

Generalizations of the classical Weyl and Colin de Verdière's formulas and the orbit method

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The classical Weyl formula expresses the leading term of the asymptotics of the counting function $N(\lambda, H)$ of the spectrum of a self-adjoint operator H in an invariant form: one can “hear” the volume of the subset of the cotangent bundle where the symbol of the operator H is less than λ . In particular, it is applicable to Schrödinger operators with electric potentials growing at infinity. The Weyl formula is formulated in an invariant form; however, it gives $+\infty$ for magnetic Schrödinger operators with magnetic tensors growing at infinity. For these operators, Colin de Verdière’s formula is known, but the form of the latter is not invariant. In this article, we suggest an invariant generalization of both Weyl’s and Colin de Verdière’s formulas for wide classes of Schrödinger operators with polynomial electric and magnetic fields. The construction is based on the orbit method due to Kirillov, and it allows one to hear the geometry of coadjoint orbits.

The aim of this article is to provide a unifying framework for various types of spectral asymptotics for Schrödinger operators.

Introduction

1.1 The Weyl Formula and Its Generalizations. Let

$$H = H(a) + V = - \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} + \sqrt{-1} a_j(x) \right)^2 + V(x) \quad [1.1]$$

be a Schrödinger operator in \mathbb{R}^n with a real continuous semi-bounded electric potential V and magnetic potential $a(x) = (a_1(x), \dots, a_n(x)) \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. The operator H admits a unique realization as a self-adjoint operator in $L_r(\mathbb{R}^n)$, with the domain containing $C_0^\infty(\mathbb{R}^n)$ (see ref. 1 and references therein). If the electric potential V grows regularly at infinity, it is well known that the spectrum of $H(0) + V$ is discrete, and the counting function of the spectrum obeys the classical Weyl formula

$$N(\lambda, H(0) + V) \sim (2\pi)^{-n} \int_{a(x, \xi) < \lambda} dx d\xi. \quad [1.2]$$

Here, $a(x, \xi) = \|\xi\|^2 + V(x)$ is the symbol of $H(0) + V$, and $f(\lambda) \sim g(\lambda)$ means that $f(\lambda)/g(\lambda) \rightarrow 1$ as $\lambda \rightarrow +\infty$. One easily rewrites 1.2 in the following form:

$$N(\lambda, H(0) + V) \sim (2\pi)^{-n} |v_n| \int_{\mathbb{R}^n} (\lambda - V(x))_+^{n/2} dx, \quad [1.3]$$

where $|v_n|$ is the volume of the unit ball of \mathbb{R}^n and $a_+ = \max\{0, a\}$ (e.g., refs. 2 and 3). The classical Weyl formula is applicable to many classes of operators and, in its initial form, was related to the (Dirichlet or Neumann) Laplacian on a bounded domain Ω , with symbol $a(x, \xi)$. For the Laplacian, Eq. 1.3 is valid with $V(x) = 0$ and the integration over \mathbb{R}^n replaced by integration over Ω ; hence, 1.3 allows one “to hear the area of the drum.” If more information about the spectrum is available, then one can “hear” much more about the geometry of a “drum” (see refs. 4 and 5).

Refs. 6–10 show that the spectrum of $H(0) + V$ can be discrete even if V does not grow in some directions, and for wide classes of degenerate potentials, the leading term of the asymptotics of $N(\lambda, H(a) + V)$ is computed. The results of these articles agree with the general “uncertainty principle” stated in ref. 11; it seems that this principle provides upper and lower bounds, but it is difficult to use it to study spectral asymptotics. Note that, in many cases, asymptotic formulas are nonclassical in the sense that they do not agree with the “classical” formula (Eq. 1.2). The following three cases are possible: the classical Weyl formula holds (the so-called “weak degeneration case”); an analog of the classical Weyl formula with the operator-valued symbol parameterized by points of a set with a measure inherited from $T^*\mathbb{R}^n$ is valid (“strong degeneration case”); and the classical Weyl formula fails, but the leading term of the asymptotics is expressed in terms of an auxiliary scalar function and no operator-valued symbol is involved (“intermediate degeneration case”). In simple strong degeneration cases, an operator-valued symbol is parameterized by the cotangent bundle over a manifold of degeneration of V , called M , and the operator-valued analog of 1.2 is of the following form:

$$N(\lambda, H(0) + V) \sim (2\pi)^{-n+r} \int_{T^*M} N(\lambda, a(x', \xi')) dx' d\xi', \quad [1.4]$$

where $r = \text{codim} M$, and for each $(x', \xi') \in T^*M$, $a(x', \xi')$ is an operator in $L_2(\mathbb{R}^r)$. Similar types of asymptotic formulas hold for many other classes of differential operators, pseudodifferential operators, and boundary value problems (see refs. 9 and 12–14 and references therein).

1.2. Colin de Verdière’s Formula. If $V = 0$ and the magnetic tensor $B = [b_{jk}]$, $b_{jk}(x) = \partial_k a_j(x) - \partial_j a_k(x)$, grows regularly at infinity, the leading term of the asymptotics was obtained in ref. 15:

$$N(\lambda, H(a)) \sim \int_{\mathbb{R}^n} v_{B(x)}(\lambda) dx, \quad [1.5]$$

where $v_B(\lambda)$ is defined as follows. Let $\text{rank } B = 2r$, and let $b_1 \geq b_2 \geq \dots \geq b_r > 0$ be the positive eigenvalues of iB . Then

$$v_B(\lambda) = (2\pi)^{-n+r} |v_{n-2r}| b_1 \cdots b_r \cdot \sum_{n_1, \dots, n_r \geq 0} (\lambda - \sum_{j=1}^r (2n_j + 1) \cdot b_j)_+^{n/2-r}.$$

Note that B, r , and the b_j ’s values depend on x . However, in the case of a Schrödinger operator with polynomial potentials, there is a dense open subset of \mathbb{R}^n of full measure on which $B(x)$ has maximal rank, so one can replace the integral in 1.5 by the integral over this subset. Then, r will remain constant throughout the integration.

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1.3. The Case of Degenerate Potentials. In the general case, only upper and lower bounds for $N(\lambda, H(a) + V)$ are known (16). They are given in terms of a function $\Psi^* = \Psi_{a,V}^*$ constructed in ref. 1; for polynomial $V(\geq 0)$ and b_{jk} ,

$$\Psi^*(x) = \sum_{\alpha} |\partial^{\alpha} V(x)|^{1/(|\alpha|+2)} + \sum_{\alpha, j, k} |\partial^{\alpha} b_{jk}(x)|^{1/(|\alpha|+2)}. \quad [1.6]$$

In the case $B \neq 0$ not growing in some directions, the leading term of the asymptotics is unknown apart from a special case of Schrödinger operator (and Dirac operator) in 2D with homogeneous potentials (14).

Note the difference among formulas 1.2, 1.4, and 1.5: the first two are written in an invariant form, whereas the last one is similar to 1.3, which is a realization of the invariant formula 1.2. This observation suggests that there should be an invariant formula of which 1.5 is a realization. Moreover, one should expect that there is a general formula, with 1.2, 1.4, and 1.5 as special cases, and that this formula should work in some cases of degenerate potentials. The following observations indicate the direction where one should look for such a formula.

The Weyl and Colin de Verdière’s Formulas: A Unifying View

2.1. Schrödinger Operators and Unitary Representations. Fix a Schrödinger operator $H = H(a) + V$ with polynomial potentials a, V . We define \mathfrak{g}_H to be the real Lie algebra generated by the polynomial differential operators $L_j = \partial/\partial x_j + \sqrt{-1}a_j(x)$ ($1 \leq j \leq n$) and $L_0 = \sqrt{-1}V(x)$. It is clear that \mathfrak{g}_H is a finite dimensional nilpotent Lie algebra. Moreover, the commutator $[\mathfrak{g}_H, \mathfrak{g}_H]$ consists only of multiplication operators, and thus $[\mathfrak{g}_H, \mathfrak{g}_H]$ is an abelian ideal of \mathfrak{g}_H .

Recall that two Schrödinger operators, $H = H(a) + V$ and $H' = H(a') + V'$, are said to be “gauge equivalent” if $V = V'$ and the corresponding magnetic tensors are the same: $B = B'$. By Poincaré’s lemma, the last condition is equivalent to the existence of a differentiable function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, such that $a'_j(x) = a_j(x) + \partial_j \phi(x)$ for all $1 \leq j \leq n$. If such a ϕ exists, it is easy to check that the unitary operator $\exp(i\phi)$ conjugates H into H' ; in particular, H and H' have the same spectrum. However, if H, H' are Schrödinger operators with polynomial potentials that are gauge equivalent, then the corresponding Lie algebras $\mathfrak{g}_H, \mathfrak{g}_{H'}$ are isomorphic, because the commutation relations in \mathfrak{g}_H depend only on $V(x), b_{jk}(x)$, and their derivatives.

By the “tautological representation” of \mathfrak{g}_H , we mean the representation of \mathfrak{g}_H on $L^2(\mathbb{R}^n)$ by (unbounded) skew-adjoint operators that takes every element of \mathfrak{g}_H to the polynomial differential operator that it represents. Note that, unlike the case of finite dimensional representations, the problem of lifting the tautological representation to a unitary representation of the connected and simply connected nilpotent Lie group $G = \exp \mathfrak{g}_H$ is not trivial. We address this issue in the theorem below.

From a more abstract point of view, let \mathfrak{g} be an arbitrary finite dimensional nilpotent Lie algebra over \mathbb{R} such that $[\mathfrak{g}, \mathfrak{g}]$ is abelian. At “sublaplacian” for \mathfrak{g} is an element $H^{\circ} \in U(\mathfrak{g})_{\mathbb{C}}$, which has the form $H^{\circ} = -(L_1^2 + \dots + L_N^2) - \sqrt{-1}L_0$, where $L_0, L_1, \dots, L_N \in \mathfrak{g}$ are linearly independent elements that generate \mathfrak{g} as a Lie algebra, and L_0 commutes with $[\mathfrak{g}, \mathfrak{g}]$. Note that we have extended the standard definition of a sublaplacian (which does not contain the L_0 term) to include the case of a Schrödinger operator with nonzero electric potential. Then, we have the following result.

Theorem 2.1. (i) Every unitary irreducible representation of $G = \exp \mathfrak{g}$ has a natural realization in a space $L^2(\mathbb{R}^n), n \leq N$, such that each element of \mathfrak{g} maps to a polynomial differential operator of order ≤ 1 ; L_0 and all elements of $[\mathfrak{g}, \mathfrak{g}]$ map to multiplication operators; and H° maps to a Schrödinger operator with polynomial potentials,

which has discrete spectrum if the image of $-\sqrt{-1}L_0$ is a polynomial that is bounded below.

(ii) Conversely, if H is a Schrödinger operator (1.1) with polynomial potentials, there exists a Schrödinger operator H_0 with polynomial potentials, which is gauge equivalent to H , such that if $\mathfrak{g} = \mathfrak{g}_{H_0}$ and $H^{\circ} \in U(\mathfrak{g})_{\mathbb{C}}$ is the element corresponding to H_0 , then the tautological representation of \mathfrak{g} on $L^2(\mathbb{R}^n)$ can be lifted to a unitary irreducible representation of $G = \exp \mathfrak{g}$, which is irreducible if and only if H has discrete spectrum.

It is important to have a concrete realization of each of the representations of \mathfrak{g} that arises from a unitary irreducible representation of G . These realizations will be discussed in detail in section 3.4.

2.2 Interpretation of Weyl and Colin de Verdière’s Formulas. Assume that $\sigma(H) = \sigma_d(H)$, so ρ is irreducible. The orbit method, due to Kirillov (17), provides a natural one-to-one correspondence between (unitary equivalence classes of) unitary irreducible representations of G and orbits of the coadjoint action of G on \mathfrak{g}^* . In particular, we let $\Omega_{\rho} \subset \mathfrak{g}^*$ denote the coadjoint orbit corresponding to ρ . Suppose that the magnetic potential $a = 0$, and that $V(x)$ grows regularly at infinity. The values of the symbol $a(x, \xi)$ appearing in the classical Weyl formula (1.2) can be interpreted as the images of H° in a family of representations of G on the 1D space $L^2(\mathbb{R}^0)$. The family is parameterized by points of the orbit Ω_{ρ} , and the measure $(2\pi)^{-n} dx d\xi$ coincides with the canonical (Kostant) measure on Ω_{ρ} .

However, assume that $V = 0$ and the magnetic tensor $B(x)$ grows regularly at infinity. It is shown in ref. 18 that the formula of Colin de Verdière 1.5 can be written in the following form:

$$N(\lambda, H) \sim \int_Q N(\lambda, H_{\Theta}) d\nu(\Theta) \text{ as } \lambda \rightarrow +\infty, \quad [2.1]$$

where H_{Θ} is the image of H° in a certain unitary irreducible representation of G on $L^2(\mathbb{R}^r)$, Q is a manifold parameterizing a family of such representations, and the measure $d\nu(\Theta)$ can be obtained in the following way. Let $\tilde{Q} \subset \mathfrak{g}^*$ be the union of the orbits corresponding to the representations parameterized by the points of Q . There is a natural “projection map” $p : \Omega_{\rho} \rightarrow \tilde{Q}$, such that the pushforward $\tilde{\nu}$ of the canonical measure on Ω_{ρ} is a G -invariant measure on \tilde{Q} . One can decompose $\tilde{\nu}$ as an integral of the canonical measures on the orbits contained in \tilde{Q} , with respect to a certain “quotient” measure on $Q = \tilde{Q}/G$. Then, we take ν to be this quotient measure.

Let us explain the case $n = 2$ in detail. The magnetic tensor must be of the following form:

$$B(x) = \begin{pmatrix} 0 & b(x) \\ -b(x) & 0 \end{pmatrix},$$

where $b(x)$ is a polynomial, and because $B(x)$ grows regularly at infinity, we may assume without loss of generality that $b(x) > 0$ for $\|x\| \gg 0$. Note that the eigenvalues of $\sqrt{-1}B(x)$ are $\pm b(x)$.

The Lie algebra \mathfrak{g} is generated by the operators $L_1 = \partial/\partial x_1 + \sqrt{-1}a_1(x)$ and $L_2 = \partial/\partial x_2 + \sqrt{-1}a_2(x)$, which satisfy $[L_1, L_2] = \sqrt{-1}b(x)$. Let us write $P_0 = \sqrt{-1}b(x)$, and let P_1, \dots, P_N be an arbitrary basis of the vector space spanned by all mixed partial derivatives of P_0 of all positive orders (i.e., not including P_0). Thus, $\{L_1, L_2, P_0, P_1, \dots, P_N\}$ is a basis of \mathfrak{g} . We can now define a projection map $p : \Omega_{\rho} \rightarrow \mathfrak{g}^*$ by $p(f)(L_1) = f(L_1), p(f)(L_2) = f(L_2), p(f)(P_0) = f(P_0), p(f)(P_j) = 0$ for $1 \leq j \leq N$. We will now show that if \tilde{Q} is taken to be the image of this map, then Q is G -stable, and the pushforward measure $\tilde{\nu} = p_*(\mu_{\rho})$ is G -invariant (where μ_{ρ} is the Kostant measure on the orbit Ω_{ρ}). Moreover, if $Q = \tilde{Q}/G$ and ν is the measure on Q induced by

$\tilde{\nu}$, then the right side of 2.1 coincides with the right side of Colin de Verdière's formula.

It follows from Proposition 3.6 and Proposition 3.7 that the orbit Ω_ρ admits a parameterization $\phi : \mathbb{R}^4 \rightarrow \mathfrak{g}^*$ given by

$$\phi(\xi_1, \xi_2, y_1, y_2)(L_i) = \xi_i \quad \text{for } i = 1, 2,$$

and

$$\phi(\xi_1, \xi_2, y_1, y_2)(P_j) = -\sqrt{-1} \cdot P_j(y_1, y_2) \quad \text{for } 0 \leq j \leq N.$$

Moreover, we have $\mu_\rho = (2\pi)^{-2} \cdot \phi_* (d\xi dy)$, where $d\xi dy$ denotes the Lebesgue measure on \mathbb{R}^4 . Let \mathfrak{a} denote the subspace of \mathfrak{g} spanned by P_1, \dots, P_N ; it is clearly an ideal of \mathfrak{g} . By definition, the image of the map p is contained in the annihilator of this ideal in \mathfrak{g}^* , which we can identify with $(\mathfrak{g}/\mathfrak{a})^*$. Now, $\mathfrak{g}/\mathfrak{a}$ has basis $\{X, Y, Z\}$, where X, Y , and Z are the images of L_1, L_2 , and P_0 under the quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$. They satisfy the relations $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$, so we see that $\mathfrak{g}/\mathfrak{a}$ is the 3D Heisenberg algebra. Let us use the basis $\{X, Y, c\}$ to identify $(\mathfrak{g}/\mathfrak{a})^*$ with \mathbb{R}^3 in the obvious way. Then, the composition $\psi = p \circ \phi : \mathbb{R}^4 \rightarrow (\mathfrak{g}/\mathfrak{a})^*$ is given by $\psi(\xi, y) = (\xi_1, \xi_2, b(y))$, and we are interested in the measure $\tilde{\nu} = (2\pi)^{-2} \psi_* (d\xi dy)$. It is well known that there are two types of coadjoint orbits in $(\mathfrak{g}/\mathfrak{a})^*$: the 2D orbits given by $f(Z) = c$, where c is a nonzero constant, and the 0D orbits [namely, points of the plane defined by $f(Z) = 0$]. In particular, we see that $\psi(\mathbb{R}^4)$ is a union of coadjoint orbits, so the set $p(\Omega_\rho)$ is G -stable. Moreover, if $c \neq 0$ is fixed, then the functions $u : f \mapsto f(X)$ and $v : f \mapsto f(Y)$ are coordinates on the coadjoint orbit defined by $f(Z) = c$, and the Kostant measure on this orbit is given by $\mu_c = (2\pi)^{-1} c^{-1} dudv$. Consequently, the pushforward measure $\tilde{\nu}$ can be decomposed as an integral of the Kostant measures μ_c in the following way:

$$\tilde{\nu} = \int_{\mathbb{R}} \mu_c d\nu(c),$$

where ν is the measure on \mathbb{R} obtained as the pushforward of the measure $(2\pi)^{-1} b(y) dy$ by the map $b : \mathbb{R}^2 \rightarrow \mathbb{R}$. [A fortiori, this formula implies that $\tilde{\nu}$ is G -invariant. Note also that we have ignored the plane $f(Z) = 0$ in the computation above, which can be done because it has measure zero with respect to $\tilde{\nu}$.] Last, the representation of $\mathfrak{g}/\mathfrak{a}$ corresponding to the orbit $f(Z) = c$ can be realized in the space $L^2(\mathbb{R})$ such that $X \mapsto \partial/\partial x$ and $Y \mapsto \sqrt{-1}cx$. Under this representation, the sublaplacian $-(X^2 + Y^2)$ maps to the operator $-\Delta_x + c^2x^2$, whose spectrum can be computed explicitly; it consists of eigenvalues of the form $(2m + 1)c$, each having multiplicity 1, where m runs over all nonnegative integers. We now have all the ingredients that are needed to make sense of the right side of 2.1, and we see that it becomes

$$(2\pi)^{-1} \cdot \int_{\mathbb{R}^2} N(\lambda, -\Delta_x + y^2x^2) \cdot b(y) dy,$$

which coincides with the right side of Colin de Verdière's formula.

The classical Weyl formula also can be written in the form 2.1, with Q parameterizing a family of 1D representations (in this case, $Q = \tilde{Q}$, so one does not need to decompose the pushforward measure).

3. Main Results and Conjectures

3.1. Generalizations: The Main Idea. It is tempting to conjecture that for any magnetic Schrödinger operator with discrete spectrum one can find a family of irreducible representations of G and the pushforward measure $d\nu(\Theta)$ on Q such that 2.1 holds. As it turns out, this construction can be realized in many, albeit not all, cases, and our first goal is to suggest a general way of construction of the

family Q and the pushforward measure $d\nu(\Theta)$. Naturally (cf. refs. 9, 12, and 13 for generalizations of the classical Weyl formula), we have two similar (but a bit different) algorithms: one for the strong degeneration case and one for the weak and intermediate degeneration case. In the intermediate degeneration case, one has to introduce additional logarithmic factors into 2.1. To verify our conjecture for several classes of magnetic Schrödinger operators, we use a modification of the variational technique from refs. 9 and 12–14.

Let us keep the same notation as described above and write μ_{Ω_ρ} for the canonical (Kostant) measure on the orbit Ω_ρ . In trying to turn the vague ideas above into a precise formula that applies to wide classes of the Schrödinger operators, one meets two considerable difficulties. The first difficulty is the fact that there seems to be no natural general way of defining a projection map $p : \Omega_\rho \rightarrow \tilde{Q} \subset \mathfrak{g}^*$, such that the pushforward $p_*(\mu_{\Omega_\rho})$ will always be a G -invariant measure. The second difficulty, which is more serious, is that in the intermediate degeneration cases, there exists an asymptotic formula of the form 2.1 (with additional logarithmic factors), but the measure ν cannot be obtained from a pushforward measure arising from a process described above.

Thus, one has to look for a different construction of the subset $\tilde{Q} \subset \mathfrak{g}^*$ and the G -invariant measure $\tilde{\nu}$ on \tilde{Q} . We suggest a construction which has the advantage of being canonical (i.e., independent of any choices). Moreover, the measure $\tilde{\nu}$ that it provides is automatically G -invariant. Thus, both problems mentioned above are solved at once. To our knowledge, no similar construction has been used previously in this or any related context.

Let us give a brief description of our idea. For each $\lambda > 0$, we let $\mu_\lambda = \mu_{\lambda, \Omega_\rho}$ denote the positive Borel measure on \mathfrak{g}^* defined by $\mu_\lambda(A) = \mu_{\Omega_\rho}(\Omega_\rho \cap \lambda A)$ for every Borel subset $A \subset \mathfrak{g}^*$. Note that μ_λ is supported on $\lambda^{-1} \cdot \Omega_\rho$, which is another coadjoint orbit in \mathfrak{g}^* . Now, Ω_ρ is closed in \mathfrak{g}^* , and there is a coordinate system on Ω_ρ , which identifies Ω_ρ with \mathbb{R}^{2n} , such that μ_{Ω_ρ} corresponds to the usual Lebesgue measure under this identification (both of these statements hold for arbitrary nilpotent Lie algebras). In particular, we see that each μ_λ can be identified with a positive linear functional on the space $C_c(\mathfrak{g}^*)$ of compactly supported continuous functions on \mathfrak{g}^* . Note also that, if A is a neighborhood of 0 in \mathfrak{g}^* , then, as $\lambda \rightarrow +\infty$, the sets $\Omega_\rho \cap \lambda A$ exhaust all of Ω_ρ ; thus, $\mu_\lambda(A) \rightarrow +\infty$. Let us now suppose that there exists a function $f(\lambda)$ such that the functionals $f(\lambda) \mu_\lambda \in C_c(\mathfrak{g}^*)^*$ have a nonzero weak-* limit $f_0 \in C_c(\mathfrak{g}^*)^*$. By the Riesz representation theorem, f_0 corresponds to a positive Borel measure μ_0 on \mathfrak{g}^* . We define $\tilde{Q} = \text{supp}(\mu_0)$, and $\tilde{\nu} = \mu_0|_{\tilde{Q}}$. Then, \tilde{Q} is a conical G -invariant subset of \mathfrak{g}^* , and the G -invariance of $\tilde{\nu}$ is automatic because each of the measures μ_λ is G -invariant.

For simplicity, we refer to the construction described above as the “scaling construction.” Because of its “homogeneous” nature, it is not surprising that in applying the construction to the computation of spectral asymptotics of Schrödinger operators, one has to require a certain homogeneity condition on the potentials. We say, somewhat imprecisely, that 1.1 is a Schrödinger operator with quasihomogeneous potentials if $V(x)$ and $B(x)$ are quasihomogeneous polynomials of the same weight; i.e., if there exists an n -tuple of positive rational numbers $\gamma = (\gamma_1, \dots, \gamma_n)$ such that for all $t \in \mathbb{R}$, $t > 0$, and all $x \in \mathbb{R}^n$, we have

$$V(t^{\gamma_1}x_1, \dots, t^{\gamma_n}x_n) = t \cdot V(x) \quad \text{and} \quad B(t^{\gamma_1}x_1, \dots, t^{\gamma_n}x_n) = t \cdot B(x).$$

We prove that in the quasihomogeneous situation in which the classical formulas of Weyl and Colin de Verdière are applicable, our construction gives the same result as the pushforward construction described above. However, in the intermediate degeneration examples that we have studied, it also produces the “correct” measure space (\tilde{Q}, ν) , even though the pushforward construction no longer applies.

We remark that our scaling construction makes sense for any nilpotent Lie algebra. Indeed, let \mathfrak{g} be a finite dimensional nilpotent Lie algebra over \mathbb{R} and $\Omega \subset \mathfrak{g}^*$ a coadjoint orbit. It is known (e.g., see chapter I of ref. 19) that Ω is a closed (in fact, Zariski closed) submanifold of \mathfrak{g}^* . Moreover, it follows from the explicit parameterization obtained in ref. 20 that there exists a polynomial map $\varphi: \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$, which is a diffeomorphism onto Ω , and such that under this diffeomorphism μ_Ω corresponds to the standard Lebesgue measure on \mathbb{R}^{2n} .

As before, for every $\lambda > 0$, we define a positive Borel measure μ_λ on \mathfrak{g}^* as follows:

$$\mu_\lambda(A) = \mu_\Omega(\Omega \cap \lambda \cdot A) = \text{meas}(\varphi^{-1}(\lambda \cdot A)),$$

where meas is the Lebesgue measure. Because φ is proper, we see that $C_c(\mathfrak{g}^*) \subset L^1(d\mu_\lambda)$ for each $\lambda > 0$. In particular, we can again identify μ_λ with a positive linear functional on $C_c(\mathfrak{g}^*)$, and the rest of our construction goes through without any changes. It is apparent from the computations of explicit examples that the scaling construction is closely related to the geometry of the embedding $\Omega \rightarrow \mathfrak{g}^*$.

The idea of applying representation-theoretic methods to the study of partial differential operators is not new (e.g., see ref. 21 and references therein). Several authors have studied extensions of the known results about Schrödinger operators to the differential operators arising from unitary representations of general nilpotent Lie groups. In ref. 22, upper and lower bounds for $N(\lambda, H)$ were obtained, where H is the image under an irreducible representation of the “sublaplacian” on a stratified nilpotent Lie algebra. Manchon (23) has generalized the approximate spectral projection method of Tulovskii and Shubin (24) to prove a Weyl-type asymptotic formula for elliptic operators associated to representations of arbitrary nilpotent Lie groups. In refs. 25 and 26, this result was generalized to arbitrary Lie groups (more precisely, to the representations corresponding to closed tempered coadjoint orbits for which Kirillov’s character formula is valid). However, note that refs. 23, 25, and 26 use the initial form of the approximate spectral projection method, which requires the high regularity of the symbol. In particular, if a degeneration of any kind is present, this form of the approximate spectral projection method does not work at all. For a general version of the approximate spectral projection method and applications to various classes of degenerate and hypoelliptic operators, see refs. 9, 12, and 13.

Most of the works relating differential operators to representation theory of nilpotent Lie groups deal only with stratified Lie algebras (21,22); i.e., Lie algebras \mathfrak{g} admitting a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_s$ as a direct sum of vector subspaces, such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$ ($\mathfrak{g}_j = (0)$ for $j > s$), and \mathfrak{g} is generated by \mathfrak{g}_1 as a Lie algebra. However, there are situations in which the Lie algebra arising from a Schrödinger operator with polynomial potentials admits no natural grading. The theory that we develop in section 3 makes no use of a grading on \mathfrak{g} .

In ref. 18, we use an example of the Schrödinger operator in 2D with zero electric potential and magnetic tensor $b(x) = x_1^2 - x_2$ (this is an example of strong degeneration, and there is no natural grading) to illustrate in detail the use of our conjectural formula. We also study the weak degeneration case for operators without either magnetic or electrical potential and deduce from our conjecture the classical Weyl formula and Colin de Verdière’s formula, respectively. In particular, we prove that, in the case of a quasihomogeneous electric potential, the classical Weyl formula holds if and only if the integral in this formula converges, and our general conjectural formula also gives the classical Weyl formula if and only if this condition is satisfied. Last, we consider the Schrödinger operator in 2D with magnetic tensor $b(x) = x_1^k x_2^l$ and zero electric potential. In the case $k \neq l$, we have the strong degeneration, and in the case $k = l$, we have the intermediate degeneration. In all cases, we derive the leading term of the asymptotics from our

conjectural formula, and we prove them by using the variational method in the form (9, 12–14).

The next subsections contain formulations of our main conjectures and statements of several representation-theoretic results that are necessary for the applications of our conjectures and also interesting in their own right. More details and complete proofs are given in ref. 18.

3.2 Preliminary Version of the Conjecture. Let us now formulate a preliminary version of our conjecture. Let H be a Schrödinger operator (1.1) with discrete spectrum and quasihomogeneous polynomial potentials, and let $\mathfrak{g} = \mathfrak{g}_H$ be the associated Lie algebra. Because we are interested in $\sigma(H)$, we may assume, by Theorem 2.1, that the tautological representation of \mathfrak{g} lifts to a unitary representation of G on $L^2(\mathbb{R}^n)$; moreover, this representation is then irreducible, from which corresponds to a coadjoint orbit $\Omega \subset \mathfrak{g}^*$. Let μ_Ω be the Kostant measure on Ω ; for the precise normalization, see Definition 3.4. Then, we have the “dilates” μ_λ of the measure μ_Ω , as defined in section 2: $\mu_\lambda(A) = \mu_\Omega(\Omega \cap \lambda \cdot A)$, for every Borel subset $A \subseteq \mathfrak{g}^*$. Furthermore, H naturally defines an element $H^\circ \in U(\mathfrak{g})_{\mathbb{C}}$, and the definition of \mathfrak{g} implies that H° is a sublaplacian for \mathfrak{g} . For any coadjoint orbit $\Theta \subset \mathfrak{g}^*$, we denote by H_Θ the image of H° in the unitary irreducible representation of G that corresponds to Θ via Kirillov’s theory. By Theorem 2.1, each H_Θ can be naturally realized as a Schrödinger operator with polynomial potentials.

Conjecture 1. *There exist a positive real number α and a nonnegative integer β such that the weak limit $\mu_0 = \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} (\log \lambda)^{-\beta} \mu_\lambda$ exists and is nonzero. Then μ_0 is automatically G -invariant; let $Q = (\text{supp } \mu_0)/G$, and let $\rho: \text{supp } \mu_0 \rightarrow Q$ be the natural projection. Let ν be the measure on Q such that for every nonnegative Borel-measurable function F on \mathfrak{g}^* , we have*

$$\int_{\mathfrak{g}^*} F d\mu_0 = \int_Q d\nu(q) \cdot \int_{\rho^{-1}(q)} F(x) d\mu_q(x),$$

where $d\mu_q$ denotes the Kostant measure corresponding to the orbit $\rho^{-1}(q)$ (the existence of ν is proved in ref. 18, proposition 2.12). Then there exists a constant $\kappa \geq 1$ such that

$$N(\lambda, H) \sim \kappa (\log \lambda)^\beta \cdot \int_Q N(\lambda, H_\Theta) d\nu(\Theta) \quad \text{as } \lambda \rightarrow +\infty. \tag{3.1}$$

Some motivation for the form of this conjecture, and especially for the appearance of the logarithmic factors in both 3.1 and the definition of μ_0 , is provided by a result of Nilsson, which we now recall. It is a special case of theorem 1 in ref. 27; the latter is, in turn, based on the results of ref. 28.

Theorem 3.1 (27, 28). *Let $P(x)$ be a real polynomial on \mathbb{R}^n such that $P(x) \rightarrow +\infty$ as $\|x\| \rightarrow \infty$, and set*

$$G(\lambda) = \text{meas}\{x \in \mathbb{R}^n | P(x) \leq \lambda\}.$$

Then, there exist positive reals c, C, α and a nonnegative integer β such that

$$C^{-1} \cdot \lambda^\alpha (\log \lambda)^\beta \leq G(\lambda) \leq C \cdot \lambda^\alpha (\log \lambda)^\beta \quad \text{for all } \lambda > c.$$

The precise relationship of this theorem to our results is explained in detail in ref. 18. Here, we remark that the explicit formulas for the measure μ and its dilates μ_λ obtained in section 3.5 imply that the growth of the measures μ_λ as $\lambda \rightarrow \infty$ is closely related to the growth of the function $G(\lambda)$ in Theorem 3.1 for a suitably defined polynomial $P(x)$.

result. Let \mathfrak{g} be a Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ an ideal. Write \mathfrak{a}^\perp for the annihilator of \mathfrak{a} in \mathfrak{g}^* . The quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ induces an isomorphism of vector spaces $(\mathfrak{g}/\mathfrak{a})^* \cong \mathfrak{a}^\perp \rightarrow \mathfrak{g}^*$. Let G be a connected Lie group with Lie algebra \mathfrak{g} , and let $A \subset G$ be the closed connected normal subgroup corresponding to \mathfrak{a} . The adjoint action of G on \mathfrak{g} leaves \mathfrak{a} stable, from which G also acts on $\mathfrak{g}/\mathfrak{a}$ and on $(\mathfrak{g}/\mathfrak{a})^*$. Then, we have the following:

Proposition 3.5. (i) *The isomorphism $(\mathfrak{g}/\mathfrak{a})^* \rightarrow \mathfrak{a}^\perp$ above is G -equivariant, and the action of G on $(\mathfrak{g}/\mathfrak{a})^*$ factors through the quotient group G/A ; thus, the G -orbits in $(\mathfrak{g}/\mathfrak{a})^*$ are the same as the coadjoint orbits of G/A in $(\mathfrak{g}/\mathfrak{a})^*$.*

(ii) *If $\Omega \subset \mathfrak{g}^*$ is any coadjoint orbit, then either $\Omega \cap \mathfrak{a}^\perp = \emptyset$, or $\Omega \subset \mathfrak{a}^\perp$. In the latter case, Ω is the image of a coadjoint orbit in $(\mathfrak{g}/\mathfrak{a})^*$. Moreover, the canonical symplectic form and the Kostant measure on Ω are the same whether we regard Ω as a coadjoint orbit for G or as a coadjoint orbit for G/A .*

(iii) *If G is simply connected and nilpotent, then the bijection between the coadjoint orbits in \mathfrak{g}^* that meet \mathfrak{a}^\perp and the coadjoint orbits in $(\mathfrak{g}/\mathfrak{a})^*$, defined above, corresponds, by Kirillov's theory, to the natural bijection between the unitary irreducible representations of G that are trivial on A , and all unitary irreducible representations of G/A .*

We return to the situation considered in Section 3.4. Thus, G is a connected and simply connected nilpotent Lie group with Lie algebra \mathfrak{g} such that $[\mathfrak{g}, \mathfrak{g}]$ is abelian. Fix a point $f_0 \in \mathfrak{g}^*$. We want to parameterize the G -orbit $G \cdot f_0 \subset \mathfrak{g}^*$. As before, we assume we are given a sublaplacian $H^\circ = -(L_1^2 + \dots + L_N^2) - \sqrt{-1} \cdot L_0$ for \mathfrak{g} , and we let \mathfrak{h} be a real polarization of \mathfrak{g} at f_0 provided by Lemma 3.2: $\mathfrak{g}(f_0) + \mathbb{R}L_0 + [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. Furthermore, we suppose that for some $1 \leq n \leq N$, L_1, \dots, L_n is a complementary basis for \mathfrak{h} in \mathfrak{g} .

From now on, we also assume that \mathfrak{h} is an abelian ideal of \mathfrak{g} . To justify this assumption, we note that because \mathfrak{h} is an ideal of \mathfrak{g} , so is $\mathfrak{a} := [\mathfrak{h}, \mathfrak{h}]$; however, by the definition of a polarization, f_0 annihilates \mathfrak{a} . Thus, f_0 induces a linear functional f_0 on $\mathfrak{g}/\mathfrak{a}$. By Proposition 3.5, the canonical inclusion $(\mathfrak{g}/\mathfrak{a})^* \rightarrow \mathfrak{g}^*$ gives an isomorphism of the coadjoint orbit of f_0 in $(\mathfrak{g}/\mathfrak{a})^*$ onto the coadjoint orbit of f_0 in \mathfrak{g}^* ; moreover, this isomorphism preserves the canonical symplectic form and the Kostant measure. Last, note that because $\mathfrak{a} \subset \mathfrak{g}(f)$ by Lemma 3.2, it is clear that $\mathfrak{h}/\mathfrak{a}$ is a maximal isotropic subspace of $\mathfrak{g}/\mathfrak{a}$ with respect to the form B_{f_0} . Thus, from the point of view of either the coadjoint orbit of f_0 , or of the corresponding unitary irreducible representation, nothing is lost by passing from \mathfrak{g} to $\mathfrak{g}/\mathfrak{a}$.

Proposition 3.6. *With the notation above, assume that \mathfrak{h} is abelian. The map $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathfrak{g}^*$ defined by $\langle \varphi(\xi, x), L_j \rangle = \xi_j$ for $1 \leq j \leq n$,*

$$\langle \varphi(\xi, x), Y \rangle = \sum_{\alpha_1, \dots, \alpha_n \geq 0} \frac{1}{\alpha_1! \cdots \alpha_n!} f_0((\text{ad } L_1)^{\alpha_1} \cdots (\text{ad } L_n)^{\alpha_n}(Y)) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ for all } Y \in \mathfrak{h},$$

is a diffeomorphism of \mathbb{R}^{2n} onto the coadjoint orbit of f_0 in \mathfrak{g}^* .

By a slight abuse of notation, we identify Ω with \mathbb{R}^{2n} by using the diffeomorphism φ , and in particular, we view (ξ, x) as coordinates on the orbit Ω . Let us define polynomials $b_{jk}(x)$ by

$$b_{jk}(x) = \langle \varphi(0, x), [L_j, L_k] \rangle;$$

note that if \mathfrak{g} arises from a Schrödinger operator with polynomial potentials, and if f_0 restricts to the linear functional $\sqrt{-1} \cdot P(x) \mapsto P(0)$ on the subspace of \mathfrak{g} consisting of multiplication operators, then the $b_{jk}(x)$ are precisely the components of the magnetic tensor of the operator. The next proposition gives an explicit formula for the Kostant measure.

Proposition 3.7. *The canonical symplectic form and the Kostant measure on the orbit Ω are given by*

$$\omega_\Omega = \sum_{j=1}^n d\xi_j \wedge dx_j + \sum_{1 \leq j < k \leq n} b_{kj}(x) dx_j \wedge dx_k \quad [3.5]$$

and

$$\mu_\Omega = (2\pi)^{-n} \cdot d\xi_1 \cdots d\xi_n dx_1 \cdots dx_n.$$

In other words, if we identify μ_Ω with its extension by zero to \mathfrak{g}^* , then we can write

$$\mu_\Omega = (2\pi)^{-n} \cdot \varphi_*(d\xi_1 \cdots d\xi_n dx_1 \cdots dx_n), \quad [3.6]$$

where φ_* denotes the pushforward by the map $\varphi : \mathbb{R}^{2n} \rightarrow \mathfrak{g}^*$.

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