the finite-dimensional representations of the algebra \mathcal{O}^λ by equipping them with suitable filtrations and studying the associated graded coherent sheaves on \tilde{X} , which will be supported on the exceptional fiber of $\tilde{X} \rightarrow X$. (The “old” functor $\text{gr}: (\mathcal{O}^\lambda\text{-mod})^{\text{filt}} \rightarrow \text{Coh}(X)$ is unsuitable for this purpose, since the associated graded of any finite-dimensional \mathcal{O}^λ -module is supported at the singular point $0 \in X$ and thus carries no information about the module itself except for its dimension.)

Recently a somewhat more canonical version of Theorem 1 for Kleinian singularities of type A has been obtained by Musson in [16]. His approach is very different from ours in that instead of using Holland’s results, he constructs a filtered \mathbb{Z} -algebra that deforms the minimal resolution by using the explicit description of the latter as a toric variety (which replaces Cassens and Slodowy’s construction). In particular, this approach does not generalize to other types of Kleinian singularities. Apart from the basic theory of \mathbb{Z} -algebras, our papers are completely disjoint, and can be read independently.

The meaning of the words “more canonical” is explained in Section 6, where we also restate Theorem 1 in a precise way (see Theorem 15). The other two important results in the paper are Theorem 5 (on the Cassens–Slodowy’s minimal resolution) and Theorem 12 (a strengthening of Gordon–Stafford’s result on Morita \mathbb{Z} -algebras).

2. Recollections on quivers

In this section we recall several constructions using quivers that are important for the formulation and the proof of our main result. To avoid any possible misunderstanding, we begin by fixing some simple terminology. An *algebra* will always mean for us an associative algebra over \mathbb{C} , and if A, B are algebras, then an (A, B) -bimodule M is required to satisfy the condition that the two induced actions of \mathbb{C} on M coincide. All tensor products, unless specified otherwise, will be taken over \mathbb{C} . With the exception of \mathbb{Z} -algebras (defined in Section 5), all rings are assumed to have a multiplicative identity, and all modules are assumed to be unital.

As above, we let (Q, I) denote an affine quiver associated to a finite nontrivial subgroup $\Gamma \subset SL_2(\mathbb{C})$. It is obtained by orienting the McKay graph of Γ in an arbitrary way. This ambiguity is inessential: as pointed out in [6, Lemma 2.2], the algebras Π^λ and \mathcal{O}^λ we define below are independent of the choice of orientation up to isomorphism. Given $\lambda \in \mathbb{C}^I$, recall [6, p. 606] that the *deformed preprojective algebra* of Q with parameter λ is defined by

$$\Pi^\lambda = \Pi^\lambda(Q) = \mathbb{C}\bar{Q} / \left(\sum_{a \in Q} [a, a^*] - \lambda \right),$$

where the parentheses denote the two-sided ideal generated by the element inside, and \mathcal{O}^λ is the spherical subalgebra $\mathcal{O}^\lambda = e_0 \Pi^\lambda e_0$. Here \bar{Q} denotes the *double* of Q , i.e., the quiver obtained from Q by adding an arrow a^* for each arrow $a \in Q$ such that the tail (respectively, head) of a^* is the head (respectively, tail) of a . We write $\mathbb{C}\bar{Q}$ for the path algebra of \bar{Q} , and $e_i \in \mathbb{C}\bar{Q}$ for the idempotent corresponding to the vertex $i \in I$; the extending vertex of Q is denoted by $0 \in I$. Finally, λ is identified with the element $\sum_{i \in I} \lambda_i e_i \in \mathbb{C}\bar{Q}$.

We define a grading on the algebra $\mathbb{C}\bar{Q}$ by assigning degree 0 to each idempotent e_i , and degree 1 to each arrow $a \in Q$ and its opposite arrow a^* ; in other words, the grading is by the length

of paths. The algebras Π^λ and \mathcal{O}^λ inherit natural filtrations from this grading. This filtration will be used throughout the paper without further explicit mention.

We let $\delta \in \mathbb{N}^I$ denote the minimal positive imaginary root for Q , and we write

$$\Lambda = \{\xi \in \mathbb{Z}^I \mid \xi \cdot \delta = 0\},$$

$$\Lambda_+ = \{\xi \in \Lambda \mid \xi \cdot \alpha \geq 0 \text{ for every positive Dynkin root } \alpha\},$$

$$\Lambda_{++} = \{\xi \in \Lambda \mid \xi \cdot \alpha > 0 \text{ for every positive Dynkin root } \alpha\}.$$

Hereafter, a *root* is an element of the root system associated to the quiver Q , which in this case can be defined as the set of all $\alpha \in \mathbb{Z}^I \setminus \{0\}$ such that $q(\alpha) \leq 1$, where q is the Tits form corresponding to Q . A root α is *Dynkin* if $\alpha \cdot \epsilon_0 = 0$, where $\epsilon_0 \in \mathbb{Z}^I$ is the standard coordinate vector corresponding to the extending vertex (in other words, the coordinate of α corresponding to the extending vertex is zero). A root α is *real* (respectively, *imaginary*) if $q(\alpha) = 1$ (respectively, $q(\alpha) = 0$); note that a Dynkin root is automatically real. There is a natural identification of Λ with the weight lattice of the finite root system associated to the Dynkin diagram obtained by deleting the extending vertex, so that Λ_+ (respectively, Λ_{++}) corresponds to the set of dominant (respectively, dominant regular) weights.

In the second half of this section we discuss geometric constructions related to affine quivers. Let us denote by $\text{Rep}(Q, \delta)$ (respectively, $\text{Rep}(\overline{Q}, \delta)$) the affine space of all representations of Q (respectively, \overline{Q}) with dimension vector $\delta = (\delta_i)_{i \in I}$. Using the trace pairing, $\text{Rep}(\overline{Q}, \delta)$ is naturally identified with the cotangent bundle $T^* \text{Rep}(Q, \delta)$. Let

$$G = PGL(\delta) = \left(\prod_{i \in I} GL(\delta_i, \mathbb{C}) \right) / \mathbb{C}^\times,$$

where \mathbb{C}^\times is embedded diagonally into the product. This is a reductive algebraic group acting by conjugation on the varieties $\text{Rep}(Q, \delta)$ and $\text{Rep}(\overline{Q}, \delta)$, and we write

$$\mu : \text{Rep}(\overline{Q}, \delta) \longrightarrow \mathfrak{g}^*$$

for the moment map for the action of G on $\text{Rep}(Q, \delta)$ (see [6, p. 606]), where

$$\mathfrak{g} = \text{Lie}(G) = \mathfrak{p}gl(\delta) = \left(\prod_{i \in I} \mathfrak{gl}(\delta_i, \mathbb{C}) \right) / \mathbb{C}$$

is the Lie algebra of G .

Using the determinant maps $\det : GL(\delta_i, \mathbb{C}) \rightarrow \mathbb{C}^\times$, we identify Λ with the group of 1-dimensional characters of G . Given $\chi \in \Lambda_{++}$, Cassens and Slodowy [4] construct a minimal resolution of the Kleinian singularity X as the projective morphism

$$\tilde{X} := \text{Proj } S \longrightarrow \text{Spec } S_0 \cong X,$$

where S is the graded algebra

$$S = \bigoplus_{n \geq 0} S_n, \quad S_n = \mathbb{C}[\mu^{-1}(0)]^{G, \chi^n}. \tag{2.1}$$

Here $\mathbb{C}[\mu^{-1}(0)]$ stands for the algebra of regular functions on the *scheme-theoretic* fiber of μ over $0 \in \mathfrak{g}^*$ and $\mathbb{C}[\mu^{-1}(0)]^{G \cdot \chi^n}$ denotes the G -eigenspace corresponding to the character χ^n . Note that each component S_n of S is itself graded, where the grading is induced by the grading on $\mathbb{C}[\mu^{-1}(0)]$, which in turn is induced by the grading of $\mathbb{C}[\text{Rep}(\overline{Q}, \delta)]$ by the degree of polynomials (we use the fact that $\text{Rep}(\overline{Q}, \delta)$ is an affine space).

The minimal resolution $\tilde{X} \rightarrow X$ is studied in more detail in Section 3, where we also prove a result (Theorem 5) on the structure of the ring S that, to the best of our knowledge, does not appear in the existing literature.

3. A study of the minimal resolution

Recall the notation $X = \mathbb{C}^2/\Gamma$ and $\mu : \text{Rep}(\overline{Q}, \delta) \rightarrow \mathfrak{g}^*$ introduced previously. By definition, we have

$$\mathbb{C}[\mu^{-1}(0)] = \frac{\mathbb{C}[\text{Rep}(\overline{Q}, \delta)]}{\mathbb{C}[\text{Rep}(\overline{Q}, \delta)] \cdot \mu^*(\mathfrak{g})}, \tag{3.1}$$

where $\mu^*(\mathfrak{g})$ denotes the linear subspace of $\mathbb{C}[\text{Rep}(\overline{Q}, \delta)]$ obtained by pulling back via μ the elements of \mathfrak{g} viewed as linear functions on \mathfrak{g}^* . By a result of Crawley-Boevey [5, Theorem 1.2], the scheme $\mu^{-1}(0)$ is in fact reduced and irreducible. (The reason for defining $\mu^{-1}(0)$ as the scheme-theoretic fiber is that (3.1) will be important for us later on.) We define

$$R = \mathbb{C}[\mu^{-1}(0)]^G.$$

It is well known that

$$\text{Spec } R = \text{Spec } \mathbb{C}[\mu^{-1}(0)]^G = \mu^{-1}(0)//G \cong X. \tag{3.2}$$

In particular, R is a normal, 2-dimensional, commutative Gorenstein domain.

Cassens and Slodowy [4, §7] explain that a minimal resolution of X can be constructed as a GIT quotient

$$\tilde{X} = \mu^{-1}(0)_{\chi}^{ss}/G \tag{3.3}$$

for any $\chi \in \Lambda_{+++}$, where $\mu^{-1}(0)_{\chi}^{ss}$ denotes the open subset of $\mu^{-1}(0)$ consisting of the points *semistable* with respect to χ . Moreover, they prove that for each such χ ,

- (A) $\mu^{-1}(0)_{\chi}^{ss} = \mu^{-1}(0)_{\chi}^s$, the set of *stable* points with respect to χ , and
- (B) the action of G on $\mu^{-1}(0)_{\chi}^s$ is *free* (recall that, a priori, the action of G on the set of stable points only needs to have finite stabilizers; in our situation, however, all stabilizers turn out to be trivial).

Furthermore, (3.2) and (3.3) lead to the description of the resolution $\tilde{X} \rightarrow X$ as the natural map

$$\text{Proj } S \longrightarrow \text{Spec } S_0,$$

where S is the graded ring defined by (2.1).

Lemma 3. *The algebra S is finitely generated.*

This is a special case of a very general statement:

Lemma 4. *Let Y be an affine scheme of finite type over \mathbb{C} , let G be a complex reductive group acting algebraically on Y , and put $T = \mathbb{C}[Y]$. For any algebraic homomorphism $\chi : G \rightarrow \mathbb{C}^\times$, the algebra $\bigoplus_{n \geq 0} T^{G, \chi^n}$ is finitely generated.*

Proof. Consider the induced action of G on $Y \times \mathbb{C}$, where the action on the first factor is the given one, and the action on \mathbb{C} is via χ . The ring $\mathbb{C}[Y \times \mathbb{C}] = T \otimes \mathbb{C}[z]$ has the obvious grading by the degree of polynomials with respect to z , and we clearly have an isomorphism of graded algebras

$$\mathbb{C}[Y \times \mathbb{C}]^G \cong \bigoplus_{n \geq 0} T^{G, \chi^n}.$$

In particular, the algebra on the right-hand side is finitely generated (here we have used the fact that G is reductive). \square

The main goal of this section is to obtain some more detailed information on the ring S , in the form of the following result.

Theorem 5. *Let $p : \mu^{-1}(0)_{\chi}^{ss} \rightarrow \tilde{X}$ denote the quotient map.*

- (1) *There exists a unique line bundle \mathcal{L} on \tilde{X} such that $p^*\mathcal{L}$ is the trivial line bundle on $\mu^{-1}(0)_{\chi}^{ss}$ equipped with the G -linearization given by the character χ . Moreover, \mathcal{L} is ample.*
- (2) *The induced map*

$$S_n = \mathbb{C}[\mu^{-1}(0)]^{G, \chi^n} \longrightarrow \Gamma(\tilde{X}, \mathcal{L}^{\otimes n})$$

is an isomorphism for sufficiently large n . In particular, S_n is a torsion-free S_0 -module of generic rank 1 for sufficiently large n .

- (3) *The multiplication map*

$$S_m \otimes S_n \longrightarrow S_{m+n}$$

is surjective for sufficiently large m and n .

It will be clear from the proof of the theorem that essentially the only properties that we use are the fact that G is reductive, statements (A) and (B) above, and the fact that $\mu^{-1}(0)_{\chi}^{ss}$ is dense in $\mu^{-1}(0)$. Thus the theorem could be stated and proved in a much more general context, where $\mu^{-1}(0)$ is replaced by any affine variety Y with an action of G satisfying properties (A) and (B), such that the set of semistable points Y_{χ}^{ss} is dense in Y .

We begin the proof of Theorem 5 by observing that (A) and (B) imply that the quotient map p is a principal G -bundle. Now it is easy to see that the notion of a G -linearization for a coherent sheaf on $\mu^{-1}(0)_{\chi}^{ss}$ is equivalent to the notion of a descent datum for the (flat) morphism p . Hence the first statement of part (1) of the theorem follows immediately from flat descent theory (see,

for example, [7, Exposé I, Théorème 4.5]). Another consequence of descent theory is that for any line bundle \mathcal{M} on \tilde{X} , we have

$$\Gamma(\tilde{X}, \mathcal{M}) = \Gamma(\mu^{-1}(0)_{\chi}^{ss}, p^* \mathcal{M})^G. \tag{3.4}$$

Indeed, the left-hand side of (3.4) coincides with $\text{Hom}(\mathcal{O}_{\tilde{X}}, \mathcal{M})$. But $p^* \mathcal{O}_{\tilde{X}}$ is the trivial line bundle on $\mu^{-1}(0)_{\chi}^{ss}$ equipped with the trivial G -linearization, and descent theory for morphisms [7] implies that

$$\text{Hom}(\mathcal{O}_{\tilde{X}}, \mathcal{M}) = \text{Hom}_{G\text{-equiv}}(p^* \mathcal{O}_{\tilde{X}}, p^* \mathcal{M}),$$

which proves (3.4).

Mumford’s construction of the quotient (3.3) shows that for some $N \in \mathbb{N}$, there exists an ample line bundle \mathcal{L}' on \tilde{X} such that $p^* \mathcal{L}'$ is the trivial line bundle on $\mu^{-1}(0)_{\chi}^{ss}$ equipped with the G -linearization given by χ^N (see [15, Theorem 1.10(ii)]). Moreover, $\mathcal{L}' = \mathcal{O}_{\tilde{X}}(N)$ for the description of \tilde{X} as $\text{Proj } S$. Now the discussion in the previous paragraph implies that $\mathcal{L}' \cong \mathcal{L}^{\otimes N}$; in particular, \mathcal{L} itself is ample, completing the proof of part (1) of the theorem.

The arguments that follow are rather standard, however, we find it easier to give them than to find specific places in the literature where these arguments are presented in exactly the form we need. Replacing N by one of its multiples if necessary, we may assume that

$$S_{jN} = (S_N)^j \quad \text{for all } j \geq 1;$$

this follows from the fact that S is finitely generated (Lemma 3) and [11, Lemma 2.1.6(v)]. Similarly, we may assume that $\mathcal{L}^{\otimes n}$ is very ample and generated by global sections for all $n \geq N$ (using [11, Proposition 4.5.10(ii)]). And, finally, we may assume that the natural map

$$S_{jN} \longrightarrow \Gamma(\tilde{X}, \mathcal{L}^{\otimes jN})$$

is an isomorphism for all $j \geq 1$. From now on we fix $N \in \mathbb{N}$ satisfying all the properties listed above.

In particular, each of the bundles

$$\mathcal{L}^{\otimes N}, \quad \mathcal{L}^{\otimes(N+1)}, \quad \dots, \quad \mathcal{L}^{\otimes(2N-1)} \tag{3.5}$$

is generated by global sections. But for any $n \in \mathbb{N}$, we have, from (3.4),

$$\Gamma(\tilde{X}, \mathcal{L}^{\otimes n}) = \Gamma(\mu^{-1}(0)_{\chi}^{ss}, \mathcal{O})^{G, \chi^n}.$$

Now recall that $\mu^{-1}(0)_{\chi}^{ss}$ is the set of points of $\mu^{-1}(0)$ where at least one element of $\mathbb{C}[\mu^{-1}(0)]^{G, \chi^N} = S_N$ does not vanish. In particular, if $\sigma \in \Gamma(\tilde{X}, \mathcal{L}^{\otimes n})$, then there exist $j \in \mathbb{N}$ and finitely many elements $f_1, \dots, f_r \in S_{jN}$ such that $f_i \sigma \in S_{n+jN}$ for each i , and the open sets $\{f_i \neq 0\}$ cover all of $\mu^{-1}(0)_{\chi}^{ss}$.

Since we are dealing with finitely many line bundles (3.5), we deduce that there exists $d_1 \in \mathbb{N}$ such that for every $0 \leq k \leq N - 1$, the line bundle $\mathcal{L}^{\otimes(N+d_1N+k)}$ is generated by finitely many sections coming from the elements of S_{N+d_1N+k} .

For every $0 \leq k \leq N - 1$, let us now choose a finite-dimensional subspace

$$V_k \subseteq S_{N+d_1N+k} \subseteq \Gamma(\tilde{X}, \mathcal{L}^{\otimes(N+d_1N+k)})$$

of sections which generate the line bundle $\mathcal{L}^{\otimes(N+d_1N+k)}$. These sections determine a surjection of coherent sheaves

$$\phi_k : \mathcal{O}_{\tilde{X}} \otimes_{\mathbb{C}} V_k \longrightarrow \mathcal{L}^{\otimes(N+d_1N+k)}. \tag{3.6}$$

Let \mathcal{N}_k denote the kernel of this surjection; it is a coherent sheaf on \tilde{X} . Since $\mathcal{L}^{\otimes N}$ is very ample, there exists $d_2 \in \mathbb{N}$ such that

$$H^1(\tilde{X}, \mathcal{L}^{\otimes jN} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{N}_k) = 0 \tag{3.7}$$

for every $j \geq d_2$ and every $0 \leq k \leq N - 1$.

We can now prove part (2) of Theorem 5. Namely, every integer $n \geq (1 + d_1 + d_2) \cdot N$ can be written as $n = jN + N + d_1N + k$ for some (uniquely determined) $j \geq d_2$ and $0 \leq k \leq N - 1$. We have a short exact sequence, induced by (3.6):

$$0 \longrightarrow \mathcal{L}^{\otimes jN} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{N}_k \longrightarrow \mathcal{L}^{\otimes jN} \otimes_{\mathbb{C}} V_k \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow 0.$$

Applying the long exact cohomology sequence and using (3.7), we see that the map

$$\Gamma(\tilde{X}, \mathcal{L}^{\otimes jN} \otimes_{\mathbb{C}} V_k) \longrightarrow \Gamma(\tilde{X}, \mathcal{L}^{\otimes n})$$

is surjective. But

$$\Gamma(\tilde{X}, \mathcal{L}^{\otimes jN} \otimes_{\mathbb{C}} V_k) = \Gamma(\tilde{X}, \mathcal{L}^{\otimes jN}) \otimes_{\mathbb{C}} V_k = S_{jN} \otimes_{\mathbb{C}} V_k \subseteq S_{jN} \otimes_{\mathbb{C}} S_{N+d_1N+k},$$

and so, a fortiori, the natural map

$$S_n \longrightarrow \Gamma(\tilde{X}, \mathcal{L}^{\otimes n})$$

is surjective for all $n \geq (1 + d_1 + d_2) \cdot N$. Also, this map is injective for all n because $\mu^{-1}(0)_{\tilde{X}}^{ss}$ is dense in $\mu^{-1}(0)$ (since $\mu^{-1}(0)$ is irreducible). Observe moreover that $\Gamma(\tilde{X}, \mathcal{L}^{\otimes n})$ is a finitely generated $\mathbb{C}[\mu^{-1}(0)]$ -module of generic rank 1, since $\tilde{X} \rightarrow X$ is a projective birational map which is an isomorphism away from the fiber over the singular point $0 \in X$. This proves part (2) of the theorem.

Finally, part (3) of Theorem 5 follows immediately from parts (1) and (2) and the following general result.

Proposition 6. *Let Y be a scheme, projective over a (commutative) Noetherian ring A , and let \mathcal{L} be an ample invertible sheaf on Y . Then the natural map*

$$\Gamma(Y, \mathcal{L}^{\otimes m}) \otimes_A \Gamma(Y, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(Y, \mathcal{L}^{\otimes(m+n)}) \tag{3.8}$$

is surjective for all sufficiently large m and n .

For a proof in the case where A is a field, we refer the reader to [14]. The proof in the general case is exactly the same.

4. Quantization of Kleinian singularities

In this section we recall some results of M.P. Holland [12] that are crucial for our construction of the quantization of the minimal resolution. We will use the notations

$$\Gamma, I, Q, \delta, G, \mathfrak{g}, \mu, \text{ etc.}$$

defined in the previous sections; in particular, the coordinates of the vector δ are denoted by δ_i , $i \in I$. If $a \in Q$ is an arrow, we write $t(a), h(a) \in I$ for the tail and head of a , respectively. The defect $\partial \in \mathbb{Z}^I$ is defined by

$$\partial_i = -\delta_i + \sum_{t(a)=i} \delta_{h(a)} \quad \text{for all } i \in I.$$

We identify $\mathbb{C}_0^I := \{\chi \in \mathbb{C}^I \mid \chi \cdot \delta = 0\}$ with the space of 1-dimensional characters of \mathfrak{g} via the various trace maps $\mathfrak{gl}(\delta_i, \mathbb{C}) \rightarrow \mathbb{C}$. If $\chi \in \mathbb{C}_0^I$, we define a filtered algebra

$$\mathcal{U}^\chi = \frac{\mathcal{D}(\text{Rep}(Q, \delta))^G}{[\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^G}, \tag{4.1}$$

where $\mathcal{D}(\text{Rep}(Q, \delta))$ is the algebra of polynomial differential operators on the affine space $\text{Rep}(Q, \delta)$, and

$$\iota : \mathfrak{g} \longrightarrow \text{Vect}(\text{Rep}(Q, \delta)) \subset \mathcal{D}(\text{Rep}(Q, \delta))$$

is the Lie algebra map induced by the G -action.

Caution. For consistency with the filtration on the algebras \mathcal{O}^λ introduced in Section 2, we need to use the Bernstein filtration on the algebra $\mathcal{D}(\text{Rep}(Q, \delta))$ (instead of the more standard order filtration), which is defined by assigning degree 1 to the linear coordinate functions and to the coordinate vector fields. Fortunately, as remarked in [12], the results of Sections 2–4 of that paper remain valid if the order filtration is replaced by the Bernstein filtration. From now on it will be implicitly assumed that the results of all constructions involving differential operators will be equipped with filtrations induced from the Bernstein filtration.

Theorem 7. (Holland) *If $\lambda \in \mathbb{C}^I$ is such that $\lambda \cdot \delta = 1$, then there is a natural isomorphism of filtered algebras*

$$\mathcal{O}^\lambda \cong \mathcal{U}^{\lambda - \partial - \epsilon_0},$$

where $\epsilon_0 \in \mathbb{Z}^I$ is the standard basis vector corresponding to the extending vertex.

Proof. See [12, Corollary 4.7]. \square

The following result will also be important to us.

Theorem 8. *There are natural isomorphisms of graded algebras*

$$\text{gr}^\bullet \mathcal{D}(\text{Rep}(Q, \delta)) \cong \mathbb{C}[T^* \text{Rep}(Q, \delta)] \cong \mathbb{C}[\text{Rep}(\overline{Q}, \delta)],$$

where the gradings on the last two come from viewing $T^* \text{Rep}(Q, \delta)$ and $\text{Rep}(\overline{Q}, \delta)$ as vector spaces (i.e., linear functions on $T^* \text{Rep}(Q, \delta)$ and $\text{Rep}(\overline{Q}, \delta)$ are assigned degree 1). In addition, if $\chi \in \mathbb{C}_0^I$, there is a natural isomorphism

$$\text{gr}^\bullet \left(\frac{\mathcal{D}(\text{Rep}(Q, \delta))}{\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})} \right) \cong \frac{\text{gr}^\bullet \mathcal{D}(\text{Rep}(Q, \delta))}{[\text{gr}^\bullet \mathcal{D}(\text{Rep}(Q, \delta))] \cdot \mathfrak{g}}$$

of graded $\mathbb{C}[T^* \text{Rep}(Q, \delta)]$ -modules.

Proof. In the first statement, the first isomorphism is just a general statement about differential operators on a vector space, and the second one follows from the identification of $T^* \text{Rep}(Q, \delta)$ with $\text{Rep}(\overline{Q}, \delta)$ (see, e.g., [12, p. 820]). For the last isomorphism, combine [12, Proposition 2.4] with the fact that the moment map $\mu : \text{Rep}(\overline{Q}, \delta) \rightarrow \mathfrak{g}^*$ is flat [6, Lemma 8.3]. \square

5. Morita \mathbb{Z} -algebras

In this section we review the basic theory of \mathbb{Z} -algebras following [9]. We also give a detailed proof of a strengthening of Lemma 5.5 of [9] that is used in our paper.

Definition 9. A lower-triangular \mathbb{Z} -algebra is an abelian group B , bigraded by \mathbb{Z} in the following way:

$$B = \bigoplus_{i \geq j \geq 0} B_{ij},$$

and equipped with an associative \mathbb{Z} -bilinear multiplication satisfying

$$B_{ij} B_{jk} \subseteq B_{ik}, \quad B_{ij} B_{lk} = 0 \quad \text{if } j \neq l.$$

In particular, each $B_i := B_{ii}$ is an associative ring in the usual sense, and hence, according to our conventions, is required to have a unit. Moreover, each B_{ij} is a (B_i, B_j) -bimodule, and the units of B_i and B_j are required to act as the identity on B_{ij} . However, B will almost never have a unit since it is defined as an infinite direct sum.

Next we consider modules over \mathbb{Z} -algebras.

Definition 10. Let B be a lower-triangular \mathbb{Z} -algebra as in the definition above. A graded B -module is a positively graded abelian group

$$M = \bigoplus_{i \geq 0} M_i$$

equipped with a left B -module structure satisfying

$$B_{ij}M_j \subseteq M_i, \quad B_{ij}M_l = 0 \quad \text{if } l \neq k.$$

In particular, each M_i is a left B_i -module, and hence, according to our conventions, is assumed to be unital.

With these definitions at hand, we can construct several categories of modules as follows. If B is a \mathbb{Z} -algebra, we define B -grmod to be the category of Noetherian graded B -modules, we define B -tors to be the full subcategory consisting of *bounded* modules (i.e., $M \in B$ -grmod such that $M_n = (0)$ for $n \gg 0$), and we define B -qgr as the Serre quotient of B -grmod by B -tors. The philosophy behind this definition is that one should think of “Proj B ” as a “noncommutative projective scheme,” and of B -qgr as the category of coherent sheaves on Proj B . It is clear that for a (nonnegatively) graded ring A , we can define the categories A -grmod, A -tors and A -qgr in a similar way. If A is commutative, Noetherian and generated by A_1 as an A_0 -algebra, then Serre’s classical theorem implies that the category of coherent sheaves on Proj A is in fact equivalent to A -qgr.

On the other hand, $A = \bigoplus_{n \geq 0} A_n$ is a graded ring, we can associate to it a lower-triangular \mathbb{Z} -algebra $B = \widehat{A}$ by defining $B_{ij} = A_{i-j}$ for $i \geq j \geq 0$. As explained in [9, §5.3], we then have a natural equivalence of categories

$$A\text{-qgr} \xrightarrow{\sim} \widehat{A}\text{-qgr}.$$

We are now ready for the key definition; note that it is *weaker* than the corresponding notion introduced in [9, §5.4].

Definition 11. A *Morita \mathbb{Z} -algebra* is a lower-triangular \mathbb{Z} -algebra

$$B = \bigoplus_{i \geq j \geq 0} B_{ij}$$

such that there exists $N \in \mathbb{N}$ for which:

- (i) the (B_i, B_j) -bimodule B_{ij} yields an equivalence

$$B_j\text{-mod} \xrightarrow{\sim} B_i\text{-mod}$$

whenever $i - j \geq N$; and

- (ii) the multiplication map

$$B_{ij} \otimes_{B_j} B_{jk} \longrightarrow B_{ik}$$

is an isomorphism whenever $i - j, j - k \geq N$.

Under these assumptions, we also say that B is *Morita equivalent* to B_0 . This terminology is explained by the following result.

Theorem 12. *Suppose that B is a Morita \mathbb{Z} -algebra such that each B_i is a left Noetherian ring, and each B_{ij} is a finitely generated left B_i -module. Then:*

- (1) *Each finitely generated graded left B -module is graded-Noetherian.*
- (2) *The association*

$$\phi : M \longmapsto \bigoplus_{n \geq 0} B_{n,0} \otimes_{B_0} M$$

induces an equivalence of categories

$$\Phi : B_0\text{-mod} \xrightarrow{\sim} B\text{-qgr}.$$

Proof. This result is an analogue of Lemma 5.5 in [9]. Our original proof followed the ideas used in [9], but was done from scratch. Following the referee’s suggestion, we present a shorter argument for part (2) of the theorem which deduces it from the result of [9].

(1) Let M be a finitely generated graded B -module. We have to show that every graded submodule of M is also finitely generated. It is clear that M is generated by finitely many homogeneous elements, so it is enough to consider the case where M is generated by one homogeneous element, say of degree a . In this case M is a graded homomorphic image of $\bigoplus_{j \geq a} B_{ja}$, so we assume, without loss of generality, that

$$M = \bigoplus_{j \geq a} B_{ja}.$$

Now let

$$L = \bigoplus_{j \geq a} L_j \subseteq M$$

be a graded submodule. We use the notation

$$B_{ij}^* = \text{Hom}_{B_i\text{-mod}}(B_{ij}, B_i),$$

which is a (B_j, B_i) -bimodule. Let $N \in \mathbb{N}$ be as in the definition of a Morita \mathbb{Z} -algebra. Then for $j \geq a + N$, we have a chain of maps of left B_a -modules

$$B_{ja}^* \otimes_{B_j} L_j \hookrightarrow B_{ja}^* \otimes_{B_j} M_j = B_{ja}^* \otimes_{B_j} B_{ja} \xrightarrow{\cong} B_a,$$

where the first map is injective because B_{ja}^* is a projective right B_j -module, and the second map is an isomorphism by definition. We let

$$X(j) \subseteq B_a$$

be the image of the composition above. Since B_a is assumed to be Noetherian, there exists an integer $b \geq a + N$ such that

$$\sum_{j \geq a+N} X(j) = \sum_{i=a+N}^b X(i) \subseteq B_a.$$

Now for $k \geq a + N$, we have

$$L_k \cong B_{ka} \otimes_{B_a} B_{ka}^* \otimes_{B_k} L_k,$$

which means that

$$L_k = B_{ka} X(k) \quad \text{for } k \geq a + N,$$

as submodules of $B_{ka} = M_k$. Thus, for $k \geq b + N$, we have

$$L_k = B_{ka} X(k) \subseteq \sum_{i=a+N}^b B_{ka} X(i) = \sum_{i=a+N}^b B_{ki} B_{ia} X(i) = \sum_{i=a+N}^b B_{ki} L_i,$$

where we have used the assumption that $B_{ki} \otimes_{B_i} B_{ia} \xrightarrow{\cong} B_{ka}$ for $k - i, i - a \geq N$. Thus we see that

$$\sum_{k \geq b+N} L_k \text{ is generated by } \sum_{i=a+N}^b L_i$$

as a B -module. Finally, for $a \leq j \leq b + N$, L_j is a B_j -submodule of the finitely generated B_j -module $M_j = B_{ja}$, and is therefore finitely generated, completing the proof of (1).

(2) Fix $N \in \mathbb{N}$ satisfying the condition of Definition 11, and consider $B^{(N)} = \bigoplus_{i \geq j \geq 0} B_{iN, jN}$. Note that this algebra satisfies the stronger version of the definition of a Morita \mathbb{Z} -algebra. In addition to the functor $\Phi : B_0\text{-mod} \rightarrow B\text{-qgr}$ introduced in the theorem, consider the functors $\Psi_N : B\text{-qgr} \rightarrow B^{(N)}\text{-qgr}$ and $\Theta_N : B^{(N)}\text{-qgr} \rightarrow B_0\text{-mod}$ induced by

$$P = \bigoplus_i P_i \mapsto P^{(N)} = \bigoplus_j P_{jN} \quad \text{and}$$

$$Q = \bigoplus_j Q_{jN} \mapsto B_{Nj,0}^* \otimes_{B_{jN}} Q_{jN} \quad \text{for } j \gg 0,$$

respectively. The functor Θ_N is well defined by Lemma 5.5 in [9], which also shows that $\Psi_N \circ \Phi$ and Θ_N are mutually quasi-inverse equivalences of categories between $B_0\text{-mod}$ and $B^{(N)}\text{-qgr}$. In addition, it is clear from Definition 11 that Ψ_N is fully faithful; since $\Psi_N \circ \Phi$ is an equivalence of categories, we see that Ψ_N must a fortiori be essentially surjective, and hence it is an equivalence of categories. Finally, since $\Psi_N \circ \Phi$ is quasi-inverse to Θ_N , and since both Ψ_N and Θ_N are equivalences of categories, so is Φ . \square

6. Quantization of the minimal resolution

In this section we use Holland’s results described in Section 4 to define the algebras $B^\lambda(\chi)$ mentioned in Theorem 1 and restate the latter in a more explicit way. We use the same notation as in Sections 2 and 4. We let $\mathfrak{X}(G)$ denote the group of the algebraic group homomorphisms $\zeta : G \rightarrow \mathbb{C}^\times$. Note that if $\zeta \in \mathfrak{X}(G)$, its differential $d\zeta : \mathfrak{g} \rightarrow \mathbb{C}$ can be thought of as an element of \mathbb{C}_0^I (see Section 4).

Given $\zeta \in \mathfrak{X}(G)$ and $\chi \in \mathbb{C}_0^I$, we define

$$P_{\chi,\zeta} = \frac{\mathcal{D}(\text{Rep}(Q, \delta))^{G,\zeta}}{[\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^{G,\zeta}}.$$

Lemma 13. *The actions of $\mathcal{D}(\text{Rep}(Q, \delta))^G$ on $\mathcal{D}(\text{Rep}(Q, \delta))^{G,\zeta}$ by left and right multiplication descend to a $(\mathcal{U}^{\chi+d\zeta}, \mathcal{U}^\chi)$ -bimodule structure on $P_{\chi,\zeta}$.*

Proof. Recalling the definition (4.1) of the algebras \mathcal{U}^χ and $\mathcal{U}^{\chi+d\zeta}$, we see that in order to prove the lemma we need to verify that each of the following four expressions:

$$\mathcal{D}(\text{Rep}(Q, \delta))^G \cdot [\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^{G,\zeta}, \tag{6.1}$$

$$[\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^{G,\zeta} \cdot \mathcal{D}(\text{Rep}(Q, \delta))^G, \tag{6.2}$$

$$[\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi - d\zeta)(\mathfrak{g})]^G \cdot \mathcal{D}(\text{Rep}(Q, \delta))^{G,\zeta}, \tag{6.3}$$

and

$$\mathcal{D}(\text{Rep}(Q, \delta))^{G,\zeta} \cdot [\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^G, \tag{6.4}$$

is contained in

$$[\mathcal{D}(\text{Rep}(Q, \delta)) \cdot (\iota - \chi)(\mathfrak{g})]^{G,\zeta}.$$

This is quite easy to see for (6.1) and (6.4). For (6.2) this is also not hard once we remember that the elements of $\mathcal{D}(\text{Rep}(Q, \delta))^G$ commute with $\iota(\mathfrak{g})$ and hence with $(\iota - \chi)(\mathfrak{g})$. The most interesting one is (6.3). Consider an element of this product, written as $(\sum L_j \cdot (\iota - \chi - d\zeta)(x_j)) \cdot M$, where $M \in \mathcal{D}(\text{Rep}(Q, \delta))^{G,\zeta}$, $L_j \in \mathcal{D}(\text{Rep}(Q, \delta))$ and $x_j \in \mathfrak{g}$. The assumption on M implies that $[\iota(x), M] = d\zeta(x) \cdot M$ for every $x \in \mathfrak{g}$. Therefore

$$\begin{aligned} \left(\sum L_j \cdot (\iota - \chi - d\zeta)(x_j)\right) \cdot M &= \sum L_j \cdot M \cdot (\iota - \chi - d\zeta)(x_j) \\ &\quad + \sum L_j \cdot d\zeta(x_j) \cdot M \\ &= \sum L_j \cdot M \cdot (\iota - \chi)(x_j), \end{aligned}$$

which completes the proof. \square

If $\lambda \in \mathbb{C}^I$ is such that $\lambda \cdot \delta = 1$, we will write

$$P_\zeta^\lambda = P_{\lambda - \partial - \epsilon_0, \zeta}.$$

By Theorem 7 and Lemma 13, if we equip P_ζ^λ with the filtration induced by the Bernstein filtration on $\mathcal{D}(\text{Rep}(Q, \delta))$, we can think of it as a filtered $(\mathcal{O}^{\lambda+d\zeta}, \mathcal{O}^\lambda)$ -bimodule. Moreover, by (3.1) and Theorem 8, we have an isomorphism of graded bimodules

$$\text{gr}^\bullet P_\zeta^\lambda \cong \mathbb{C}[\mu^{-1}(0)]^{G, \zeta}. \tag{6.5}$$

The following fact will be used implicitly in Section 7; it is needed in order to justify the use of Proposition 21(1).

Lemma 14. *The filtration on P_ζ^λ induced by the Bernstein filtration on differential operators is good in the sense of [1, Definition 2.19].*

Proof. In view of (6.5) and the remark after the proof of Proposition 2.22 in [1], it suffices to show that $\mathbb{C}[\mu^{-1}(0)]^{G, \zeta}$ is a finitely generated $\mathbb{C}[\mu^{-1}(0)]^G$ -module for every ζ . Now the algebra $\bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G, \zeta^n}$ is finitely generated by the argument given in the proof of Lemma 3, and therefore our claim follows from Lemma 2.1.6(i) in [11]. \square

Observe now that differentiation of characters induces an isomorphism of abelian groups $d: \mathfrak{X}(G) \xrightarrow{\cong} \Lambda \subseteq \mathbb{C}_0^I$; by abuse of notation, if $\xi \in \Lambda$, we will write

$$P_\xi^\lambda = P_\zeta^\lambda \quad \text{and} \quad \mathbb{C}[\mu^{-1}(0)]^{G, \xi} = \mathbb{C}[\mu^{-1}(0)]^{G, \zeta}, \tag{6.6}$$

where $\zeta \in \mathfrak{X}(G)$ is such that $d\zeta = \xi$. Now, given $\chi \in \Lambda_{++}$, we define a lower-triangular \mathbb{Z} -algebra $B(\lambda, \chi)$ by

$$B(\lambda, \chi)_{ij} = \begin{cases} \mathcal{O}^{\lambda+j\cdot\chi} & \text{if } i = j \geq 0, \\ P_{(i-j)\cdot\chi}^{\lambda+j\cdot\chi} & \text{if } i > j \geq 0. \end{cases}$$

All the structure maps of this \mathbb{Z} -algebra are induced by the multiplication of elements of $\mathcal{D}(\text{Rep}(Q, \delta))^G$ (cf. Lemma 13), and all compatibility conditions follow immediately from the associativity of this multiplication. Observe that $B(\lambda, \chi)$ is naturally filtered by the Bernstein filtration on differential operators. (We leave the formulation of the general notion of a filtered \mathbb{Z} -algebra B to the reader: each component B_{ij} should be positively filtered, and all the structure maps should be compatible with the filtrations. See also [10].)

We are now ready to state our main result:

Theorem 15. *Let $\lambda \in \mathbb{C}^I$ be such that $\lambda \cdot \delta = 1$ and $\lambda \cdot \alpha \neq 0$ for every Dynkin root α (i.e., the algebra \mathcal{O}^λ has finite global dimension, cf. Remark 2). Given $\chi \in \Lambda_{++}$, there exists $\xi \in \Lambda_{++}$ such that the lower-triangular filtered \mathbb{Z} -algebra*

$$B^\lambda(\chi) := B(\lambda + \xi, \chi),$$

where $B(\lambda + \xi, \chi)$ is constructed above, satisfies the properties:

- (1) $\mathcal{O}^\lambda\text{-mod}$ is naturally equivalent to $B^\lambda(\chi)\text{-qgr}$, in a way compatible with filtrations, and
- (2) there is a natural isomorphism $\text{gr}^\bullet B^\lambda(\chi) \cong \widehat{S}$, where \widehat{S} is the \mathbb{Z} -algebra associated to the graded algebra S defined by (2.1).

Part (2) of the theorem follows trivially from the definitions and from (6.5). The proof of part (1) occupies Section 7. The words “in a way compatible with filtrations” mean that the functor $\mathcal{O}^\lambda\text{-mod} \rightarrow B^\lambda(\chi)\text{-qgr}$ defining the equivalence admits a natural extension to a functor defined between the corresponding categories of filtered modules, and the extension is also an equivalence. This will in fact be obvious from the proof we give. The two statements of the theorem together make it obvious in what sense \mathcal{O}^λ deserves to be called a quantization of \widetilde{X} , namely, the “noncommutative projective scheme” $\text{Proj } B^\lambda(\chi)$ (constructed using quantum Hamiltonian reduction) deforms \widetilde{X} by part (2), and the category of coherent sheaves on $\text{Proj } B^\lambda(\chi)$ is equivalent to the category of finitely generated \mathcal{O}^λ -modules by part (1).

Note that the construction of $B^\lambda(\chi)$ depends on the choice of ξ . However, this dependence is not very serious, since two different choices of ξ lead to naturally Morita equivalent \mathbb{Z} -algebras, which is why ξ is omitted from the notation. Furthermore, if \mathcal{O}^λ has no nonzero finite-dimensional modules, one can take $\xi = 0$, and we conjecture that one can always take $\xi = 0$ as long as λ is dominant (see Remark 24).

More importantly, $B^\lambda(\chi)$ also depends on the choice of $\chi \in \Lambda_{+++}$. For this reason our quantization of the minimal resolution \widetilde{X} may be called “non-canonical.” A more canonical version of the quantization would consist of replacing $B^\lambda(\chi)$ by a “lower-triangular Λ -algebra,” bigraded by Λ_+ instead of \mathbb{Z}_+ , and realizing \widetilde{X} as a suitable “multi- Proj ” of the Λ_+ -graded ring $\bigoplus_{\chi \in \Lambda_+} \mathbb{C}[\mu^{-1}(0)]^{G,\chi}$. This idea was implemented for Kleinian singularities of type A by Musson in [16], using different methods. Note, however, that such a construction cannot be obtained by a straightforward modification of the results of the present paper. The most apparent reason for this is that we repeatedly make crucial use of the following simple fact: given a natural number N , every integer $n \geq 2N - 1$ can be written as a sum of integers that lie between N and $2N - 1$. However, this fact has no suitable analogue for lattices other than \mathbb{Z} , in the sense that if $\text{rk } \Lambda \geq 2$, then Λ_{+++} is not finitely generated as a monoid. This issue will be addressed in [3].

7. Proof of the main theorem

In this section we prove part (1) of Theorem 15. The idea of the proof can be summarized as follows. First we need to reduce the proof to a situation where Theorem 12 can be applied. To verify that $B^\lambda(\chi)$ is a Morita \mathbb{Z} -algebra we study the associated graded modules of the bimodules

$$P_{ij} := P_{(i-j)\chi}^{\lambda+\xi+j\chi}, \quad i > j \geq 0.$$

by means of Theorem 5. Finally we pass from the results on $\text{gr}^\bullet P_{ij}$ to the corresponding results on P_{ij} for an appropriate choice of $\xi \in \Lambda_{+++}$.

7.1. Affine Weyl groups and shift functors

The first step is accomplished by

Proposition 16. *Let $\lambda \in \mathbb{C}^I$ and $\xi \in \Lambda$ be such that $\lambda \cdot \delta = 1$ and the algebras \mathcal{O}^λ and $\mathcal{O}^{\lambda+\xi}$ both have finite global dimension. Then there exists an equivalence of categories*

$$\mathcal{O}^\lambda\text{-mod} \xrightarrow{\sim} \mathcal{O}^{\lambda+\xi}\text{-mod}, \tag{7.1}$$

compatible with filtrations in the obvious sense.

Proof. Under the assumptions of the proposition, the results of [6, Corollary 6.4, Theorem 9.5 and Corollary 9.6] imply that the functors $\Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda}$ - and $e_0 \Pi^{\lambda+\xi} \otimes_{\Pi^{\lambda+\xi}}$ - provide equivalence of categories

$$\mathcal{O}^\lambda\text{-mod} \xrightarrow{\sim} \Pi^\lambda\text{-mod} \quad \text{and} \quad \Pi^{\lambda+\xi}\text{-mod} \xrightarrow{\sim} \mathcal{O}^{\lambda+\xi}\text{-mod}$$

that are compatible with filtrations. Hence, if we can show that there is an equivalence $\Pi^\lambda\text{-mod} \xrightarrow{\sim} \Pi^{\lambda+\xi}\text{-mod}$ that is also compatible with filtrations, we can define (7.1) as the composition of these three equivalences.

Now let E denote the affine space of all $\lambda \in \mathbb{C}^I$ such that $\lambda \cdot \delta = 1$. Each simple reflection s_i corresponding to a vertex $i \in I$ of Q defines an automorphism $r_i : E \rightarrow E$, and the reflection functors of [6, §5] provide an equivalence $\Pi^\lambda\text{-mod} \xrightarrow{\sim} \Pi^{r_i \lambda}\text{-mod}$ for every $\lambda \in E$. It is clear from the construction given in [6] that this equivalence is compatible with filtrations. On the other hand, if $\phi : Q \rightarrow Q$ is an automorphism of the underlying graph of Q , it also induces an automorphism $\phi^* : E \rightarrow E$, and it is easy to see that there is a natural isomorphism $\Pi^\lambda \rightarrow \Pi^{\phi^* \lambda}$ of filtered algebras, for any $\lambda \in E$. Thus we have reduced the proof of the proposition to Lemma 17. \square

Lemma 17. *Given $\xi \in \Lambda$, the map $\lambda \mapsto \lambda + \xi$ can be written as a composition of simple reflections and automorphisms of the graph Q .*

Proof. We use some standard facts about root systems and affine Weyl groups that can be found in [2, Chapter VI]. Consider the vector space $V = (\mathbb{Z}^I / \mathbb{Z}\delta) \otimes_{\mathbb{Z}} \mathbb{C}$; it is well known that the image of the set of real roots for Q under the projection map $\mathbb{Z}^I \rightarrow \mathbb{Z}^I / \mathbb{Z}\delta$ is a reduced root system \mathcal{R} in the space V , in the sense of [2, Chapter VI, §1.4]. Now V^* is naturally identified with $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, and E can be viewed as an affine space for the vector space V^* . Let W_{ext} denote the group of automorphisms of E generated by the translations by the elements of Λ and by the Weyl group W_{fin} of the root system \mathcal{R} ; sometimes W_{ext} is called the *extended Weyl group* of the root system \mathcal{R} . It follows from the results of [2, Chapter VI, §§2.1, 2.3], that W_{ext} has the alternate description as the group of automorphisms of E generated by the *affine Weyl group* W_{aff} (which by definition is generated by the simple reflections corresponding to all vertices of Q ; it is called simply the Weyl group of Q in [6]) and the group of automorphisms of the graph Q . This proves the lemma. \square

Remark 18. As we will see below, our proof of part (1) of Theorem 15 relies heavily on the fact that the “shift functors”

$$P_\xi^\lambda \otimes_{\mathcal{O}^\lambda} : \mathcal{O}^\lambda\text{-mod} \longrightarrow \mathcal{O}^{\lambda+\xi}\text{-mod}, \tag{7.2}$$

where the bimodules P_{ξ}^{λ} have been defined by (6.6), are equivalences of categories for “sufficiently large” λ and ξ . Even though the proof of Proposition 16 provides a definition of shift functors, it seems impractical to use this result for quantization of minimal resolutions Kleinian singularities, since it is hard to compute explicitly the associated graded spaces of the bimodules defining the equivalences (7.1), and to control the compositions of these equivalences. It is not known to us if the shift functors of Proposition 16 are isomorphic to the shift functors (7.2).

7.2. Auxiliary general results

In view of Proposition 16, we are reduced to showing that $B^{\lambda}(\chi)$ is a Morita \mathbb{Z} -algebra in the sense of Definition 11 for “sufficiently large” ξ . Most of the work will go into verifying the first condition; the second one will be easily checked at the end of the section. Our argument is based in part on the following characterization of Morita equivalence, which follows immediately from the dual basis lemma.

Proposition 19. *Let A, B be rings, and let P be an (A, B) -bimodule, finitely generated both as a left A -module and as a right B -module, such that the natural ring homomorphisms*

$$A \longrightarrow \text{End}_B(P) \quad \text{and} \quad B \longrightarrow \text{End}_A(P)^{op}$$

are isomorphisms. If P is projective both as a left A -module and as a right B -module, then the functor $P \otimes_B -$ gives an equivalence of categories between $B\text{-mod}$ and $A\text{-mod}$.

In order to apply it we need the following general geometric statement.

Proposition 20. *Let X be a normal affine irreducible algebraic surface over \mathbb{C} , and \mathcal{E} a torsion-free coherent sheaf on X . Then $\text{Ext}^n(\mathcal{E}, \mathcal{O}_X)$ is finite-dimensional for all $n \geq 1$. In addition, if \mathcal{E} has generic rank 1, then*

$$\text{End}_{\mathcal{O}_X}(\mathcal{E}) = \Gamma(X, \mathcal{O}_X). \tag{7.3}$$

Proof. Note that X has at worst finitely many singular points because it is a normal surface. As usual, we write $\mathcal{E}^{\vee} = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ for the dual sheaf. The canonical map $\mathcal{E} \rightarrow \mathcal{E}^{\vee\vee}$ is known to be an isomorphism away from the singular points of X and possibly another finite set of points; moreover, $\mathcal{E}^{\vee\vee}$ is locally free away from the singular points of X . This implies that \mathcal{E} itself is also locally free away from a finite set of points. Thus for $n \geq 1$, the sheaf $\mathcal{E}x^n(\mathcal{E}, \mathcal{O}_X)$ has finite support. On the other hand, since X is affine, the category of coherent \mathcal{O}_X -modules is equivalent to the category of finitely generated $\mathbb{C}[X]$ -modules, and in particular $\text{Ext}^n(\mathcal{E}, \mathcal{O}_X) \cong \Gamma(X, \mathcal{E}x^n(\mathcal{E}, \mathcal{O}_X))$. This implies the first statement.

For the second statement, let $S \subset X$ be the finite set of points where \mathcal{E} is not locally free. Then

$$\Gamma(X, \mathcal{O}_X) \subseteq \text{End}_{\mathcal{O}_X}(\mathcal{E}) \hookrightarrow \text{End}_{\mathcal{O}_{X \setminus S}}(\mathcal{E}|_{X \setminus S}) = \Gamma(X \setminus S, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X),$$

where the second inclusion follows from the assumption that \mathcal{E} is torsion-free, and the last equality follows from the assumption that X is normal. This completes the proof. \square

7.3. Morita equivalence and the bimodules P_{ij}

Let $N \in \mathbb{N}$ be fixed, so that the statements of the last two parts of Theorem 5 hold for all $m, n \geq N$. The key point is that, whereas the bimodules P_{ij} themselves depend on ξ , the associated graded modules do not. In fact, Theorem 8 implies that $\text{gr}^\bullet P_{ij} \cong S_{i-j}$ as R -bimodules, using the notation of Section 3. Now to verify that the natural maps

$$\mathcal{O}^{\lambda+\xi+i\chi} \xrightarrow{\cong} \text{End}_{\text{mod-}\mathcal{O}^{\lambda+\xi+j\chi}}(P_{ij})$$

and

$$\mathcal{O}^{\lambda+\xi+j\chi} \xrightarrow{\cong} \text{End}_{\mathcal{O}^{\lambda+\xi+i\chi}\text{-mod}}(P_{ij})^{op}$$

are isomorphisms, it is enough (as in [8, §3], in the proof of Theorem 1.5(iv)) to check the corresponding statement at the level of associated graded modules. But this follows immediately from part (2) of Theorem 5 and the second statement of Proposition 20. Hence we only need to make sure that P_{ij} is projective as a left and as a right module. To this end we prove

Proposition 21. *Let A be a finitely generated connected filtered algebra such that $\text{gr}^\bullet A$ is commutative and Gorenstein.*

- (1) *If M is a finitely generated left A -module equipped with a good filtration, then $\text{Ext}_A^j(M, A)$ is a filtered right A -module, and $\text{gr}^\bullet \text{Ext}_A^j(M, A)$ is a subquotient of $\text{Ext}_{\text{gr}^\bullet A}^j(\text{gr}^\bullet M, \text{gr}^\bullet A)$ for every $j \geq 0$.*
- (2) *If, in addition, A has finite global dimension and $\text{Ext}_A^j(M, A) = (0)$ for all $j \geq 1$, then M is projective.*

Proof. Part (1) is Proposition 3.1 in [1]. For (2), we use induction on the projective dimension of M , which is finite by assumption. If the projective dimension is 0, we are done. Otherwise there is an exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0 \tag{7.4}$$

with P finitely generated and projective. The projective dimension of N is one less than that of M , and the long exact sequence of Ext's shows that $\text{Ext}_A^j(N, A) = (0)$ for all $j \geq 1$, whence N is projective by induction. In particular, our assumption on M implies that $\text{Ext}_A^1(M, N) = (0)$, so the sequence (7.4) splits, and thus M is projective. \square

We need one more

Proposition 22. *Let $\lambda \in \mathbb{C}^I$ be such that $\lambda \cdot \delta = 1$ and \mathcal{O}^λ has finite global dimension. Given $d \in \mathbb{N}$, the algebra $\mathcal{O}^{\lambda+\xi}$ has no nonzero modules of dimension $\leq d$ for all sufficiently large $\xi \in \Lambda_+$.*

Proof. Obviously, it suffices to prove the statement above with the word “nonzero” replaced by the word “simple.” We now recall [6, Corollary 6.4] that the deformed preprojective algebra Π^λ defined in Section 2 is Morita equivalent to \mathcal{O}^λ , the equivalence being given explicitly by

$$M \mapsto e_0 \Pi^\lambda \otimes_{\Pi^\lambda} M = e_0 M,$$

where $e_0 \in \Pi^\lambda$ denotes the idempotent corresponding to the extending vertex. Note that this functor acts as $\alpha \mapsto \epsilon_0 \cdot \alpha$ on the dimension vectors of the modules.

Now we know from [6, Theorem 7.4] that the dimension vectors of simple finite-dimensional Π^λ -modules are among the positive roots α satisfying $\lambda \cdot \alpha = 0$ (observe that all such roots are necessarily real and non-Dynkin). Also, given $\xi \in \Lambda$, the map

$$\alpha \mapsto \alpha - (\xi \cdot \alpha)\delta$$

establishes a bijection of *finite* sets

$$\{\text{roots } \alpha \text{ such that } \lambda \cdot \alpha = 0\} \xrightarrow{\sim} \{\text{roots } \beta \text{ such that } (\lambda + \xi) \cdot \beta = 0\}.$$

If α is a real root, let us write $\alpha' = \alpha - (\epsilon_0 \cdot \alpha)\delta$, which is a Dynkin root. Observe that if $\xi \in \Lambda$, then we have $\xi \cdot \alpha = \xi \cdot \alpha'$. Thus, if $\beta = \alpha - (\xi \cdot \alpha)\delta$, then

$$\epsilon_0 \cdot \beta = \epsilon_0 \cdot \alpha - \xi \cdot \alpha'. \tag{7.5}$$

Now, using finiteness, choose $N \in \mathbb{N}$ such that $|\epsilon_0 \cdot \alpha| \leq N$ for every root α with $\lambda \cdot \alpha = 0$, and choose $\xi_0 \in \Lambda_+$ such that

$$\xi_0 \cdot \psi > N + d \quad \text{for every positive Dynkin root } \psi.$$

It follows that if $\xi \geq \xi_0$ and α, β are related as above, then (7.5) implies

$$|\epsilon_0 \cdot \beta| \geq |\xi \cdot \alpha'| - |\epsilon_0 \cdot \alpha| > N + d - N = d.$$

Finally, note that $\mathcal{O}^{\lambda+\xi}$ has finite global dimension for sufficiently large ξ , since, according to Remark 2, we only need to choose ξ large enough so that $(\lambda + \xi) \cdot \alpha \neq 0$ for every Dynkin root α . It then follows from the discussion above that the possible dimensions of the simple finite-dimensional $\mathcal{O}^{\lambda+\xi}$ -modules are among the integers $\epsilon_0 \cdot \beta$, where β is a positive root with $(\lambda + \xi) \cdot \beta = 0$, which completes the proof of the proposition. \square

Remark 23. We have stated and proved the proposition above for left \mathcal{O}^λ -modules. However, the same result also holds for right modules. Indeed, if (Q, I) is any quiver, it is easy to see that for any $\lambda \in \mathbb{C}^I$ there is a natural isomorphism between $\Pi^{-\lambda}(Q)$ and the opposite algebra of $\Pi^\lambda(Q)$ that preserves the idempotents corresponding to the vertices of Q . In particular, it follows from [6, Theorem 7.4] that the dimension vectors of simple left $\Pi^\lambda(Q)$ -modules and simple right $\Pi^\lambda(Q)$ -modules are the same.

7.4. Completion of the proof

We keep the same natural number N as in the previous subsection, and we recall that the associated graded modules $\text{gr}^\bullet P_{ij}$ depend only on $i - j$ and not on ξ . For simplicity, let us write $\mathcal{O}_i = \mathcal{O}^{\lambda + \xi + i\chi}$. Now for $n = N, N + 1, \dots, 2N - 1$, the first statement of Propositions 20 and 21(1) (which can be used because $\text{gr}^\bullet \mathcal{O}_i \cong \mathbb{C}[X] \cong \mathbb{C}[x, y]^T$ is commutative and Gorenstein) imply that if $i - j = n$, then the modules

$$\text{Ext}_{\mathcal{O}_i\text{-mod}}^\ell(P_{ij}, \mathcal{O}_i) \quad \text{and} \quad \text{Ext}_{\text{mod-}\mathcal{O}_j}^\ell(P_{ij}, \mathcal{O}_j) \quad (\ell = 1, 2) \tag{7.6}$$

have dimension which is uniformly bounded by an integer which is independent of ξ , and depends only on $i - j$ but not on i or j separately. Furthermore,

$$\text{Ext}_{\mathcal{O}_i\text{-mod}}^\ell(P_{ij}, \mathcal{O}_i) = \text{Ext}_{\text{mod-}\mathcal{O}_j}^\ell(P_{ij}, \mathcal{O}_j) = (0) \quad \text{for } \ell \geq 3,$$

because the global dimensions of \mathcal{O}_i and \mathcal{O}_j are at most 2 (see [6, Theorem 1.6]). In particular, by Proposition 22 and Remark 23, we can choose and fix a large enough ξ for which the modules (7.6) are necessarily zero if $i - j \in \{N, N + 1, \dots, 2N - 1\}$. Thus, by Proposition 21(2), we have now shown that, for the ξ that we have chosen, the bimodules P_{ij} induce Morita equivalences between the algebras \mathcal{O}_j and \mathcal{O}_i whenever $i - j \in \{N, N + 1, \dots, 2N - 1\}$.

We finish the argument as follows. With the notation above, it is obvious that any integer $m \geq 2N - 1$ can be written as a sum $m = m_1 + \dots + m_k$, where each $m_s \in \{N, N + 1, \dots, 2N - 1\}$. Thus, for $i - j = m$, we see from Theorem 5(3) (by passing to the associated graded modules as usual) that the bimodule P_{ij} is a homomorphic image of the tensor product

$$\mathcal{P} := P_{j+m_1, j} \otimes P_{j+m_2, j+m_1} \otimes \dots \otimes P_{i, j+m_{k-1}}$$

over the appropriate algebras \mathcal{O}_{j+m_s} . Since each factor in this tensor product induces a Morita equivalence by the argument above, so does the whole tensor product. Hence, using [13, Lemma 3.5.8] (as in the proof of [13, Proposition 3.3.3(1)]), the surjection $\mathcal{P} \twoheadrightarrow P_{ij}$ must necessarily be an isomorphism; in particular, P_{ij} also induces a Morita equivalence between \mathcal{O}_j and \mathcal{O}_i .

As the last step, note that the same argument shows that the natural map $P_{ij} \otimes_{\mathcal{O}_j} P_{jk} \rightarrow P_{ik}$ is an isomorphism whenever $i - j, j - k \geq 2N - 1$, and the proof is complete.

Remark 24. Note that if \mathcal{O}^λ has no nonzero finite-dimensional modules (by [6, Theorem 0.3], this happens if and only if $\lambda \cdot \alpha \neq 0$ for all non-Dynkin roots α), then the modules (7.6) are automatically zero, and the choice of ξ is unnecessary. We conjecture that, in fact, the modules P_{ij} induce Morita equivalences between the algebras \mathcal{O}_i and \mathcal{O}_j provided λ is dominant in the sense that $\text{Re}(\lambda \cdot \alpha) > 0$ for every positive Dynkin root α .

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