

# Distribution of the First Particle in Discrete Orthogonal Polynomial Ensembles

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**Abstract:** We show that the distribution function of the first particle in a discrete orthogonal polynomial ensemble can be obtained through a certain recurrence procedure, if the (difference or  $q$ -) log-derivative of the weight function is rational. In a number of classical special cases the recurrence procedure is equivalent to the difference and  $q$ -Painlevé equations of [10, 17].

Our approach is based on the formalism of discrete integrable operators and discrete Riemann–Hilbert problems developed in [3, 4].

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## 1. Introduction

*1.1.* The basic problem considered in this paper is the following. Let  $\mathfrak{X}$  be a locally finite subset of  $\mathbb{R}$  and  $w : \mathfrak{X} \rightarrow \mathbb{R}_{>0}$  be a positive-valued function on  $\mathfrak{X}$  with finite moments:

$$\sum_{x \in \mathfrak{X}} |x|^n w(x) < \infty, \quad n = 0, 1, \dots$$

Fix a positive integer  $k$  (the number of particles) and consider the probability measure on all  $k$ -point subsets of  $\mathfrak{X}$  given by

$$\text{Prob}\{x_1, \dots, x_k\} = \text{const} \cdot \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \prod_{i=1}^k w(x_i). \tag{1.1}$$

We are interested in the distribution of  $\max\{x_1, \dots, x_k\}$  with respect to this measure.

The problem is motivated by random matrix theory on one side, and by combinatorial and representation theoretic models on the other one.

In random matrix theory, probability measures of the form

$$\text{const} \cdot \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \prod_{i=1}^k w(x_i) dx_i$$

on  $k$ -point subsets of  $\mathbb{R}$ , with  $w(x)$  being a smooth function on a subinterval of  $\mathbb{R}$ , play a prominent role. Most computations for such models are conveniently done by means of the orthogonal polynomials associated with the weight function  $w(x)$ . On this ground, these measures are often called *orthogonal polynomial ensembles*. See [14, 15] and references therein for a further discussion.

The problem of describing the distribution of the  $\max\{x_i\}$  in the continuous setting for the classical weights has been solved in the following sense: the distribution function was explicitly written in terms of a specific solution of one of the six (2<sup>nd</sup> order nonlinear ordinary differential) Painlevé equations. It was done in [20] for the Hermite weight  $w(x) = \exp(-x^2)$ ,  $x \in \mathbb{R}$ , and for the Laguerre weight  $x^a \exp(-x)$ ,  $x > 0$ ; in [20, 9] for the Jacobi weight  $(1 - x)^a(1 + x)^b$ ,  $x \in (-1, 1)$ ; and in [22, 5] for the quasi-Jacobi weight  $(1 - ix)^s(1 + ix)^{\bar{s}}$ ,  $x \in \mathbb{R}$ .<sup>1</sup> Thus, it is natural to ask what would be an analog of these results when we take  $w$  to be a classical *discrete* weight function.

On the other hand, in recent years the random variable  $\max\{x_i\}$  with  $x_i$ 's distributed according to (1.1) with certain specific weights, came up as the main quantity of interest in a number of problems originating in combinatorics, first-passage percolation, representation theory, and growth processes, see e.g. [7, 11, 12, 2] and references therein.

*1.2.* In order to state our first result we need to introduce more notation. Let us denote the points of  $\mathfrak{X}$  by  $\pi_s$ ,  $s = 0, 1, \dots, N$ , with  $\pi_0 < \pi_1 < \dots$  or  $\pi_0 > \pi_1 > \dots$ . Here  $N = |\mathfrak{X}| - 1$  may be finite or infinite. We use the following two basic assumptions:

- There exists an affine transformation  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\sigma \pi_{s+1} = \pi_s$  for all  $s$ ,  $0 \leq s < N$ .
- There exist polynomials  $P(x)$  and  $Q(x)$  such that

$$\frac{w(\pi_{s-1})}{w(\pi_s)} = \frac{P(\pi_s)}{Q(\pi_s)}, \quad 1 \leq s \leq N,$$

and  $P(\pi_0) = 0$ .

The orthogonality data for a number (but not all) hypergeometric polynomials of the Askey scheme satisfy these assumptions, see Sect. 7 below for details.<sup>2</sup>

<sup>1</sup> Continuous problems of this type have been extensively studied. We refer to the introduction of [5] for a brief review and references.

<sup>2</sup> For some classical families of polynomials both assumptions are satisfied but the orthogonality set  $\mathfrak{X}$  is not locally finite. We extend our results to those cases, see Sect. 4 below.

We prove that under the two conditions above, there exists a certain recurrence procedure which computes the *gap probability*

$$D_s = \text{Prob} \{x_i \notin \{\pi_s, \pi_{s+1}, \dots\} \text{ for all } i\}$$

$$= \begin{cases} \text{Prob}\{\max\{x_i\} < \pi_s\}, & \text{if } \pi_0 < \pi_1 < \dots, \\ \text{Prob}\{\min\{x_i\} > \pi_s\}, & \text{if } \pi_0 > \pi_1 > \dots, \end{cases}$$

with  $x_i$ 's distributed according to (1.1). In fact, the recurrence procedure produces a sequence  $(A_s, M_s(\zeta))$ , where  $A_s$  is a nilpotent 2 by 2 matrix and  $M_s(\zeta)$  is a matrix polynomial

$$M_s(\zeta) = M_s^{(l)}\zeta^l + \dots + M_s^{(0)}, \quad M_s^{(i)} \in \text{Mat}(2, \mathbb{C}),$$

of degree  $l = \max\{\text{deg } P, \text{deg } Q\}$ . The elementary step of the recurrence is provided by the equality

$$\left(I + \frac{A_s}{\sigma\zeta - \pi_s}\right) M_s(\zeta) = M_{s+1}(\zeta) \left(I + \frac{A_{s+1}}{\zeta - \pi_{s+1}}\right). \tag{1.2}$$

It is not hard to see that if  $\det M_s(\pi_{s+1}) \neq 0$  (which is always the case in our setting) then (1.2) defines  $(A_{s+1}, M_{s+1})$  uniquely provided that we know  $(A_s, M_s)$ . However, the existence of  $(A_{s+1}, M_{s+1})$  is not obvious and needs to be proved.<sup>3</sup> Again, in our setting it always holds.

We then show that the ratio

$$\left(\frac{D_{s+3}}{D_{s+2}} - \frac{D_{s+2}}{D_{s+1}}\right) \left(\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s}\right)^{-1}$$

is an explicit rational function of  $(A_s, M_s^{(l)}, \dots, M_s^{(0)})$  and  $(A_{s+1}, M_{s+1}^{(l)}, \dots, M_{s+1}^{(0)})$ .

Since  $D_s = \text{Prob}\{\max\{x_i\} < \pi_s\}$  is nonzero only if  $s \geq k$  (recall that  $k$  is the number of  $x_i$ 's in (1.1)), it is enough to provide the initial conditions  $D_k, D_{k+1}, D_{k+2}, A_k, M_k(\zeta)$  in order to be able to compute  $D_s$  for arbitrary  $s$ . These initial conditions are readily expressed in terms of  $\{\pi_s\}$  and  $\{w(\pi_s)\}$ , see Sect. 6 below.

For certain classical weights  $w$  the recurrence relation (1.2) can be substantially simplified. To illustrate the situation, let us consider  $\mathfrak{X} = \mathbb{Z}_{\geq 0}$  and  $w(x) = a^x/x!$ , where  $a > 0$  is a parameter. This weight function corresponds to the Charlier orthogonal polynomials.

In this case,  $A_s$  and  $M_s(\zeta)$  can be parameterized by three scalar sequences  $a_s, b_s, c_s$  as follows:

$$M_s(\zeta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} b_s & b_s c_s \\ a/c_s & a \end{bmatrix}, \quad A_s = (k + b_s) \begin{bmatrix} -1 & -a_s c_s \\ 1/(a_s c_s) & 1 \end{bmatrix}. \tag{1.3}$$

Then the equality (1.2) leads to the following recurrence relations:

$$a_{s+1} = \frac{(b_s + a a_s)(k + b_s + a a_s)}{a a_s (s + 1 + b_s + a a_s)}, \tag{1.4}$$

$$b_{s+1} = \frac{s + 1}{1 - a_{s+1}} - (s + 1 + k + b_s + a a_s), \tag{1.5}$$

$$c_{s+1} = \frac{a a_s}{k + b_s + a a_s} c_s. \tag{1.6}$$

<sup>3</sup> In fact, for 2 by 2 matrices one can easily see that  $(A_{s+1}, M_{s+1}^{(l)}, \dots, M_{s+1}^{(0)})$  are rational functions of  $(A_s, M_s^{(l)}, \dots, M_s^{(0)})$ .

The connection of these sequences and the distribution  $D_s$  is given by

$$\left(\frac{D_{s+3}}{D_{s+2}} - \frac{D_{s+2}}{D_{s+1}}\right) \left(\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s}\right)^{-1} = \frac{c_s(b_s + aa_s)(b_{s+1} + aa_{s+1})}{a(s+2)a_{s+1}^2 c_{s+1}}.$$

The corresponding initial conditions can be found in Subsect. 11.2.

Under the change of variables

$$f_s = a_s^{-1}, \quad g_s = aa_s + b_s + s + 1,$$

(1.4)–(1.5) turn into

$$f_s f_{s+1} = \frac{ag_s}{(g_s - s - 1)(g_s + k - s - 1)},$$

$$g_s + g_{s+1} = \frac{a}{f_{s+1}} - \frac{s+1}{1-f_{s+1}} - k + 2s + 3.$$

This recurrence is immediately identified with the difference Painlevé IV equation (dPIV) of [17].

1.3. It turns out that the situation for the Charlier weight described above is rather typical. We are also able to reduce (1.2) to scalar rational recurrence relations for other classical weight functions. Our results are summarized in the following table.

The family of orthogonal polynomials	The corresponding orthogonality sets	The weight defining the families	The corresponding ( $q$ -)difference equations
Charlier polynomials	$\mathfrak{X} = \mathbb{Z}_{\geq 0}$	$\omega(x) = a^x/x!$ $a > 0$	difference Painlevé IV, see Sect. 11.2
Meixner polynomials	$\mathfrak{X} = \mathbb{Z}_{\geq 0}$	$\omega(x) = \frac{(\beta)_x}{x!} c^x$ $\beta > 0, 0 < c < 1$	difference Painlevé V, see Sect. 11.4
Krawtchouk polynomials	$\mathfrak{X} = \{0, 1, \dots, N\}$	$\omega(x) = \binom{N}{x} p^x (1-p)^{N-x}$ $N \in \mathbb{Z}_{\geq 0}, 0 < p < 1$	difference Painlevé V, see Sect. 11.5
$q$ -Charlier polynomials	$\mathfrak{X} = \{q^{-x}   x \in \mathbb{Z}_{\geq 0}\}$	$\omega(x) = \frac{a^x}{(q; q)_x} q^{\binom{x}{2}}$ $a > 0$	degeneration of $q - P_{VI}$ , see Sect. 7.5, Sect. 10.3 and Sect. 11.7
Alternative $q$ -Charlier polynomials	$\mathfrak{X} = \{q^x   x \in \mathbb{Z}_{\geq 0}\}$	$\omega(x) = \frac{a^x}{(q; q)_x} q^{\binom{x+1}{2}}$ $a > 0$	see the next paragraph
Little $q$ -Laguerre (or Wall) polynomials	$\mathfrak{X} = \{q^x   x \in \mathbb{Z}_{\geq 0}\}$	$\omega(x) = \frac{(aq)^x}{(q; q)_x}$ $0 < a < q^{-1}$	degeneration of $q - P_{VI}$ , see Sect. 7.5, Sect. 10.3 and Sect. 11.7
Little $q$ -Jacobi polynomials	$\mathfrak{X} = \{q^x   x \in \mathbb{Z}_{\geq 0}\}$	$\omega(x) = \frac{(bq; q)_x}{(q; q)_x} (aq)^x$ $0 < a < q^{-1}, b < q^{-1}$	$q$ -Painlevé VI, see Sect. 7.5, Sect. 10.1 and Sect. 11.7
$q$ -Krawtchouk polynomials	$\mathfrak{X} = \{q^{-x}   x = 0, \dots, N\}$	$\omega(x) = \frac{(q^{-N}; q)_x}{(q; q)_x} (-p)^{-x}$ $N \in \mathbb{Z}_{\geq 0}, p > 0$	$q$ -Painlevé VI, see Sect. 7.5, Sect. 10.1 and Sect. 11.7

It is remarkable that in almost all the cases we can solve explicitly, we end up with one of the equations of Sakai’s hierarchy which was constructed out of purely algebraic geometric considerations, see [17]. (We were not able to see such a reduction in the

alternative q-Charlier case, but we do not claim that there is none.) So far we have not found a conceptual explanation for this fact.

One can notice, however, that recurrence relations originating from (1.2) must have some kind of singularity confinement property. (This property was the starting point of Sakai’s work.) For example, the parameterization (1.3) does not make much sense if, say,  $A_s$  has a zero (2.1) element. Then the values of  $a_s$  and  $c_s$  are not well-defined. In terms of the recurrence relations, this is reflected by vanishing of one of the denominators in (1.4)–(1.6). However, the matrix sequence  $\{(A_s, M_s)\}$  does not feel this singularity, which means that the sequences  $\{a_s\}, \{b_s\}, \{c_s\}$  can be “continued through” their singular values. Of course, all Sakai’s equations have this kind of singularity confinement by construction.

Let us also point out that it is not clear at this point whether the weights of higher hypergeometric polynomials of the Askey scheme will also lead to one of Sakai’s equations. All the cases that we were able to solve explicitly have *linear* matrices  $M_s(\zeta)$  in (1.2), while, say, for the Hahn weight  $M_s(\zeta)$  is quadratic. Handling such cases seems to be a problem of the next level of difficulty. It remains an interesting open problem to derive explicit rational recurrence relations for  $\deg M_s = 2$ .

For Charlier and Meixner weights,  $D_s$  can also be written as Toeplitz determinants with symbols

$$(1+z)^k \exp(az^{-1}) \quad \text{and} \quad (1+z)^k(1+bz^{-1})^c \tag{1.7}$$

respectively, see [21, 12, 6]. Here  $a, b, c$  are parameters. Among previous results on the subject let us mention

- the derivation of dPII for Toeplitz determinants with the symbol  $\exp(\theta(z+z^{-1}))$ , see [2, 4, 1] (note also the derivation of the same equation for the quantity closely related to these Toeplitz determinants in [16, 21]);
- derivation of dPV for Toeplitz determinants with symbol  $(1+z)^k(1+bz^{-1})^c$  and  $k$  being not necessarily integral in [4];
- derivation of rational recurrence relations for Toeplitz determinants with symbols of the form

$$\exp(P_1(z) + P_2(z^{-1})) z^\gamma (1-d_1z)^{\gamma'_1} (1-d_2z)^{\gamma'_2} (1-d_1^{-1}z^{-1})^{\gamma''_1} (1-d_2^{-1}z^{-1})^{\gamma''_2}, \tag{1.8}$$

where  $P_1$  and  $P_2$  are polynomials with  $|\deg P_1 - \deg P_2| \leq 1$ ,  $\gamma'_1, \gamma'_2, \gamma''_1, \gamma''_2, d_1, d_2$  are constants, see [1]. Interestingly enough, for the symbols (1.7), which are special cases of (1.8), the relations of [1] do not seem to have much in common with those of [4] and the present paper.

1.4. The methods used in this paper are based on the formalism of discrete integrable operators and discrete Riemann–Hilbert problem (DRHP) developed in [3, 4]. The first step is to represent  $D_s$  as a Fredholm determinant of an integrable operator:  $D_s = \det(1 - K_s)$ , where  $K_s$  is an operator in  $\ell^2(\{\pi_s, \pi_{s+1}, \dots\})$  with the kernel

$$K_s(x, y) = \frac{1}{\|p_{k-1}\|_{\ell^2(\mathfrak{X}, w)}^2} \frac{p_{k-1}(x)p_k(y) - p_k(x)p_{k-1}(y)}{x - y} \sqrt{w(x)w(y)}.$$

Here  $p_k$  and  $p_{k-1}$  are monic  $k^{\text{th}}$  and  $(k-1)^{\text{st}}$  orthogonal polynomials on  $\mathfrak{X}$  with respect to the weight function  $w$ . Using the results of [3, 4], the computation of such a Fredholm

determinant can be reduced to solving a DRHP on  $\{\pi_0, \dots, \pi_{s-1}\}$  with a jump matrix easily expressible in terms of  $w$ . Our assumptions on  $\mathfrak{X}$  and  $w$ , see above, then allow us to obtain a *Lax pair* for the solution  $m_s(\zeta)$  of this DRHP, which has the form

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - \pi_s} \right) m_s(\zeta), \quad m_s(\sigma\zeta) = M_s(\zeta)m_{s+1}(\zeta)D^{-1}(\zeta). \quad (1.9)$$

Here  $M_s(\zeta)$  is a matrix polynomial and  $D(\zeta)$  is a fixed diagonal matrix polynomial. The compatibility condition for this pair of equations is exactly (1.2).

The paper is organized as follows. In Sect. 2 we reduce the problem of computing discrete orthogonal polynomials with a given weight to a DRHP. In Sect. 3 we derive the Lax pair (1.9). In Sect. 4 we show how to express the Fredholm determinants  $D_s$  in terms of  $(A_s, M_s)$ . In Sect. 5 we prove that the compatibility condition (1.2) always has a unique solution. In Sect. 6 we derive the initial conditions  $A_k, M_k, D_k, D_{k+1}$ . In Sect. 7 we write down explicitly the Lax pairs for 14 families of discrete hypergeometric orthogonal polynomials of the Askey scheme. In Sect. 8 we solve (1.2) in terms of rational recurrences for the matrix elements of  $A_s$  and  $M_s$  for  $\deg M_s = 1$ . In Sect. 9 we show how to reduce (1.2) to dPIV and dPV equations in the case when  $\deg M_s = 1$  and  $\sigma\zeta = \zeta - 1$ . In Sect. 10 we reduce (1.2) to  $q$ -PVI or its degeneration in the case when  $\deg M_s = 1$  and  $\sigma\zeta = q^{\pm 1}\zeta$ . Finally, in Sect. 11 we solve (1.2) in terms of difference and  $q$ -Painlevé equations for 7 families of classical orthogonal polynomials. At the end of the paper we also provide a few plots of the “density function” (difference or  $q$ -derivative of  $D_s$ ) obtained using the formulas of Sect. 11.

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1.5. The following notation is used throughout our paper. For an integer  $k$ , we write  $\mathbb{Z}_{\geq k} = \{k, k + 1, k + 2, \dots\}$ . If  $a, q \in \mathbb{C}$  and  $k \in \mathbb{Z}_{\geq 0}$ , one defines the *Pochhammer symbol* and its  $q$ -analogue (often also called the  $q$ -shifted factorial) by

$$(a)_0 := 1, \quad (a)_k := a(a + 1)(a + 2) \cdots (a + k - 1) \text{ if } k \geq 1$$

and

$$(a; q)_0 := 1, \quad (a; q)_k := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{k-1}) \text{ if } k \geq 1,$$

respectively. One usually writes

$$(a_1, \dots, a_r)_k = \prod_{j=1}^r (a_j)_k \quad \text{and} \quad (a_1, \dots, a_r; q)_k = \prod_{j=1}^r (a_j; q)_k.$$

If  $r, s \in \mathbb{Z}_{\geq 0}$  and  $a_1, \dots, a_r, b_1, \dots, b_s, z, q \in \mathbb{C}$ , the *hypergeometric series* and the *basic hypergeometric series* are defined by

$${}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r)_k}{(b_1, \dots, b_s)_k} \frac{z^k}{k!}$$

and

$${}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} (-1)^{(1+s-r)k} q^{\binom{k}{2}} \frac{z^k}{(q; q)_k},$$

respectively.

## 2. Discrete Riemann-Hilbert Problems and Orthogonal Polynomials

2.1. In this section we explain how solutions of discrete Riemann-Hilbert problems (DRHP) for jump matrices of a special type can be expressed in terms of the corresponding orthogonal polynomials. Let  $\mathfrak{X}$  be a discrete locally finite subset of  $\mathbb{C}$ , and let  $w : \mathfrak{X} \rightarrow \text{Mat}(2, \mathbb{C})$  be a function. As in [3, 4], we say that an analytic function

$$m : \mathbb{C} \setminus \mathfrak{X} \longrightarrow \text{Mat}(2, \mathbb{C})$$

solves the DRHP  $(\mathfrak{X}, w)$  if  $m$  has simple poles at the points of  $\mathfrak{X}$  and its residues at these points are given by the *jump* (or *residue*) condition

$$\text{Res}_{\zeta=x} m(\zeta) = \lim_{\zeta \rightarrow x} (m(\zeta)w(x)), \quad x \in \mathfrak{X}. \tag{2.1}$$

**Lemma 2.1.** *If  $m(\zeta)$  is a solution of the DRHP  $(\mathfrak{X}, w)$  and the matrix  $w(x)$  is nilpotent for all  $x \in \mathfrak{X}$ , then the function  $\det m(\zeta)$  is entire. If, in addition,  $\det m(\zeta) \rightarrow 1$  as  $\zeta \rightarrow \infty$ , then  $\det m(\zeta) \equiv 1$ .*

*Proof.* For each  $x \in \mathfrak{X}$ , the jump condition (2.1) implies that the function  $m(\zeta) \cdot (I - (\zeta - x)^{-1}w(x))$  is analytic in a neighborhood of  $x$ . Since  $w(x)$  is nilpotent, this product has the same determinant as  $m(\zeta)$ , which shows that  $\det m(\zeta)$  has no pole at  $x$ . The second statement of the lemma follows from Liouville’s theorem.  $\square$

2.2. We now assume that the matrix  $w(x)$  has the following form:

$$w(x) = \begin{pmatrix} 0 & \omega(x) \\ 0 & 0 \end{pmatrix}, \tag{2.2}$$

where  $\omega : \mathfrak{X} \rightarrow \mathbb{C}$  is a function. Recall that a collection  $\{P_n(\zeta)\}_{n=0}^\infty$  of complex polynomials is called the collection of *orthogonal polynomials associated to the weight function*  $\omega$  if

- $P_n$  is a polynomial of degree  $n$  for all  $n = 1, 2, \dots$ , and  $P_0 \equiv \text{const}$ ;
- if  $m \neq n$ , then

$$\sum_{x \in \mathfrak{X}} P_m(x)P_n(x)\omega(x) = 0.$$

We will always take  $P_n$  to be *monic*:  $P_n(x) = x^n + \text{lower terms}$ .

In order for the definition to make sense, we assume that all moments of the weight function  $\omega$  are finite, i.e.,

$$\text{the series } \sum_{x \in \mathfrak{X}} |\omega(x)x^j| \text{ converges for all } j \geq 0. \tag{2.3}$$

Under this condition, one can consider the following inner product on the space  $\mathbb{C}[\zeta]$  of all complex polynomials:

$$(f(\zeta), g(\zeta))_\omega := \sum_{x \in \mathfrak{X}} f(x)g(x)\omega(x).$$

It is clear that there exists a collection of orthogonal polynomials  $\{P_n(\zeta)\}$  associated to  $\omega$  such that  $(P_n, P_n)_\omega \neq 0$  for all  $n$  if and only if the restriction of  $(\cdot, \cdot)_\omega$  to the space  $\mathbb{C}[\zeta]^{\leq d}$  of polynomials of degree at most  $d$  is nondegenerate for all  $d \geq 0$ . If this condition holds, we say that the weight function  $\omega$  is *nondegenerate*. In this case, it is also clear that the collection  $\{P_n\}$  is unique.

*Remark 2.2.* If the set  $\mathfrak{X}$  is finite, one has to modify the definitions above. Indeed, if  $\mathfrak{X}$  consists of  $N + 1$  points ( $N \in \mathbb{Z}_{\geq 0}$ ), the inner product  $(\cdot, \cdot)_\omega$  is necessarily degenerate on  $\mathbb{C}[\zeta]^{\leq d}$  for all  $d > N$ . So, instead, we require that  $(\cdot, \cdot)_\omega$  be nondegenerate on  $\mathbb{C}[\zeta]^{\leq d}$  for  $0 \leq d \leq N$ , and we are only interested in a collection  $\{P_n(\zeta)\}_{n=0}^N$  of orthogonal polynomials of degrees up to  $N$ . On the other hand, the condition (2.3) is empty in this case.

*Remark 2.3.* If the values of the weight function  $\omega$  are real and strictly positive and the orthogonality set  $\mathfrak{X}$  is contained in  $\mathbb{R}$ , then  $\omega$  is automatically nondegenerate.

2.3. The first basic result of the paper is

**Theorem 2.4 (Solution of DRHP).** *Let  $\mathfrak{X}$  be a discrete locally finite subset of  $\mathbb{C}$  and  $\omega : \mathfrak{X} \rightarrow \mathbb{C}$  be a nondegenerate weight function satisfying (2.3). Let  $\{P_n(\zeta)\}_{n=0}^N$  be the collection of monic orthogonal polynomials associated to  $\omega$ , where  $N = \text{card}(\mathfrak{X}) - 1 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . Assume that the jump matrix  $w(x)$  is given by (2.2). Then for any  $k = 1, 2, \dots, N$ , the DRHP  $(\mathfrak{X}, w)$  has a unique solution  $m_{\mathfrak{X}}(\zeta)$  satisfying the asymptotic condition*

$$m_{\mathfrak{X}}(\zeta) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} = I + O\left(\frac{1}{\zeta}\right) \quad \text{as } \zeta \rightarrow \infty, \tag{2.4}$$

where  $I$  is the identity matrix. If we write

$$m_{\mathfrak{X}}(\zeta) = \begin{pmatrix} m_{\mathfrak{X}}^{11}(\zeta) & m_{\mathfrak{X}}^{12}(\zeta) \\ m_{\mathfrak{X}}^{21}(\zeta) & m_{\mathfrak{X}}^{22}(\zeta) \end{pmatrix},$$

then  $m_{\mathfrak{X}}^{11}(\zeta) = P_k(\zeta)$  and  $m_{\mathfrak{X}}^{21}(\zeta) = (P_{k-1}, P_{k-1})_\omega^{-1} \cdot P_{k-1}(\zeta)$ .

*Remark 2.5.* The asymptotic condition (2.4) needs to be made more precise. Indeed, if  $\mathfrak{X}$  is infinite, then the LHS of (2.4) has poles accumulating at infinity. In this case, we require that the asymptotics be uniform on a sequence of expanding contours (e.g., circles whose radii tend to  $\infty$ ) whose distance from  $\mathfrak{X}$  remains bounded away from zero. A similar remark applies to all asymptotic formulas below.

*Remark 2.6.* A result for continuous weight functions similar to Theorem 2.4 was proved in [8].

*Proof.* Fix a natural number  $k \leq N$  and define a matrix-valued function

$$m = \begin{pmatrix} m^{11} & m^{12} \\ m^{21} & m^{22} \end{pmatrix} : \mathbb{C} \longrightarrow \text{Mat}(2, \mathbb{C})$$

by

$$\begin{aligned} m^{11}(\zeta) &= P_k(\zeta), \\ m^{12}(\zeta) &= \sum_{x \in \mathfrak{X}} \frac{P_k(x)\omega(x)}{\zeta - x}, \\ m^{21}(\zeta) &= c \cdot P_{k-1}(\zeta), \\ m^{22}(\zeta) &= c \cdot \sum_{x \in \mathfrak{X}} \frac{P_{k-1}(x)\omega(x)}{\zeta - x}, \end{aligned}$$

where  $c$  is the unique constant for which  $m^{22}(\zeta) = \zeta^{-k} + O(\zeta^{-k-1})$  as  $\zeta \rightarrow \infty$  (we will show below that such a  $c$  exists, and in fact  $c = (P_{k-1}, P_{k-1})_{\omega}^{-1}$ ). We first observe that (2.3) implies that the series for  $m^{12}(\zeta)$  converges uniformly and absolutely on compact subsets of  $\mathbb{C} \setminus \mathfrak{X}$ , and hence  $m^{12}(\zeta)$  is analytic on the complement of  $\mathfrak{X}$ . Moreover, since for each  $x \in \mathfrak{X}$ , the series

$$\sum_{y \in \mathfrak{X} \setminus \{x\}} \frac{P_k(y)\omega(y)}{\zeta - y}$$

converges uniformly and absolutely on compact subsets of  $(\mathbb{C} \setminus \mathfrak{X}) \cup \{x\}$ , we see that  $m^{12}(\zeta)$  has a simple pole at  $x$ , with the residue given by

$$\operatorname{Res}_{\zeta=x} m^{12}(\zeta) = P_k(x)\omega(x).$$

Similarly,  $m^{22}(\zeta)$  is analytic away from  $\mathfrak{X}$ , with a simple pole at each  $x \in \mathfrak{X}$  satisfying

$$\operatorname{Res}_{\zeta=x} m^{22}(\zeta) = c \cdot P_{k-1}(x)\omega(x).$$

This shows that our matrix  $m(\zeta)$  satisfies the jump condition (2.1). To verify the asymptotic condition (and find the constant  $c$ ), note that

$$m^{11}(\zeta) \cdot \zeta^{-k} = 1 + O(1/\zeta), \quad m^{21}(\zeta) \cdot \zeta^{-k} = O(1/\zeta), \quad \text{as } \zeta \rightarrow \infty.$$

Next, we write

$$\frac{1}{\zeta - x} = \frac{1}{\zeta} \cdot \left(1 - \frac{x}{\zeta}\right)^{-1} = \frac{1}{\zeta} + \frac{x}{\zeta^2} + \dots + \frac{x^{k-1}}{\zeta^k} + O(\zeta^{-k-1}). \tag{2.5}$$

By the definition of orthogonal polynomials, we have

$$\sum_{x \in \mathfrak{X}} P_k(x)\omega(x)x^i = 0, \quad 0 \leq i \leq k - 1,$$

and hence substituting (2.5) into the definition of  $m^{12}(\zeta)$  yields

$$m^{12}(\zeta) = O(\zeta^{-k-1}) \quad \text{as } \zeta \rightarrow \infty.$$

Also, (2.5) gives

$$m^{22}(\zeta) = c \cdot \sum_{x \in \mathfrak{X}} P_{k-1}(x)\omega(x)x^{k-1} \cdot \zeta^{-k} + O(\zeta^{-k-1}) \quad \text{as } \zeta \rightarrow \infty.$$

Hence, if we set

$$c = \left( \sum_{x \in \mathfrak{X}} P_{k-1}(x)\omega(x)x^{k-1} \right)^{-1} = (P_{k-1}, P_{k-1})_{\omega}^{-1},$$

then the matrix  $m(\zeta)$  satisfies all the conditions of the theorem.

To prove uniqueness, let  $m_{\mathfrak{X}}$  be an arbitrary solution of the DRHP  $(\mathfrak{X}, w)$  satisfying the asymptotic condition (2.4). Since the functions  $m$  and  $m_{\mathfrak{X}}$  have the same (simple) poles and satisfy the same residue conditions at these poles, it is clear that the function  $f(\zeta) = m(\zeta)^{-1} \cdot m_{\mathfrak{X}}(\zeta)$  is entire (note that  $m(\zeta)$  is invertible by Lemma 2.1). The asymptotic conditions on the matrices  $m$  and  $m_{\mathfrak{X}}$  imply that  $f(\zeta) \rightarrow I$  as  $\zeta \rightarrow \infty$ . By Liouville's theorem,  $f \equiv I$ , which means that  $m = m_{\mathfrak{X}}$ .  $\square$

### 3. Lax Pairs for Solutions of DRHP

3.1. Let  $\mathfrak{X}$  be a discrete locally finite subset of  $\mathbb{C}$ , let  $w : \mathfrak{X} \rightarrow \text{Mat}(2, \mathbb{C})$  be a function, and fix a natural number  $k < \text{card}(\mathfrak{X})$ . If  $w$  arises from a nondegenerate weight function  $\omega$  on  $\mathfrak{X}$  with finite moments, as in Sect. 2, then from Theorem 2.4 we know that the DRHP  $(\mathfrak{X}, w)$  admits a unique solution  $m_{\mathfrak{X}}(\zeta)$  with the asymptotics of  $\text{diag}(\zeta^k, \zeta^{-k})$  at infinity.

*Convention.* From now on, we assume that the weight function under consideration is everywhere real and strictly positive, and the orthogonality set is contained in  $\mathbb{R}$ .

If  $\mathfrak{Z} \subseteq \mathfrak{X}$  is any subset of cardinality  $> k$ , it follows from Remark 2.3 and our convention that the restriction of  $\omega$  to  $\mathfrak{Z}$  is also nondegenerate, whence by Theorem 2.4, the DRHP  $(\mathfrak{Z}, w|_{\mathfrak{Z}})$  has a unique solution  $m_{\mathfrak{Z}}(\zeta)$  such that  $m_{\mathfrak{Z}}(\zeta) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \rightarrow I$  as  $\zeta \rightarrow \infty$ .

Let us now assume that the set  $\mathfrak{X}$  is parameterized as  $\mathfrak{X} = \{\pi_x\}_{x=0}^N$ , where  $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . For every  $s \geq 0$ , we consider the subset  $\mathfrak{Z}_s := \{\pi_x\}_{x \leq s-1} \subseteq \mathfrak{X}$ . If  $s > k$ , then  $\text{card}(\mathfrak{Z}_s) > k$ , so by the previous paragraph, we have the corresponding solution  $m_s(\zeta) := m_{\mathfrak{Z}_s}(\zeta)$ . It can also be shown that even though  $\text{card}(\mathfrak{Z}_k) = k$ , the DRHP  $(\mathfrak{Z}_k, w|_{\mathfrak{Z}_k})$  still has a unique solution  $m_k(\zeta)$ . Indeed, uniqueness can be proved by the same argument as in Theorem 2.4, and in Proposition 6.1 below we give an explicit formula for  $m_k(\zeta)$ . Note also that  $m_s(\zeta)$  is defined for all  $s \geq k$ ; in particular, if  $N$  is finite, we have  $m_s(\zeta) = m_{\mathfrak{X}}(\zeta)$  for all  $s > N$ .

Our next basic assumption is:

$$\begin{aligned} &\text{there exists an affine transformation } \sigma : \mathbb{C} \longrightarrow \mathbb{C} \text{ such that } \sigma\pi_{x+1} = \pi_x \\ &\text{for all } 0 \leq x < N. \end{aligned} \tag{3.1}$$

The *Lax pair* in our setting will consist of two equations, one of which relates  $m_{s+1}(\zeta)$  with  $m_s(\zeta)$  and the other one relates  $m_s(\sigma\zeta)$  with  $m_{s+1}(\zeta)$ . We will denote the derivative of  $\sigma$  (which is a constant) by  $\eta$ , so that

$$\sigma(\zeta_1) - \sigma(\zeta_2) = \eta \cdot (\zeta_1 - \zeta_2) \quad \text{for all } \zeta_1, \zeta_2 \in \mathbb{C}. \tag{3.2}$$

The cases of special interest are those when  $\mathfrak{X}$  is the orthogonality set for one of the families of discrete hypergeometric orthogonal polynomials of the Askey scheme. In this situation,  $\mathfrak{X}$  is a subset of either a one-dimensional lattice or a one-dimensional  $q$ -lattice in  $\mathbb{C}$ , and  $\sigma$  is given by either  $\sigma\zeta = \zeta - 1$  or  $\sigma\zeta = q^{\pm 1}\zeta$ . These cases will be treated in greater detail in Sect. 7; for now we concentrate on the general theory.

3.2. The main result of this section is

**Theorem 3.1 (Lax pair).** *For each  $s = k, k + 1, k + 2, \dots$ , let  $\mathfrak{Z}_s = \{\pi_x \in \mathfrak{X} \mid x \leq s - 1\}$ , and let  $m_s(\zeta)$  be the unique solution of the DRHP  $(\mathfrak{Z}_s, w|_{\mathfrak{Z}_s})$  such that  $m_s(\zeta) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \rightarrow I$  as  $\zeta \rightarrow \infty$ , where  $w$  is given by<sup>4</sup>*

$$w(\pi_x) = \begin{pmatrix} 0 & \omega(x) \\ 0 & 0 \end{pmatrix}.$$

<sup>4</sup> To simplify notation, we assume that the weight function  $\omega(x)$  is always defined for  $x \in \mathbb{Z}_{\geq 0}$ ,  $x \leq N$ .

(a) For each  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , there exists a constant nilpotent matrix  $A_s$  such that

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - \pi_s} \right) m_s(\zeta). \tag{3.3}$$

(b) Assume that there exist entire functions  $d_1(\zeta)$ ,  $d_2(\zeta)$  such that

$$\frac{\omega(x-1)}{\omega(x)} = \eta \cdot \frac{d_1(\pi_x)}{d_2(\pi_x)}, \quad \text{for all } 1 \leq x \leq N,$$

$$d_1(\pi_0) = 0, \quad \text{and} \quad d_2(\sigma^{-1}\pi_N) = 0 \text{ if } N \text{ is finite.}$$

Let

$$D(\zeta) = \begin{pmatrix} d_1(\zeta) & 0 \\ 0 & d_2(\zeta) \end{pmatrix}.$$

Then for every  $s \in \mathbb{Z}_{\geq k}$ , we have

$$m_s(\sigma\zeta) = M_s(\zeta)m_{s+1}(\zeta)D^{-1}(\zeta), \tag{3.4}$$

where  $M_s(\zeta)$  is entire<sup>5</sup>.

(c) With the assumptions of part (b), suppose that the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  are polynomials of degree at most  $n$  in  $\zeta$ , and write  $d_1(\zeta) = \lambda_1 \zeta^n + (\text{lower terms})$ ,  $d_2(\zeta) = \lambda_2 \zeta^n + (\text{lower terms})$ . Set  $\kappa_1 = \eta^k \lambda_1$ ,  $\kappa_2 = \eta^{-k} \lambda_2$ . Then the matrix  $M_s(\zeta)$  is polynomial of degree at most  $n$  in  $\zeta$ , with the coefficient of  $\zeta^n$  equal to  $\text{diag}(\kappa_1, \kappa_2)$ .

Equations (3.3) and (3.4) constitute the Lax pair.

*Remark 3.2.* As follows from the proof below, the condition  $d_2(\sigma^{-1}\pi_N) = 0$  in part (b) of the theorem is only required to assert that  $M_s(\zeta)$  is entire for  $s > N$ . If  $d_2(\sigma^{-1}\pi_N) \neq 0$ , then (b) and (c) still hold for all  $s$ ,  $k \leq s \leq N$ .

*Proof.* (a) Fix  $s \in \mathbb{Z}_{\geq 0}$ ,  $s \leq N$ , and consider the matrix-valued function  $N(\zeta) := m_{s+1}(\zeta)m_s^{-1}(\zeta)$  (recall that  $m_s(\zeta)$  is invertible by Lemma 2.1). It is clear that  $N(\zeta)$  has only one simple pole, at  $\zeta = \pi_s$ . Hence the function  $N(\zeta) - (\zeta - \pi_s)^{-1}A_s$  is entire, where  $A_s = \text{Res}_{\zeta=\pi_s} N(\zeta)$ . But  $m_s(\zeta)$  and  $m_{s+1}(\zeta)$  have the same asymptotics at infinity, so  $N(\zeta) - (\zeta - \pi_s)^{-1}A_s \rightarrow I$  as  $\zeta \rightarrow \infty$ . By Liouville's theorem,  $N(\zeta) - (\zeta - \pi_s)^{-1}A_s \equiv I$ , which gives (3.3). Taking determinants of both sides of (3.3) and using Lemma 2.1 gives  $\det(I + (\zeta - \pi_s)^{-1}A_s) = 1$  for all  $\zeta$ , which forces the matrix  $A_s$  to be nilpotent.

(b) We have  $M_s(\zeta) = m_s(\sigma\zeta)D(\zeta)m_{s+1}^{-1}(\zeta)$ . Therefore it is clear that  $M_s(\zeta)$  is analytic away from  $\{\pi_0, \pi_1, \dots, \pi_s\}$ . Let  $1 \leq x \leq s$ . Then for  $\zeta$  near  $\pi_x$ , we can write

$$m_s(\sigma\zeta) = H_1(\zeta) \left( I + \frac{w(x-1)}{\sigma\zeta - \pi_{x-1}} \right) \quad \text{and} \quad m_{s+1}^{-1}(\zeta) = \left( I - \frac{w(x)}{\zeta - \pi_x} \right) H_2(\zeta),$$

where  $H_1$  and  $H_2$  are analytic and invertible matrix-valued functions defined in a neighborhood of  $\pi_x$ . So  $M_s(\zeta)$  is analytic near  $\pi_x$  if and only if so is the product

$$\left( I + \frac{w(x-1)}{\sigma\zeta - \pi_{x-1}} \right) D(\zeta) \left( I - \frac{w(x)}{\zeta - \pi_x} \right) = \begin{pmatrix} d_1(\zeta) & \frac{\eta^{-1}d_2(\zeta)\omega(x-1) - d_1(\zeta)\omega(x)}{\zeta - \pi_x} \\ 0 & d_2(\zeta) \end{pmatrix},$$

<sup>5</sup> Again, note that  $M_s(\zeta)$  is defined for all  $s \in \mathbb{Z}_{\geq k}$ , in particular for  $s > N$ , even if  $N$  is finite.

i.e., if and only if

$$\frac{\omega(x-1)}{\omega(x)} = \eta \cdot \frac{d_1(\pi_x)}{d_2(\pi_x)}.$$

Similarly, we note that  $m_s(\sigma\zeta)$  is analytic near  $\pi_0$ , whereas

$$m_{s+1}^{-1}(\zeta) = \left( I - \frac{w(0)}{\zeta - \pi_0} \right) H(\zeta),$$

with  $H$  analytic and invertible near  $\pi_0$ , which implies that  $M_s(\zeta)$  is analytic near  $\pi_0$  if and only if  $d_1(\pi_0) = 0$ . Finally, if  $N$  is finite, we have to make sure that  $M_s(\zeta)$  has no pole at  $\sigma^{-1}\pi_N$  for  $s \geq N + 1$ . A necessary and sufficient condition for that is  $d_2(\sigma^{-1}\pi_N) = 0$ .

(c) Using the asymptotic conditions on the matrices  $m_s(\sigma\zeta)$  and  $m_{s+1}(\zeta)$ , we obtain, as  $\zeta \rightarrow \infty$ ,

$$\begin{aligned} M_s(\zeta) &= m_s(\sigma\zeta)D(\zeta)m_{s+1}^{-1}(\zeta) \\ &= \left( I + O\left(\frac{1}{\zeta}\right) \right) \begin{pmatrix} \eta^k \zeta^k & 0 \\ 0 & \eta^{-k} \zeta^{-k} \end{pmatrix} \\ &\quad \begin{pmatrix} \lambda_1 \zeta^n + O(\zeta^{n-1}) & 0 \\ 0 & \lambda_2 \zeta^n + O(\zeta^{n-1}) \end{pmatrix} \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \left( I + O\left(\frac{1}{\zeta}\right) \right) \\ &= \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \cdot \zeta^n + O(\zeta^{n-1}). \end{aligned}$$

Since  $M_s(\zeta)$  is entire by part (b), this completes the proof.  $\square$

3.3. To conclude this section, we remark that the method by which we have obtained the second equation of the Lax pair can be applied to derive a (second-order) difference equation for the orthogonal polynomials corresponding to the weight function  $\omega$ . More precisely, one proves, by the same argument as Theorem 3.1, the following

**Proposition 3.3.** *Under the assumptions of Theorem 3.1(b), the function  $M(\zeta) = m_{\mathfrak{X}}(\sigma\zeta) D(\zeta)m_{\mathfrak{X}}^{-1}(\zeta)$  is entire. The analogue of Theorem 3.1(c) also holds for the matrix  $M(\zeta)$ .*

Now we have  $m_{\mathfrak{X}}(\sigma\zeta) = M(\zeta)m_{\mathfrak{X}}(\zeta)D(\zeta)^{-1}$ . Considering the (1, 1) and (2, 1) elements of both sides and using the explicit formula for  $m_{\mathfrak{X}}(\zeta)$  given in Theorem 2.4, we obtain a system of equations of the form

$$P_k(\sigma\zeta) = (M^{11}(\zeta)P_k(\zeta) + cM^{12}(\zeta)P_{k-1}(\zeta))d_1(\zeta)^{-1}, \tag{3.5}$$

$$c \cdot P_{k-1}(\sigma\zeta) = (M^{21}(\zeta)P_k(\zeta) + cM^{22}(\zeta)P_{k-1}(\zeta))d_1(\zeta)^{-1}, \tag{3.6}$$

where  $c = (P_{k-1}, P_{k-1})_{\omega}^{-1}$  and the  $M^{ij}(\zeta)$  are the elements of the matrix  $M(\zeta)$ . If we find  $c \cdot P_{k-1}(\zeta)$  from the first equation and substitute it into the second one, we will get a relation between  $P_k(\zeta)$ ,  $P_k(\sigma\zeta)$  and  $P_k(\sigma^2\zeta)$ . Even though the functions  $M^{ij}(\zeta)$  may involve unknown parameters, one might hope that the equation we obtain in the end will only involve known parameters. We will work out an explicit example in Subsect. 11.3, for the weight function corresponding to the Charlier polynomials. We will see that the relation we get is exactly the standard difference equation for Charlier polynomials.

### 4. Recurrence Relation for Fredholm Determinants

4.1. We remain in the general setting of Sect. 3. Thus we consider a discrete locally finite subset  $\mathfrak{X} = \{\pi_x\}_{x=0}^N \subseteq \mathbb{R}$ , where  $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , and we are also given a strictly positive weight function  $\omega : \{x \in \mathbb{Z}_{\geq 0} \mid x \leq N\} \rightarrow \mathbb{C}$  whose moments are finite. For all  $s \in \mathbb{Z}_{\geq 0}$ ,  $s \leq N$ , we let

$$\mathfrak{I}_s = \{\pi_x\}_{x=0}^{s-1}, \quad \mathfrak{J}_s = \mathfrak{X} \setminus \mathfrak{I}_s.$$

Finally, we fix a natural number  $k \leq N$  and let

$$m_{\mathfrak{X}}(\zeta) = \begin{pmatrix} m_{\mathfrak{X}}^{11}(\zeta) & m_{\mathfrak{X}}^{12}(\zeta) \\ m_{\mathfrak{X}}^{21}(\zeta) & m_{\mathfrak{X}}^{22}(\zeta) \end{pmatrix}$$

be the unique solution of the discrete Riemann-Hilbert problem  $(\mathfrak{X}, w)$  with the asymptotics of  $\text{diag}(\zeta^k, \zeta^{-k})$  at infinity provided by Theorem 2.4.

Let  $\alpha, \beta : \{x \in \mathbb{Z}_{\geq 0} \mid x \leq N\} \rightarrow \mathbb{C}$  be two functions such that  $\alpha(x)\beta(x) = \omega(x)$  for all  $x$ . Consider the following kernel on  $\mathfrak{X} \times \mathfrak{X}$ :

$$K(\pi_x, \pi_y) = \begin{cases} \alpha(x)\beta(y) \frac{\phi(\pi_x)\psi(\pi_y) - \psi(\pi_x)\phi(\pi_y)}{\pi_x - \pi_y}, & x \neq y, \\ \alpha(x)\beta(x)(\phi'(\pi_x)\psi(\pi_x) - \psi'(\pi_x)\phi(\pi_x)), & x = y, \end{cases} \tag{4.1}$$

where  $\phi(\zeta) = m_{\mathfrak{X}}^{11}(\zeta) = P_k(\zeta)$  and  $\psi(\zeta) = m_{\mathfrak{X}}^{21}(\zeta) = (P_{k-1}, P_{k-1})_{\omega}^{-1} \cdot P_{k-1}(\zeta)$ . Up to the factor of  $\alpha(x)\beta(y)$ , this is precisely the Christoffel-Darboux kernel for the family of orthogonal polynomials corresponding to the weight function  $\omega$ . We will not need the specific form of the functions  $\alpha$  and  $\beta$  in our computations. Note also that changing  $\alpha$  and  $\beta$  while keeping their product fixed results in conjugation of the kernel  $K$  and hence has no effect on the Fredholm determinants studied below.

4.2. For each  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , we let  $K_s$  be the restriction of  $K$  to  $\mathfrak{J}_s \times \mathfrak{J}_s$ . We also denote by  $K$  and  $K_s$  the operators on  $l^2(\mathfrak{X})$  and  $l^2(\mathfrak{J}_s)$  defined by the kernels  $K$  and  $K_s$ , respectively. The main goal of our paper is to derive a recurrence relation for the Fredholm determinants

$$D_s = \det(1 - K_s), \quad s \in \mathbb{Z}_{\geq k}, \quad s \leq N.$$

The resulting equation will be in terms of the elements of the matrices  $A_s$  and  $M_s(\zeta)$  from the Lax pair (3.3), (3.4). Note that the Fredholm determinants are always well defined because  $K$  is a finite rank operator, as follows from the Christoffel-Darboux formula (see, e.g., [18]):

$$\frac{K(\pi_x, \pi_y)}{\alpha(x)\beta(y)} = \sum_{m=0}^{k-1} \frac{P_m(\pi_x)P_m(\pi_y)}{(P_m, P_m)_{\omega}}.$$

For a probabilistic interpretation of these determinants, see Subsect. 4.5 below.

**Lemma 4.1.** *For each  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , the operator  $1 - K_s$  is invertible, and  $D_s = \det(1 - K_s) \neq 0$ .*

*Proof.* The operator  $K$  is the orthogonal projection onto the subspace of  $l^2(\mathfrak{X})$  spanned by the polynomials  $P_0, \dots, P_{k-1}$ . In particular, it has finite rank, its only eigenvalues are 0 and 1, and the eigenvectors corresponding to the eigenvalue 1 are linear combinations of  $P_0, \dots, P_{k-1}$ , i.e., the polynomials of degree  $\leq k - 1$ . It follows that the operator  $K_s$  also has finite rank, hence  $1 - K_s$  is not invertible if and only if 1 is an eigenvalue of  $K_s$ . Suppose that  $k \leq s \leq N$  and there exists  $f \in l^2(\mathfrak{Q}_s)$  such that  $K_s f = f$ . We extend  $f$  by zero to  $\mathfrak{X}$ , and we denote the extension by  $\tilde{f} \in l^2(\mathfrak{X})$ . Now  $K\tilde{f}|_{\mathfrak{Q}_s} = K_s f = f = \tilde{f}|_{\mathfrak{Q}_s}$  and  $\tilde{f}|_{\mathfrak{Z}_s} = 0$ . This implies that  $K\tilde{f}|_{\mathfrak{Z}_s} = 0$  because otherwise we would have  $\|K\tilde{f}\|_\omega > \|\tilde{f}\|_\omega$ , which is impossible because  $K$  is a projection operator. This shows that  $K\tilde{f} = \tilde{f}$ , whence by the remarks above,  $\tilde{f}$  is a polynomial of degree  $\leq k - 1$ . But it vanishes on the set  $\mathfrak{Z}_s$  of cardinality  $\geq k$ , which implies that  $\tilde{f}$  is identically equal to zero. Hence,  $1 - K_s$  is an invertible operator, and  $\det(1 - K_s) \neq 0$ .  $\square$

4.3. For each  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , we have the unique solution  $m_s(\zeta)$  of the DRHP  $(\mathfrak{Z}_s, w|_{\mathfrak{Z}_s})$  having the same asymptotics at infinity as  $m_{\mathfrak{X}}(\zeta)$ . As in Sect. 3, we assume that there exist entire functions  $d_1(\zeta), d_2(\zeta)$  such that

$$d_1(\pi_0) = 0 \quad \text{and} \quad \frac{\omega(x - 1)}{\omega(x)} = \eta \cdot \frac{d_1(\pi_x)}{d_2(\pi_x)} \quad \text{for all } 1 \leq x \leq N. \tag{4.2}$$

The assumption that  $d_2(\sigma^{-1}\pi_N) = 0$  if  $N$  is finite is not essential for us now since it was only used in Sect. 3 to show that the function  $M_s(\zeta)$  is entire for  $s > N$ , whereas here we are only interested in the case  $k \leq s \leq N$  (cf. Remark 3.2). Note that  $d_1(\pi_x)d_2(\pi_x) \neq 0$  for all  $1 \leq x \leq N$ . We let

$$D(\zeta) = \begin{pmatrix} d_1(\zeta) & 0 \\ 0 & d_2(\zeta) \end{pmatrix} \quad \text{and} \quad M_s(\zeta) = m_s(\sigma\zeta)D(\zeta)m_{s+1}^{-1}(\zeta).$$

By part (b) of Theorem 3.1, the function  $M_s(\zeta)$  is entire for all  $k \leq s \leq N$ . Using part (a) of this theorem, we obtain the following general form of the Lax pair for the solutions  $m_s(\zeta)$ :

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - \pi_s} \right) m_s(\zeta), \quad A_s = \begin{pmatrix} p_s & q_s \\ r_s & -p_s \end{pmatrix}, \quad p_s^2 = -q_s r_s; \tag{4.3}$$

$$m_s(\sigma\zeta) = M_s(\zeta)m_{s+1}(\zeta)D(\zeta)^{-1}. \tag{4.4}$$

In Sect. 5, we will explain how one can obtain a recurrence relation for the matrix elements of  $M_s(\zeta)$  and  $A_s$ . In the present section, we assume that the function  $M_s(\zeta)$  is known, and derive a recurrence relation for the Fredholm determinants using this function and the parameters  $p_s, q_s$ .

For all  $s, k \leq s \leq N$ , we put

$$m_s(\pi_s) = \begin{pmatrix} m_s^{11} & m_s^{12} \\ m_s^{21} & m_s^{22} \end{pmatrix}, \quad M_s(\pi_{s+1}) = \begin{pmatrix} \mu_s^{11} & \mu_s^{12} \\ \mu_s^{21} & \mu_s^{22} \end{pmatrix}, \quad M'_s(\pi_{s+1}) = \begin{pmatrix} v_s^{11} & v_s^{12} \\ v_s^{21} & v_s^{22} \end{pmatrix}, \tag{4.5}$$

where  $M'_s(\zeta) = \frac{d}{d\zeta} M_s(\zeta)$ . Then we have the following recurrence relations for the matrix elements  $m_s^{ij}$  and for the Fredholm determinants  $D_s$  in terms of the parameters  $p_s, q_s, \mu_s^{ij}, v_s^{ij}$ .

**Theorem 4.2 (Recurrence relation for Fredholm determinants).** *Assume that  $p_s \neq 0$  for all  $s \geq k$ , and hence also  $q_s, r_s \neq 0$  for  $s \geq k$ .*

(a) *For each  $s \in \mathbb{Z}_{\geq k}, s \leq N$ , we have*

$$m_{s+1}^{11} = d_2(\pi_{s+1})^{-1} \left( \mu_s^{22} + \frac{p_s}{q_s} \mu_s^{12} \right) m_s^{11}. \tag{4.6}$$

(b) *For each  $s \in \mathbb{Z}_{\geq k}, s \leq N - 2$ , we have*

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} = \frac{\omega(s) \cdot u_s \cdot (m_s^{11})^2}{\eta \cdot d_1(\pi_{s+1}) \cdot d_2(\pi_{s+1})}, \tag{4.7}$$

where

$$\begin{aligned} u_s = & (v_s^{21} \mu_s^{22} - v_s^{22} \mu_s^{21}) + \frac{p_s}{q_s} \cdot (v_s^{21} \mu_s^{12} - v_s^{12} \mu_s^{21} + v_s^{11} \mu_s^{22} - v_s^{22} \mu_s^{11}) \\ & + \frac{p_s^2}{q_s^2} \cdot (v_s^{11} \mu_s^{12} - v_s^{12} \mu_s^{11}) \end{aligned}$$

for all  $s$ .

*Remark 4.3.* We will show later (see Proposition 5.5 and Remark 6.3) that under the assumptions of Subsect. 1.2,  $p_s$  is nonzero for all  $s \geq k$ .

*Proof.* (a) First we take the residues of both sides of (4.3) at  $\zeta = \pi_s$  and use the jump condition on the LHS:

$$\lim_{\zeta \rightarrow \pi_s} m_{s+1}(\zeta)w(\pi_s) = A_s m_s(\pi_s).$$

Since the first column of the matrix  $w(\pi_s)$  is zero, we deduce that the first column of the matrix  $A_s m_s(\pi_s)$  is zero, whence  $m_s^{21} = -(p_s/q_s)m_s^{11}$ . Next we substitute  $\zeta = \pi_{s+1}$  in (4.4) and rewrite it as follows:

$$m_{s+1}(\pi_{s+1}) = M_s^{-1}(\pi_{s+1})m_s(\pi_s)D(\pi_{s+1}).$$

Since  $\det M_s(\zeta) = \det D(\zeta)$  by Lemma 2.1, the last equation can be written explicitly as

$$\begin{pmatrix} m_{s+1}^{11} & m_{s+1}^{12} \\ m_{s+1}^{21} & m_{s+1}^{22} \end{pmatrix} = \frac{1}{d_1(\pi_{s+1})d_2(\pi_{s+1})} \begin{pmatrix} \mu_s^{22} & -\mu_s^{12} \\ -\mu_s^{21} & \mu_s^{11} \end{pmatrix} \begin{pmatrix} m_s^{11} & m_s^{12} \\ m_s^{21} & m_s^{22} \end{pmatrix} \begin{pmatrix} d_1(\pi_{s+1}) & 0 \\ 0 & d_2(\pi_{s+1}) \end{pmatrix}.$$

Equating the (1, 1) and (2, 1) elements of both sides yields

$$m_{s+1}^{11} = d_2(\pi_{s+1})^{-1}(\mu_s^{22}m_s^{11} - \mu_s^{12}m_s^{21}) = d_2(\pi_{s+1})^{-1} \left( \mu_s^{22} + \frac{p_s}{q_s} \mu_s^{12} \right) m_s^{11}$$

and

$$m_{s+1}^{21} = d_2(\pi_{s+1})^{-1}(-\mu_s^{21}m_s^{11} + \mu_s^{11}m_s^{21}) = -d_2(\pi_{s+1})^{-1} \left( \mu_s^{21} + \frac{p_s}{q_s} \mu_s^{11} \right) m_s^{11}.$$

(b) For all  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , we put  $R_s = K_s(1 - K_s)^{-1}$  (note that this is well defined by Lemma 4.1), so that  $R_s + 1 = (1 - K_s)^{-1}$ , and hence  $(R_s + 1)(s, s) = \det(1 - K_{s+1})/\det(1 - K_s)$ , i.e.,

$$R_s(s, s) = \frac{D_{s+1}}{D_s} - 1. \tag{4.8}$$

Now Theorem 2.3(ii) in [4] gives (Situation 2.2 ibid. explains why this theorem is applicable here)

$$R_s(s, s) = g^t(s)m_s^{-1}(\pi_s)m'_s(\pi_s)f(s),$$

where  $f(s) = (\alpha(s), 0)^t$  and  $g(s) = (0, -\beta(s))^t$ . For the purpose of our calculation, we rewrite this formula as follows:

$$R_s(s, s) = -\omega(s) \cdot e_2^t m_s^{-1}(\pi_s) m'_s(\pi_s) e_1, \tag{4.9}$$

where  $e_1 = (1, 0)^t$  and  $e_2 = (0, 1)^t$ . Similarly, we get

$$R_{s+1}(s + 1, s + 1) = -\omega(s + 1) \cdot e_2^t m_{s+1}^{-1}(\pi_{s+1}) m'_{s+1}(\pi_{s+1}) e_1. \tag{4.10}$$

We substitute  $\zeta = \pi_{s+1}$  into (4.4) again:

$$m_s(\pi_s) = M_s(\pi_{s+1})m_{s+1}(\pi_{s+1})D(\pi_{s+1})^{-1}, \tag{4.11}$$

and we differentiate (4.4) at  $\zeta = \pi_{s+1}$  to obtain

$$m'_s(\pi_s) = \eta^{-1} \frac{d}{d\zeta} \Big|_{\zeta=\pi_{s+1}} \left( M_s(\zeta)m_{s+1}(\zeta)D(\zeta)^{-1} \right). \tag{4.12}$$

If the derivative in (4.12) falls onto the third factor, the contribution of the corresponding term to the RHS of (4.9) is (by (4.11))

$$-\eta^{-1} \omega(s) \cdot e_2^t D(\pi_{s+1}) \frac{d}{d\zeta} \Big|_{\zeta=\pi_{s+1}} \left( D(\zeta)^{-1} \right) e_1 = \text{const} \cdot (0, 1) \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

If the derivative in (4.12) falls onto the second factor, the contribution of the corresponding term to the RHS of (4.9) is, by (4.11), (4.2) and (4.10),

$$\begin{aligned} & -\eta^{-1} \omega(s) \cdot e_2^t D(\pi_{s+1}) m_{s+1}^{-1}(\pi_{s+1}) m'_{s+1}(\pi_{s+1}) D(\pi_{s+1})^{-1} e_1 \\ &= -\eta^{-1} \omega(s) \frac{d_2(\pi_{s+1})}{d_1(\pi_{s+1})} \cdot e_2^t m_{s+1}^{-1}(\pi_{s+1}) m'_{s+1}(\pi_{s+1}) e_1 \\ &= -\omega(s + 1) \cdot e_2^t m_{s+1}^{-1}(\pi_{s+1}) m'_{s+1}(\pi_{s+1}) e_1 \\ &= R_{s+1}(s + 1, s + 1). \end{aligned}$$

Therefore, if we subtract  $R_{s+1}(s + 1, s + 1)$  from both sides of (4.9), we get the following equation:

$$\begin{aligned} & R_{s+1}(s + 1, s + 1) - R_s(s, s) \\ &= \eta^{-1} \omega(s) \cdot e_2^t m_s^{-1}(\pi_s) M'_s(\pi_{s+1}) m_{s+1}(\pi_{s+1}) D(\pi_{s+1})^{-1} e_1. \end{aligned} \tag{4.13}$$

Since  $\det m_s \equiv 1$  for all  $s$  by Lemma 2.1, the RHS of (4.13) equals

$$\begin{aligned} & \eta^{-1} \omega(s) \cdot (0, 1) \begin{pmatrix} m_s^{22} & -m_s^{12} \\ -m_s^{21} & m_s^{11} \end{pmatrix} \begin{pmatrix} v_s^{11} & v_s^{12} \\ v_s^{21} & v_s^{22} \end{pmatrix} \\ & \times \begin{pmatrix} m_{s+1}^{11} & m_{s+1}^{12} \\ m_{s+1}^{21} & m_{s+1}^{22} \end{pmatrix} \begin{pmatrix} d_1(\pi_{s+1})^{-1} & 0 \\ 0 & d_2(\pi_{s+1})^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & = \eta^{-1} \omega(s) d_1(\pi_{s+1})^{-1} (v_s^{21} m_s^{11} m_{s+1}^{11} + v_s^{22} m_s^{11} m_{s+1}^{21} - v_s^{11} m_s^{21} m_{s+1}^{11} - v_s^{12} m_s^{21} m_{s+1}^{21}). \end{aligned}$$

Substituting the formulas for  $m_s^{21}, m_{s+1}^{11}, m_{s+1}^{21}$  derived in part (a) into the last expression and using (4.8), we find that (4.13) is equivalent to (4.7).  $\square$

**Corollary 4.4.** *With the notation of Theorem 4.2, we have*

$$u_s \left( \frac{D_{s+3}}{D_{s+2}} - \frac{D_{s+2}}{D_{s+1}} \right) = \frac{(\mu_s^{22} + (p_s/q_s) \cdot \mu_s^{12})^2}{\eta \cdot d_1(\pi_{s+2}) \cdot d_2(\pi_{s+2})} \cdot u_{s+1} \left( \frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} \right) \quad (4.14)$$

for all  $s \in \mathbb{Z}_{\geq k}, s \leq N - 3$ .

*Proof.* Immediate from (4.7) and (4.6).  $\square$

4.4. It turns out that it is possible to extend the results of Theorem 4.2 to some cases where the orthogonality set  $\mathfrak{X}$  is discrete but not locally finite, i.e., has an accumulation point. The motivation for such an extension is the following. Even though the DRHP  $(\mathfrak{X}, w)$ , as it is stated in Sect. 2, is not well posed if  $\mathfrak{X}$  is not locally finite (for then we need to impose additional conditions near the accumulation point of the poles), the definition (4.1) of the kernel  $K$  still makes sense: the polynomials  $\phi(\zeta) = P_k(\zeta)$  and  $\psi(\zeta) = (P_{k-1}, P_{k-1})_{\omega}^{-1} \cdot P_{k-1}(\zeta)$  are well-defined. In fact, there exist several classical families of basic hypergeometric orthogonal polynomials for which the orthogonality set is discrete but not locally finite (see e.g. [13], Chapter 3). On the other hand, the solutions  $m_s(\zeta)$  of “restricted” DRHPs, and hence all the quantities derived from them, are also defined for a non-locally finite orthogonality set  $\mathfrak{X}$ , because their definitions involve only finite subsets of  $\mathfrak{X}$ . In particular, we can still consider the corresponding Lax pair as in Subsect. 4.3 and the scalar sequences  $p_s, q_s, m_s^{ij}, \mu_s^{ij}, v_s^{ij}$  defined by (4.3) and (4.5). It is therefore natural to ask whether the recurrence relation (4.7) remains valid in the case where  $\mathfrak{X}$  is not locally finite. We will see in Subsect. 4.6 that it does.

4.5. Let us recall the following probability-theoretic interpretation of the Fredholm determinants  $D_s$ , see e.g. [19]. In general, if  $\mathfrak{X}$  is a discrete, not necessarily locally finite subset of  $\mathbb{R}$  of cardinality  $N + 1$  ( $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ ),  $\omega : \mathfrak{X} \rightarrow \mathbb{R}$  is a strictly positive weight function whose moments are finite,  $\{P_n(\zeta)\}_{n=0}^N$  is the corresponding family of orthogonal polynomials and  $k$  is a natural number,  $k \leq N$ , we can consider a probability distribution  $\mathfrak{P}$  on the set of all subsets of  $\mathfrak{X}$  of cardinality  $k$ , defined by

$$\mathfrak{P}(\{x_1, \dots, x_k\}) = \frac{1}{Z} \cdot \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \cdot \prod_{i=1}^k \omega(x_i). \quad (4.15)$$

Here  $Z$  is the unique constant for which the measure of the set of all subsets of  $\mathfrak{X}$  of cardinality  $k$  is equal to 1:

$$Z = \sum_{\{x_1, \dots, x_k\} \subseteq \mathfrak{X}} \prod_{1 \leq i < j \leq k} (x_i - x_j)^2 \cdot \prod_{i=1}^k \omega(x_i). \tag{4.16}$$

Now if  $K$  denotes the kernel (4.1) defined in Subsect. 4.1, then for any subset  $\mathfrak{Y} \subseteq \mathfrak{X}$ , we have

$$\det(1 - K|_{\mathfrak{Y} \times \mathfrak{Y}}) = \sum_{x_i \notin \mathfrak{Y}} \mathfrak{P}(\{x_1, \dots, x_k\}) \tag{4.17}$$

(the sum on the RHS is taken over all subsets  $\{x_1, \dots, x_k\} \subseteq \mathfrak{X}$  of cardinality  $k$  which are disjoint from  $\mathfrak{Y}$ ).

4.6. We now assume that  $\mathfrak{X} = \{\pi_x\}_{x=0}^\infty \subset \mathbb{R}$  is discrete but not necessarily locally finite. We consider, as before, the subsets  $\mathfrak{Y}_s = \{\pi_x\}_{x=s}^\infty$  of  $\mathfrak{X}$ , and we are interested in the sequence  $\{D_s\}_{s=k}^\infty$  defined by

$$D_s = \det(1 - K_s), \quad K_s = K|_{\mathfrak{Y}_s \times \mathfrak{Y}_s},$$

where  $K$  is the kernel (4.1). The proof of Lemma 4.1 is still valid and gives  $D_s \neq 0$  for all  $s \in \mathbb{Z}_{\geq k}$ . Thus we see from the discussion of Subsect. 4.4 that both sides of (4.7) are at least well defined in this situation.

**Proposition 4.5.** *All formulas of Theorem 4.2 and Corollary 4.4 remain valid in the present situation.*

*Proof.* It is obvious from the proof of Theorem 4.2 that (4.6) remains valid. Now let us fix  $s \in \mathbb{Z}_{\geq k}$  and prove that formula (4.7) also holds in the present situation (then (4.14) follows automatically). Let  $L \in \mathbb{Z}$ ,  $L \geq s + 2$ . We write  $\mathfrak{X}^{(L)} = \{\pi_x\}_{x=0}^L \subset \mathfrak{X}$ . Since  $\mathfrak{X}^{(L)}$  is finite, all of the discussion of Subsects. 4.1–4.3 is valid for  $\mathfrak{X}^{(L)}$  in place of  $\mathfrak{X}$ . Since  $L \geq s + 2$ , it is clear that replacing  $\mathfrak{X}$  by  $\mathfrak{X}^{(L)}$  (and keeping the same weight function) has no effect on  $m_s(\zeta)$ ,  $p_s$ ,  $q_s$ ,  $m_s^{ij}$ ,  $\mu_s^{ij}$ ,  $\nu_s^{ij}$ . But of course, the quantities  $D_s$  do change. Let  $D_s^{(L)}$  denote the Fredholm determinants defined as in Subsect. 4.2 for  $\mathfrak{X}^{(L)}$  in place of  $\mathfrak{X}$ . Then Theorem 4.2(b) gives

$$\frac{D_{s+2}^{(L)}}{D_{s+1}^{(L)}} - \frac{D_{s+1}^{(L)}}{D_s^{(L)}} = \frac{\omega(s) \cdot u_s \cdot (m_s^{11})^2}{\eta \cdot d_1(\pi_{s+1}) \cdot d_2(\pi_{s+1})}.$$

It remains to observe that  $D_{s+1}^{(L)} / D_s^{(L)} = D_{s+1} / D_s$  for all  $s$ . Indeed, the discussion of Subsect. 4.5 gives the formula

$$D_s = \sum_{0 \leq i_1 < i_2 < \dots < i_k \leq s-1} \mathfrak{P}(\{\pi_{i_1}, \dots, \pi_{i_k}\}), \tag{4.18}$$

where  $\mathfrak{P}$  is given by (4.15). If  $\mathfrak{P}^{(L)}$  denotes the probability distribution on the set of all subsets of  $\mathfrak{X}^{(L)}$  of cardinality  $k$  defined similarly to  $\mathfrak{P}$ , then we also have

$$D_s^{(L)} = \sum_{0 \leq i_1 < i_2 < \dots < i_k \leq s-1} \mathfrak{P}^{(L)}(\{\pi_{i_1}, \dots, \pi_{i_k}\}). \tag{4.19}$$

We note that the summations in (4.18) and (4.19) are over the same index set, and the only difference between the definitions of  $\mathfrak{P}$  and  $\mathfrak{P}^{(L)}$  is in the normalization constant  $Z$ . Of course, when we take the ratios  $D_{s+1}^{(L)}/D_s^{(L)}$  and  $D_{s+1}/D_s$ , the normalization constants cancel each other, completing the proof.  $\square$

### 5. Compatibility Conditions for Lax Pairs

5.1. In this section we study the compatibility conditions for the Lax pairs of the form considered in Sects. 3, 4:

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - \pi_s} \right) m_s(\zeta), \tag{5.1}$$

$$m_s(\sigma\zeta) = M_s(\zeta)m_{s+1}(\zeta)D(\zeta)^{-1}. \tag{5.2}$$

The general notation and conventions are those of Sects. 3, 4. As in the second part of Sect. 4, we do not assume that the orthogonality set  $\mathfrak{X}$  is locally finite: we have already remarked in Subject. 4.4 that all of our arguments related to Lax pairs only involve finite subsets of  $\mathfrak{X}$ .

**Lemma 5.1.** Fix  $s \in \mathbb{Z}_{\geq 0}$ ,  $s \leq N$ , let  $A \in \text{Mat}(2, \mathbb{C})$ , and define

$$m(\zeta) = (I + (\zeta - \pi_s)^{-1}A)m_s(\zeta).$$

Then  $m(\zeta) = m_{s+1}(\zeta)$  if and only if the matrix  $A$  satisfies the following two conditions:

$$A \cdot m_s(\pi_s)w(\pi_s) = 0, \tag{5.3}$$

$$m_s(\pi_s)w(\pi_s) + A \cdot m'_s(\pi_s)w(\pi_s) = A \cdot m_s(\pi_s). \tag{5.4}$$

In particular, for a fixed  $s$ , there is a unique matrix  $A$  satisfying (5.3) and (5.4), namely,  $A = A_s$ .

*Proof.* By the uniqueness of  $m_{s+1}(\zeta)$ , we only have to verify that  $m(\zeta)$  satisfies the same residue conditions as  $m_{s+1}(\zeta)$  if and only if (5.3) and (5.4) hold (note that the asymptotics of  $m(\zeta)$  and  $m_{s+1}(\zeta)$  as  $\zeta \rightarrow \infty$  are clearly the same). Now if  $0 \leq x \leq s - 1$ , then since  $(I + (\zeta - \pi_s)^{-1}A)$  is analytic near  $\pi_x$ , it is clear that the residue condition at  $\pi_x$  for  $m_s(\zeta)$  implies one for  $m(\zeta)$ . Thus, we only need to consider the residue condition at the pole  $\zeta = \pi_s$ . Since  $m_s(\zeta)$  is analytic near  $\pi_s$ , we have  $\text{Res}_{\zeta=\pi_s} m(\zeta) = A \cdot m_s(\pi_s)$ . On the other hand, the limit

$$\lim_{\zeta \rightarrow \pi_s} m(\zeta)w(\pi_s) = \lim_{\zeta \rightarrow \pi_s} (I + (\zeta - \pi_s)^{-1}A)m_s(\zeta)w(\pi_s)$$

exists if and only if (5.3) holds. Moreover, if (5.3) holds, this limit equals  $m_s(\pi_s)w(\pi_s) + A \cdot m'_s(\pi_s)w(\pi_s)$ , so  $m(\zeta)$  satisfies the required residue condition at  $\zeta = \pi_s$  if and only if (5.4) holds.  $\square$

**Theorem 5.2 (Compatibility conditions for Lax pairs).** Fix  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N - 1$ .

(a) We have

$$\left( I + \frac{A_s}{\sigma\zeta - \pi_s} \right) M_s(\zeta) = M_{s+1}(\zeta) \left( I + \frac{A_{s+1}}{\zeta - \pi_{s+1}} \right). \tag{5.5}$$

(b) Conversely, assume that  $M : \mathbb{C} \rightarrow \text{Mat}(2, \mathbb{C})$  is an analytic function,  $A \in \text{Mat}(2, \mathbb{C})$  is a nilpotent matrix, and

$$\left( I + \frac{A_s}{\sigma\zeta - \pi_s} \right) M_s(\zeta) = M(\zeta) \left( I + \frac{A}{\zeta - \pi_{s+1}} \right), \tag{5.6}$$

where  $M_s(\zeta)$  and  $A_s$  are defined by (5.1), (5.2). Then  $M(\zeta) = M_{s+1}(\zeta)$  and  $A = A_{s+1}$ .

Equation (5.5) is the compatibility condition for the Lax pair (5.1), (5.2).

*Remark 5.3.* This theorem provides a recipe for computing  $A_{s+1}$  and  $M_{s+1}(\zeta)$  if we know  $A_s$  and  $M_s(\zeta)$ . Indeed, one simply needs to find the unique solution of the compatibility condition which satisfies  $A_{s+1}^2 = 0$ .

*Proof.* (a) If we replace  $\zeta$  by  $\sigma\zeta$  in (5.1) and then substitute (5.2) into the result, we obtain

$$m_{s+1}(\sigma\zeta) = \left( I + \frac{A_s}{\sigma\zeta - \pi_s} \right) M_s(\zeta) m_{s+1}(\zeta) D(\zeta)^{-1}. \tag{5.7}$$

On the other hand, if we substitute (5.1) into (5.2) and then replace  $s$  by  $s + 1$ , we get

$$m_{s+1}(\sigma\zeta) = M_{s+1}(\zeta) \left( I + \frac{A_{s+1}}{\zeta - \pi_{s+1}} \right) m_{s+1}(\zeta) D(\zeta)^{-1}. \tag{5.8}$$

Comparing (5.7) and (5.8) yields (5.5).

(b) From (5.5) and (5.6), we find that

$$M(\zeta) \left( I + \frac{A}{\zeta - \pi_{s+1}} \right) = M_{s+1}(\zeta) \left( I + \frac{A_{s+1}}{\zeta - \pi_{s+1}} \right). \tag{5.9}$$

Using Lemma 2.1 and (5.2) with  $s$  replaced by  $s + 1$ , we see that  $\det M_{s+1}(\zeta) \equiv \det D(\zeta)$ , and hence  $M_{s+1}(\zeta)$  is invertible near  $\pi_{s+1}$ . Now since  $A^2 = 0$ , we can rewrite (5.9) as

$$M_{s+1}^{-1}(\zeta) \cdot M(\zeta) = \left( I + \frac{A_{s+1}}{\zeta - \pi_{s+1}} \right) \cdot \left( I - \frac{A}{\zeta - \pi_{s+1}} \right). \tag{5.10}$$

The RHS is analytic for  $\zeta \neq \pi_{s+1}$ , and the LHS is analytic near  $\pi_{s+1}$ . Thus, both sides of (5.10) are entire functions. But the RHS tends to  $I$  as  $\zeta \rightarrow \infty$ , so by Liouville’s theorem, both sides are equal to  $I$  for all  $\zeta$ . This proves that  $M(\zeta) = M_{s+1}(\zeta)$  and  $A = A_{s+1}$ .  $\square$

5.2. Recall that the formulas of Theorem 4.2 have been derived under the assumption that the parameter  $p_s$  does not vanish for all  $s \geq k$ . Let us now establish the non-vanishing of  $p_s$  for the weight functions  $\omega$  and orthogonality sets  $\mathfrak{X}$  satisfying the assumptions of Subject. 1.2. We need the following well-known fact. For the reader's convenience, we also provide a proof.

**Lemma 5.4 (Zeroes of discrete orthogonal polynomials).** *Let  $\mathfrak{Z} \subset \mathbb{C}$  be a finite subset, and assume that  $\mathfrak{Z}$  is contained in a closed interval  $[a, b] \subseteq \mathbb{R} \subseteq \mathbb{C}$  such that  $a$  is the minimal element of  $\mathfrak{Z}$  and  $b$  is the maximal element of  $\mathfrak{Z}$ . Let  $\omega : \mathfrak{Z} \rightarrow \mathbb{R}$  be a strictly positive weight function. Then there exists a unique family  $\{P_n(\zeta)\}_{n=0}^L$  of orthogonal polynomials corresponding to  $\omega$ , where  $L = \text{card}(\mathfrak{Z}) - 1$ . The coefficients of each  $P_n(\zeta)$  are real. Moreover, for any  $0 \leq n \leq L$ , all zeroes of  $P_n(\zeta)$  are real and are contained in the open interval  $(a, b)$ .*

*Proof.* Since  $\omega$  is strictly positive, the restriction of the corresponding inner product  $(\cdot, \cdot)_\omega$  to the space  $\mathbb{R}[\zeta]^{\leq d}$  of real polynomials of degree at most  $d$  is nondegenerate for each  $0 \leq d \leq L$ . Thus, there exists a unique family of real orthogonal polynomials  $\{P_n(\zeta)\}_{n=0}^L$  corresponding to  $\omega$ , and  $(P_n, P_n)_\omega \neq 0$  for  $0 \leq n \leq L$ . Now fix  $1 \leq n \leq L$ , and assume that  $P_n(\zeta)$  has fewer than  $n$  zeroes in the open interval  $(a, b)$ . Let  $z_1, \dots, z_m$  be the zeroes of  $P_n(\zeta)$  in  $(a, b)$ , listed with their multiplicities, and let  $Q(\zeta) = (\zeta - z_1) \cdots (\zeta - z_m)$ . Then, since  $P_n(\zeta)$  and  $Q(\zeta)$  are real polynomials, we have either  $P_n(\zeta)Q(\zeta) \geq 0$  for all  $\zeta \in [a, b]$ , or  $P_n(\zeta)Q(\zeta) \leq 0$  for all  $\zeta \in [a, b]$ . In addition, since the degree of  $P_n(\zeta)$  is less than the cardinality of  $\mathfrak{Z}$ , there exists  $z \in \mathfrak{Z}$  such that  $P_n(z)Q(z) \neq 0$ . This implies that  $P_n(\zeta)$  and  $Q(\zeta)$  are not orthogonal with respect to  $\omega$ . Since the degree of  $Q(\zeta)$  is less than that of  $P_n(\zeta)$ , we have a contradiction with the definition of orthogonal polynomials.  $\square$

Now we can prove

**Proposition 5.5.** *With the notation and conventions of Sects. 3 and 4, assume that either  $\pi_0 < \pi_1 < \pi_2 \cdots$ , or  $\pi_0 > \pi_1 > \pi_2 > \cdots$ . Then  $p_s \neq 0$  for all  $s > k, s \leq N$ .*

*Proof.* Fix  $s > k, s \leq N$ . By Lemma 5.4, there exists a family  $\{P_n(\zeta)\}_{n=0}^{s-1}$  of polynomials orthogonal on  $\{\pi_0, \dots, \pi_{s-1}\}$  with the weight function given by the restriction of  $\omega$  to  $\{0, 1, \dots, s-1\}$ . Moreover, as  $\pi_s$  lies outside the interval between  $\pi_0$  and  $\pi_{s-1}$ , we have  $P_n(\pi_s) \neq 0$  for all  $0 \leq n \leq s-1$ . Now since  $k \leq s-1$ , we know from Theorem 2.4 that the first column of the matrix  $m_s(\zeta)$  has the form  $(P_k(\zeta), c \cdot P_{k-1}(\zeta))^t$ , where  $c$  is a nonzero constant. On the other hand, by Lemma 5.1, we have

$$m_s(\pi_s)w(\pi_s) = A_s \cdot (m_s(\pi_s) - m'_s(\pi_s)w(\pi_s)). \tag{5.11}$$

If  $p_s = 0$ , then because the matrix  $A_s$  is nilpotent by Theorem 3.1(a), we have either  $q_s = 0$  or  $r_s = 0$ , i.e., either the first or the second row of  $A_s$  is zero. By (5.11), this implies that either the  $(1, 2)$  or the  $(2, 2)$  element of the matrix  $m_s(\pi_s)w(\pi_s)$  is zero. This contradicts  $P_k(\pi_s), P_{k-1}(\pi_s) \neq 0$ .  $\square$

### 6. Initial Conditions for Recurrence Relations

6.1. In this section we derive the initial conditions for the recurrence relations (4.7) and (5.5). We keep the general notation and conventions of Sects. 4 and 5. Recall in particular that  $k$  is a natural number that controls the asymptotics at infinity of the solutions of all DRHPs that we consider. As in Sect. 5, we do not assume that the orthogonality set  $\mathfrak{X}$  is locally finite.

**Proposition 6.1.** *The solution  $m_k(\zeta)$  of the DRHP  $(\{\pi_0, \dots, \pi_{k-1}\}, w|_{\{\pi_0, \dots, \pi_{k-1}\}})$  with the asymptotics  $m_k(\zeta) \sim \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix}$  as  $\zeta \rightarrow \infty$  is given by*

$$m_k(\zeta) = \begin{pmatrix} (\zeta - \pi_0)(\zeta - \pi_1) \cdots (\zeta - \pi_{k-1}) & 0 \\ (\zeta - \pi_0)(\zeta - \pi_1) \cdots (\zeta - \pi_{k-1}) \sum_{m=0}^{k-1} \frac{\rho_m}{\zeta - \pi_m} (\zeta - \pi_0)^{-1} (\zeta - \pi_1)^{-1} \cdots (\zeta - \pi_{k-1})^{-1} \end{pmatrix}, \tag{6.1}$$

where

$$\rho_m = \omega(m)^{-1} \cdot \prod_{\substack{0 \leq j \leq k-1 \\ j \neq m}} (\pi_m - \pi_j)^{-2} \tag{6.2}$$

for all  $0 \leq m \leq k - 1$ .

*Proof.* Let  $m(\zeta)$  be the matrix defined by the RHS of (6.1). It is clear that  $m(\zeta)$  has the required asymptotics at infinity. Hence we only have to show that (6.2) is the (unique) choice of constants  $\rho_m$  which makes  $m(\zeta)$  satisfy the required residue conditions. Now if  $0 \leq x \leq k - 1$ , then the (2, 2) element of the matrix  $\text{Res}_{\zeta=\pi_x} m(\zeta)$  equals

$$\prod_{\substack{0 \leq l \leq k-1 \\ l \neq x}} (\pi_x - \pi_l)^{-1},$$

and the (2, 2) element of the matrix  $\lim_{\zeta \rightarrow \pi_x} m(\zeta)w(\pi_x)$  equals

$$\omega(x) \cdot \rho_x \cdot \prod_{\substack{0 \leq l \leq k-1 \\ l \neq x}} (\pi_x - \pi_l).$$

The other elements of both matrices are zero. Equating the last two expressions yields (6.2).  $\square$

6.2. Now we use Lemma 5.1 to find the matrix  $A_k$ .

**Proposition 6.2.** *The elements of the matrix*

$$A_k = \begin{pmatrix} p_k & q_k \\ r_k & -p_k \end{pmatrix}$$

are given by the following formulas:

$$q_k = \left\{ \rho_k + \sum_{m=0}^{k-1} \frac{\rho_m}{(\pi_k - \pi_m)^2} \right\}^{-1} = \left\{ \sum_{m=0}^k \left[ \omega(m)^{-1} \prod_{\substack{0 \leq j \leq k \\ j \neq m}} (\pi_m - \pi_j)^{-2} \right] \right\}^{-1}, \tag{6.3}$$

where

$$\rho_k = \omega(k)^{-1} \cdot \prod_{j=0}^{k-1} (\pi_k - \pi_j)^{-2};$$

$$p_k = -q_k \cdot \sum_{m=0}^{k-1} \frac{\rho_m}{\pi_k - \pi_m}, \tag{6.4}$$

and

$$r_k = -q_k \cdot \left\{ \sum_{m=0}^{k-1} \frac{\rho_m}{\pi_k - \pi_m} \right\}^2. \tag{6.5}$$

*Remark 6.3.* It follows from (6.2), (6.3) and (6.4) that if the orthogonality set  $\mathfrak{X}$  is contained in  $\mathbb{R}$  and either  $\pi_0 > \pi_1 > \pi_2 > \dots$  or  $\pi_0 < \pi_1 < \pi_2 < \dots$  (and the weight function  $\omega$  is strictly positive), then  $\rho_m > 0$  for  $0 \leq m \leq k - 1$ ,  $q_k > 0$ , and hence  $p_k \neq 0$ .

*Proof.* It follows from Lemma 5.1 that  $A_k$  is the unique matrix satisfying the following system of equations:

$$A_k \cdot m_k(\pi_k)w(\pi_k) = 0, \tag{6.6}$$

$$m_k(\pi_k)w(\pi_k) + A_k \cdot m'_k(\pi_k)w(\pi_k) = A_k \cdot m_k(\pi_k). \tag{6.7}$$

Substituting (6.1) into (6.6) yields (6.4). It remains to prove (6.3), for (6.5) then follows from  $p_k^2 = -q_k r_k$ . To this end, we consider the (1, 2) elements of both sides of (6.7). The first summand on the LHS contributes  $\omega(k) \cdot \prod_{j=0}^{k-1} (\pi_k - \pi_j)$  to the (1, 2) element. Now we consider the second summand. We can rewrite it as

$$\begin{aligned} & \frac{d}{d\zeta} \Big|_{\zeta=\pi_k} (A_k \cdot m_k(\zeta)w(\pi_k)) \\ &= \frac{d}{d\zeta} \Big|_{\zeta=\pi_k} \left\{ (\zeta - \pi_0) \cdots (\zeta - \pi_{k-1}) \cdot \omega(k) \cdot A_k \cdot \left( 0 \sum_{m=0}^{k-1} \frac{1}{\zeta - \pi_m} \right) \right\}. \end{aligned}$$

If the derivative falls onto the factor  $(\zeta - \pi_0) \cdots (\zeta - \pi_{k-1})$ , the corresponding term is zero because of (6.6). Hence the whole expression equals

$$\begin{aligned} & (\pi_k - \pi_0) \cdots (\pi_k - \pi_{k-1}) \cdot A_k \cdot \omega(k) \cdot \left( 0 \sum_{m=0}^{k-1} \frac{1}{(\pi_k - \pi_m)^2} \right) \\ &= \omega(k) \cdot \prod_{j=0}^{k-1} (\pi_k - \pi_j) \cdot \left( 0 \sum_{m=0}^{k-1} \frac{1}{(\pi_k - \pi_m)^2} \right). \end{aligned}$$

Finally, the (1, 2) element of  $A_k \cdot m_k(\pi_k)$  equals  $q_k \cdot \prod_{j=0}^{k-1} (\pi_k - \pi_j)^{-1}$ . Thus, comparing the (1, 2) elements of both sides of (6.7) yields

$$\left\{ \left[ \prod_{j=0}^{k-1} (\pi_k - \pi_j) \right] \cdot \omega(k) \cdot \sum_{m=0}^{k-1} \frac{\rho_m}{(\pi_k - \pi_m)^2} + \prod_{j=0}^{k-1} (\pi_k - \pi_j)^{-1} \right\} \cdot q_k$$

$$= \omega(k) \cdot \prod_{j=0}^{k-1} (\pi_k - \pi_j),$$

which gives (6.3).  $\square$

*Remark 6.4.* The two propositions we have just proved give explicit formulas for the matrices  $m_k(\zeta)$  and  $A_k$ . Using these formulas, we can also find the functions  $m_{k+1}(\zeta)$  and  $M_k(\zeta)$ . Indeed, from the Lax pair (3.3) and (3.4), we have

$$m_{k+1}(\zeta) = \left( I + \frac{A_k}{\zeta - \pi_k} \right) m_k(\zeta), \tag{6.8}$$

$$M_k(\zeta) = m_k(\sigma\zeta)D(\zeta)m_{k+1}^{-1}(\zeta) = m_k(\sigma\zeta)D(\zeta)m_k^{-1}(\zeta) \left( I - \frac{A_k}{\zeta - \pi_k} \right). \tag{6.9}$$

Even though this gives an explicit formula for  $M_k(\zeta)$ , it is cumbersome to use it in practice. In the case where the matrix  $D(\zeta)$  is linear in  $\zeta$ , a more explicit version is available:

**Proposition 6.5.** *Assume that  $d_1(\zeta) = \lambda_1\zeta + \mu_1$ ,  $d_2(\zeta) = \lambda_2\zeta + \mu_2$ , where  $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$  are constants (some of which could be zero). Then*

$$M_k(\zeta) = \begin{pmatrix} \eta^k(\lambda_1\zeta + \mu_1) + \eta^k\lambda_1(\pi_0 - \pi_k - p_k) & -\eta^k\lambda_1q_k \\ -\eta^{-k}\lambda_2r_k + (\eta^{k-1}\lambda_1 - \eta^{-k}\lambda_2)\sum_{m=0}^{k-1}\rho_m\eta^{-k}(\lambda_2\zeta + \mu_2) + \eta^{-k}\lambda_2(p_k + \pi_k - \pi_0) & \end{pmatrix}, \tag{6.10}$$

where  $p_k, q_k, r_k, \rho_m$  are given by (6.4), (6.3), (6.5) and (6.2).

*Proof.* Since  $M_k(\zeta)$  is an entire function, it suffices by Liouville’s theorem to show that (6.10) holds up to terms of order  $\zeta^{-1}$  as  $\zeta \rightarrow \infty$ . To that end, note that by (6.1) and (3.2), we have

$$m_k(\sigma\zeta) = \begin{pmatrix} \eta^k(\zeta - \pi_1) \cdots (\zeta - \pi_k) & 0 \\ \eta^{k-1}(\zeta - \pi_1) \cdots (\zeta - \pi_k) \sum_{m=0}^{k-1} \frac{\rho_m}{\zeta - \pi_{m+1}} & \eta^{-k}(\zeta - \pi_1)^{-1} \cdots (\zeta - \pi_k)^{-1} \end{pmatrix}. \tag{6.11}$$

But it follows from (6.9) that

$$M_k(\zeta) = \left\{ m_k(\sigma\zeta) \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \right\} \cdot D(\zeta) \cdot \left\{ m_k(\zeta) \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \right\}^{-1} \cdot \left( I - \frac{A_k}{\zeta - \pi_k} \right).$$

Substituting (6.1) and (6.11) into the last formula, we arrive at (6.10).  $\square$

6.3. The final result of this section is the computation of the Fredholm determinants  $D_k$  and  $D_{k+1}$ . We use the probability-theoretic interpretation of the Fredholm determinants  $D_s$  given in subsect. 4.5. One can show (this is a standard random matrix theory argument, see e.g. [14]) that the constant  $Z$  given by (4.16) is equal to the product of the norms squared of the first  $k$  monic orthogonal polynomials:

$$Z = \prod_{i=0}^{k-1} (P_i, P_i)_\omega. \tag{6.12}$$

Now we prove

**Proposition 6.6.** *With the notation of Sect. 4, let  $\mathfrak{X} \subset \mathbb{R}$  be a discrete set, let  $\{P_n(\zeta)\}$  be the family of orthogonal polynomials corresponding to a strictly positive weight function  $\omega : \mathfrak{X} \rightarrow \mathbb{R}$ , and let  $Z$  be given by (6.12). Then*

$$D_k = \frac{1}{Z} \cdot \prod_{0 \leq i < j \leq k-1} (\pi_i - \pi_j)^2 \cdot \prod_{l=0}^{k-1} \omega(l), \tag{6.13}$$

$$D_{k+1} = \omega(k) \cdot q_k^{-1} \cdot D_k \cdot \prod_{l=0}^{k-1} (\pi_k - \pi_l)^2, \tag{6.14}$$

where  $q_k$  is given by (6.3).

*Proof.* Recall that for all  $s \in \mathbb{Z}_{\geq k}, s \leq N$ , we have defined  $\mathfrak{Z}_s = \{\pi_x\}_{x=0}^{s-1}$ ,  $\mathfrak{Y}_s = \mathfrak{X} \setminus \mathfrak{Z}_s$ , and  $D_s = \det(1 - K|_{\mathfrak{Y}_s \times \mathfrak{Y}_s})$ . Hence a subset of  $\mathfrak{X}$  is disjoint from  $\mathfrak{Y}_s$  if and only if it is contained in  $\mathfrak{Z}_s$ . There exists only one subset of  $\mathfrak{Z}_k$  of cardinality  $k$ , namely,  $\mathfrak{Z}_k = \{\pi_0, \dots, \pi_{k-1}\}$  itself. Applying (4.17) yields (6.13). Next, there are  $k + 1$  subsets of  $\mathfrak{Z}_{k+1}$  of cardinality  $k$ , namely, those of the form  $\mathfrak{Z}_{k+1} \setminus \{\pi_m\}$  for  $0 \leq m \leq k$ . Applying (4.17) gives

$$D_{k+1} = \frac{1}{Z} \cdot \prod_{l=0}^k \omega(l) \cdot \left\{ \sum_{m=0}^k \left[ \frac{1}{\omega(m)} \prod_{\substack{0 \leq i < j \leq k \\ i, j \neq m}} (\pi_i - \pi_j)^2 \right] \right\}.$$

Using (6.2), (6.3) and (6.13), we see that the last equation is equivalent to (6.14).  $\square$

## 7. Lax Pairs for Discrete Orthogonal Polynomials of the Askey Scheme

7.1. In this section we specialize to weight functions appearing in the orthogonality relations for the hypergeometric orthogonal polynomials and the basic hypergeometric orthogonal polynomials of the Askey scheme. We use [13] as our main reference for the orthogonal polynomials of the Askey scheme. We are only interested in those families for which the orthogonality set is discrete. Since our ultimate goal is to derive a recurrence relation for the associated Fredholm determinants, we do not impose the local finiteness condition on  $\mathfrak{X}$ . However, the basic assumptions of Subsect. 1.2 have to be satisfied in order to use our approach (in its present form). These assumptions are not satisfied for the following families of discrete orthogonal polynomials: the Racah polynomials ([13],

§1.2), the dual Hahn polynomials ([13], §1.6), the  $q$ -Racah polynomials ([13], §3.2), the big  $q$ -Jacobi polynomials ([13], §3.5), the big  $q$ -Legendre polynomials ([13], §3.5.1), the dual  $q$ -Hahn polynomials ([13], §3.7), the big  $q$ -Laguerre polynomials ([13], §3.11), the dual  $q$ -Krawtchouk polynomials ([13], §3.17), the Al-Salam-Carlitz I polynomials ([13], §3.24), and the discrete  $q$ -Hermite I polynomials ([13], §3.28).

7.2. Now we list the families of hypergeometric and basic hypergeometric orthogonal polynomials for which our results do apply. Instead of writing out the whole Lax pair in each case, we only give the orthogonality set  $\mathfrak{X}$ , the weight function  $\omega(x)$ , the affine transformation  $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ , and the corresponding entire functions  $d_1(\zeta)$ ,  $d_2(\zeta)$  which satisfy the assumption of Theorem 3.1(b). For the basic hypergeometric polynomials, we assume from now on that  $0 < q < 1$ . This restriction ensures that  $\mathfrak{X} \subset \mathbb{R}$  and the weight function is strictly positive and has finite moments.

- Hahn polynomials ([13], §1.5):  $\mathfrak{X} = \{0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \binom{\alpha + x}{x} \binom{\beta + N - x}{N - x}, \quad \text{where } \alpha, \beta > -1 \text{ or } \alpha, \beta < -N;$$

$$\sigma\zeta = \zeta - 1, \quad d_1(\zeta) = \zeta(\zeta - \beta - N - 1), \quad d_2(\zeta) = (\zeta - N - 1)(\zeta + \alpha).$$

- Meixner polynomials ([13], §1.9):  $\mathfrak{X} = \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{(\beta)_x}{x!} c^x, \quad \text{where } \beta > 0 \text{ and } 0 < c < 1;$$

$$\sigma\zeta = \zeta - 1, \quad d_1(\zeta) = \zeta, \quad d_2(\zeta) = c\zeta + c(\beta - 1).$$

- Krawtchouk polynomials ([13], §1.10):  $\mathfrak{X} = \{0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \binom{N}{x} p^x (1 - p)^{N-x}, \quad \text{where } 0 < p < 1;$$

$$\sigma\zeta = \zeta - 1, \quad d_1(\zeta) = \zeta, \quad d_2(\zeta) = \frac{p}{p-1}(\zeta - N - 1).$$

- Charlier polynomials ([13], §1.12):  $\mathfrak{X} = \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{a^x}{x!}, \quad \text{where } a > 0;$$

$$\sigma\zeta = \zeta - 1, \quad d_1(\zeta) = \zeta, \quad d_2(\zeta) = a.$$

- $q$ -Hahn polynomials ([13], §3.6):  $\mathfrak{X} = \{q^{-x} \mid x = 0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{(\alpha q; q)_x (q^{-N}; q)_x}{(q; q)_x (\beta^{-1} q^{-N}; q)_x} (\alpha\beta q)^{-x}, \quad \text{where } 0 < \alpha, \beta < q^{-1} \text{ or } \alpha, \beta > q^{-N};$$

$$\sigma\zeta = q\zeta, \quad d_1(\zeta) = \alpha\beta(\zeta - 1)(\zeta - \beta^{-1}q^{-N-1}), \quad d_2(\zeta) = (\zeta - \alpha)(\zeta - q^{-N-1}).$$

- Little  $q$ -Jacobi polynomials ([13], §3.12):  $\mathfrak{X} = \{q^x | x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{(bq; q)_x}{(q; q)_x} (aq)^x, \quad \text{where } 0 < a < q^{-1} \text{ and } b < q^{-1};$$

$$\sigma\zeta = q^{-1}\zeta, \quad d_1(\zeta) = \zeta - 1, \quad d_2(\zeta) = a(b\zeta - 1).$$

- $q$ -Meixner polynomials ([13], §3.13):  $\mathfrak{X} = \{q^{-x} | x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{(bq; q)_x}{(q; q)_x (-bcq; q)_x} c^x q^{\binom{x}{2}}, \quad \text{where } 0 < b < q^{-1} \text{ and } c > 0;$$

$$\sigma\zeta = q\zeta, \quad d_1(\zeta) = (\zeta - 1)(\zeta + bc), \quad d_2(\zeta) = c(\zeta - b).$$

- Quantum  $q$ -Krawtchouk polynomials ([13], §3.14):  $\mathfrak{X} = \{q^{-x} | x = 0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{(pq; q)_{N-x}}{(q; q)_x (q; q)_{N-x}} (-1)^{N-x} q^{\binom{x}{2}}, \quad \text{where } p > q^{-N};$$

$$\sigma\zeta = q\zeta, \quad d_1(\zeta) = (\zeta - 1)(pq^{N+1}\zeta - 1), \quad d_2(\zeta) = 1 - q^{N+1}\zeta.$$

- $q$ -Krawtchouk polynomials ([13], §3.15):  $\mathfrak{X} = \{q^{-x} | x = 0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{(q^{-N}; q)_x}{(q; q)_x} (-p)^{-x}, \quad \text{where } p > 0;$$

$$\sigma\zeta = q\zeta, \quad d_1(\zeta) = p(\zeta - 1), \quad d_2(\zeta) = q^{-N} - q\zeta.$$

- Affine  $q$ -Krawtchouk polynomials ([13], §3.16):  $\mathfrak{X} = \{q^{-x} | x = 0, \dots, N\}$ ,  $N \in \mathbb{Z}_{\geq 0}$ ;

$$\omega(x) = \frac{(pq; q)_x (q; q)_N}{(q; q)_x (q; q)_{N-x}} (pq)^{-x}, \quad \text{where } 0 < p < q^{-1};$$

$$\sigma\zeta = q\zeta, \quad d_1(\zeta) = p(\zeta - 1), \quad d_2(\zeta) = (\zeta - p)(q^{N+1}\zeta - 1).$$

- Little  $q$ -Laguerre/Wall polynomials ([13], §3.20):  $\mathfrak{X} = \{q^x | x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{(aq)^x}{(q; q)_x}, \quad \text{where } 0 < a < q^{-1};$$

$$\sigma\zeta = q^{-1}\zeta, \quad d_1(\zeta) = \zeta - 1, \quad d_2(\zeta) = -a.$$

- Alternative  $q$ -Charlier polynomials ([13], §3.22):  $\mathfrak{X} = \{q^x \mid x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{a^x}{(q; q)_x} q^{\binom{x+1}{2}}, \quad \text{where } a > 0;$$

$$\sigma \zeta = q^{-1} \zeta, \quad d_1(\zeta) = \zeta - 1, \quad d_2(\zeta) = -\frac{a}{q} \zeta.$$

- $q$ -Charlier polynomials ([13], §3.23):  $\mathfrak{X} = \{q^{-x} \mid x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{a^x}{(q; q)_x} q^{\binom{x}{2}}, \quad \text{where } a > 0;$$

$$\sigma \zeta = q \zeta, \quad d_1(\zeta) = \zeta - 1, \quad d_2(\zeta) = a.$$

- Al-Salam-Carlitz II polynomials ([13], §3.25):  $\mathfrak{X} = \{q^{-x} \mid x \in \mathbb{Z}_{\geq 0}\}$ ;

$$\omega(x) = \frac{q^{x^2} a^x}{(q; q)_x (aq; q)_x}, \quad \text{where } a > 0;$$

$$\sigma \zeta = q \zeta, \quad d_1(\zeta) = (\zeta - 1)(\zeta - a), \quad d_2(\zeta) = a.$$

7.3. The next three sections (Sects. 8–10) deal with the various possible ways of “solving” the compatibility conditions for the Lax pairs listed above. By a “solution” of a compatibility condition of the form (5.5) we mean a collection of formulas which allow us to express the entries of the matrices  $M_{s+1}^{(i)}, A_{s+1}$  as rational functions of the entries of the matrices  $M_s^{(i)}, A_s$ , where  $M_s(\zeta) = M_s^{(l)} \zeta^l + \dots + M_s^{(0)}, M_s^{(i)} \in \text{Mat}(2, \mathbb{C})$ , for all  $s$ . The most general case where we have been able to solve the compatibility condition explicitly is the one where the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  are either linear or constant; this is described in Sect. 8. The resulting formulas can be used for practical computations, but they do not appear to be related to any known systems of difference equations. In certain more specialized cases we have been able to reduce the compatibility condition to one of the equations of H. Sakai’s hierarchy in [17].

7.4. In Sect. 9 we solve the compatibility condition (5.5) in the case where the orthogonality set has the form  $\mathfrak{X} = \{x \in \mathbb{Z}_{\geq 0} \mid x \leq N\}$  ( $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ ) and the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  are linear, by a method different from the one used in Sect. 8. We show that if both  $d_1$  and  $d_2$  are nonconstant, then the compatibility condition is (generally) equivalent to the  $d - P_V$  equation of H. Sakai [17], and if  $d_2$  is constant, then the compatibility condition is (generally) equivalent to the  $d - P_{IV}$  equation *ibid.* (see Theorem 9.3). This result allows us to write down explicit solutions for the recurrence relations corresponding to the Meixner polynomials, the Krawtchouk polynomials, and the Charlier polynomials — see Sect. 11.

7.5. As for the basic hypergeometric orthogonal polynomials, we show in Sect. 10 (see Theorem 10.1(b)) that in the case where the orthogonality set has the form  $\mathfrak{X} = \{q^{-s}\}_{s=0}^N$  ( $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ ) and the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  are linear and nonconstant, the compatibility condition for the corresponding Lax pair is equivalent to the  $q - P_{VI}$  system of M. Jimbo and H. Sakai, for a certain choice of the parameters. Since the case where  $\mathfrak{X} = \{q^s \mid s \in \mathbb{Z}_{\geq 0}\}$  is reduced to the former one after replacing  $q$  by  $q^{-1}$ , this situation suits the following families of basic hypergeometric orthogonal polynomials: the little  $q$ -Jacobi polynomials and the  $q$ -Krawtchouk polynomials (it does *not* suit the alternative  $q$ -Charlier polynomials because the constant terms of the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  must be nonzero, cf. Theorem 10.3). If one of the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  is linear and the other one is constant, it is possible to reduce the corresponding compatibility condition to a degeneration of the  $q - P_{VI}$  system. This process is described in Subsect. 10.3. It allows us to solve the compatibility conditions for the little  $q$ -Laguerre/Wall polynomials and the  $q$ -Charlier polynomials.

It turns out, however, that the method of solving the compatibility condition by reducing it to the  $q - P_{VI}$  system (or its degeneration) is rather difficult to carry out in practice. So to find a recurrence relation for the Fredholm determinants associated to classical families of basic hypergeometric orthogonal polynomials, we prefer to use the more general formulas of Sect. 8 (see Sect. 11). The disadvantage of the formulas of Sect. 8, as compared to  $q - P_{VI}$ , is the fact that the recurrence step substantially involves more than two sequences, while for  $q - P_{VI}$  two sequences suffice, cf. Theorems 8.2 and 10.3 below.

### 8. Solution of the Compatibility Condition: The General Case

8.1. In this section we “solve” (in the sense of Subsect. 7.3) the compatibility condition

$$\left(I + \frac{\eta^{-1}A_s}{\zeta - \pi_{s+1}}\right) M_s(\zeta) = M_{s+1}(\zeta) \left(I + \frac{A_{s+1}}{\zeta - \pi_{s+1}}\right) \tag{8.1}$$

derived in Sect. 5 (cf. Eq. (5.5)), in the case where the matrix  $D(\zeta)$  defined in Theorem 3.1(b) depends linearly on  $\zeta$ . Our method is based on the following simple observation.

**Proposition 8.1.** *If  $M_s(\zeta) = \Lambda \cdot \zeta + C_s$  for all  $s$ , where  $C_s$  does not depend on  $\zeta$  and  $\Lambda$  is a fixed matrix independent both of  $\zeta$  and of  $s$ , then under the assumption that  $A_{s+1}^2 = 0$ , the compatibility condition (8.1) is equivalent to the following system of linear equations:*

$$(\pi_{s+1}\Lambda + C_s + \eta^{-1}A_s\Lambda) \cdot A_{s+1} = \eta^{-1}A_s \cdot (\pi_{s+1}\Lambda + C_s), \tag{8.2}$$

$$C_{s+1} = C_s + \eta^{-1}A_s\Lambda - \Lambda A_{s+1}. \tag{8.3}$$

*Proof.* Comparing the asymptotics of both sides of (8.1) as  $\zeta \rightarrow \infty$  yields (8.3). If we take the residues of both sides of (8.1) at  $\zeta = \pi_{s+1}$ , we obtain

$$\eta^{-1}A_s \cdot (\pi_{s+1}\Lambda + C_s) = (\pi_{s+1}\Lambda + C_{s+1}) \cdot A_{s+1}.$$

Substituting (8.3) into the last equation and using the assumption that  $A_{s+1}^2 = 0$  gives (8.2).  $\square$

8.2. We now note that if the matrix  $\pi_{s+1}\Lambda + C_s + \eta^{-1}A_s\Lambda$  is invertible, then the system (8.2), (8.3) already has a unique solution  $(A_{s+1}, C_{s+1})$ . Since the compatibility condition (8.1) always has a unique solution by Theorem 5.2(b), it follows from Proposition 8.1 that in this case, the solution of (8.2), (8.3) is also the solution of (8.1). Even though we cannot prove that the matrix  $\pi_{s+1}\Lambda + C_s + \eta^{-1}A_s\Lambda$  is invertible in general, the explicit computations we have carried out for five families of basic hypergeometric orthogonal polynomials (see Sect. 11) show that the result below (Theorem 8.2) has practical significance.

8.3. We introduce the following notation. Suppose that  $d_1(\zeta) = \lambda_1\zeta + \mu_1$ ,  $d_2(\zeta) = \lambda_2\zeta + \mu_2$ , where  $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$  are constants. Then it follows from Theorem 3.1(c) that the assumption of Proposition 8.1 is satisfied for  $\Lambda = \text{diag}(\kappa_1, \kappa_2)$ , where  $\kappa_1 = \eta^k \lambda_1$  and  $\kappa_2 = \eta^{-k} \lambda_2$ . Let us write

$$A_s = \begin{pmatrix} p_s & q_s \\ r_s & -p_s \end{pmatrix} \quad \text{and} \quad C_s = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix}.$$

Finally, define  $\epsilon_s = \det(\pi_{s+1}\Lambda + C_s + \eta^{-1}A_s\Lambda)$ . Now we can state

**Theorem 8.2.** *We have*

$$\epsilon_s = d_1(\pi_{s+1})d_2(\pi_{s+1}) + \eta^{-1}\kappa_1(p_s\delta_s - r_s\beta_s) - \eta^{-1}\kappa_2(p_s\alpha_s + q_s\gamma_s). \tag{8.4}$$

If  $\epsilon_s \neq 0$ , then the following formulas hold:

$$p_{s+1} = -\eta^{-1}p_s^{-1}\epsilon_s^{-1} \cdot (p_s\beta_s + q_s\delta_s + \kappa_2\pi_{s+1}q_s) \cdot (r_s\alpha_s - p_s\gamma_s + \kappa_1\pi_{s+1}r_s), \tag{8.5}$$

$$q_{s+1} = \eta^{-1}q_s^{-1}\epsilon_s^{-1} \cdot (p_s\beta_s + q_s\delta_s + \kappa_2\pi_{s+1}q_s)^2, \tag{8.6}$$

$$r_{s+1} = \eta^{-1}r_s^{-1}\epsilon_s^{-1} \cdot (r_s\alpha_s - p_s\gamma_s + \kappa_1\pi_{s+1}r_s)^2, \tag{8.7}$$

$$\alpha_{s+1} = \alpha_s + \eta^{-1}\kappa_1p_s - \kappa_1p_{s+1}, \tag{8.8}$$

$$\beta_{s+1} = \beta_s + \eta^{-1}\kappa_2q_s - \kappa_1q_{s+1}, \tag{8.9}$$

$$\gamma_{s+1} = \gamma_s + \eta^{-1}\kappa_1r_s - \kappa_2r_{s+1}, \tag{8.10}$$

$$\delta_{s+1} = \delta_s - \eta^{-1}\kappa_2p_s + \kappa_2p_{s+1}, \tag{8.11}$$

$$u_s = -\kappa_2\gamma_s + \frac{p_s}{q_s} \cdot (\kappa_1\delta_s - \kappa_2\alpha_s) + \frac{p_s^2}{q_s^2} \cdot \kappa_1\beta_s, \tag{8.12}$$

where  $u_s$  is defined in Theorem 4.2(b).

We omit the proof, as it consists entirely of straightforward computations. We first derive (8.4) using the identity  $\det M_s(\zeta) \equiv \det D(\zeta)$  which follows from Lemma 2.1. If  $\epsilon_s \neq 0$ , we rewrite (8.2) as

$$A_{s+1} = \eta^{-1} \cdot (\pi_{s+1}\Lambda + C_s + \eta^{-1}A_s\Lambda)^{-1} \cdot A_s \cdot (\pi_{s+1}\Lambda + C_s).$$

Writing out the matrix product on the RHS explicitly yields (8.5)–(8.7). Then (8.3) gives (8.8)–(8.11). Finally, (8.12) follows immediately from the definition of  $u_s$ .

### 9. The Fifth and the Fourth Discrete Painlevé Equations

9.1. In this section we assume that the orthogonality set is of the form  $\mathfrak{X} = \{x \in \mathbb{Z}_{\geq 0} \mid x \leq N\}$  ( $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ ), so that, with the notation of Sect. 3,  $\sigma\zeta = \zeta - 1$  and  $\eta = 1$ . Our goal is to prove that if the functions  $d_1(\zeta)$  and  $d_2(\zeta)$  are linear, then the compatibility condition (5.5) is equivalent to either the fifth or the fourth discrete Painlevé equation of H. Sakai [17].

Recall from Theorem 3.1 that since  $\pi_0 = 0$ , we must have  $d_1(0) = 0$ , and because only the ratio  $d_1(\zeta)/d_2(\zeta)$  matters, we may assume without loss of generality that  $d_1(\zeta) = \zeta$ , and write  $d_2(\zeta) = \xi\zeta + \tau$  for some  $\xi, \tau \in \mathbb{C}$  (unless otherwise explicitly stated, we do not exclude the possibility  $\xi = 0$ ). By Theorem 3.1(c), we can write

$$M_s(\zeta) = \Lambda\zeta + C_s, \quad \text{where } \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \text{ and } C_s = \begin{pmatrix} C_s^{11} & C_s^{12} \\ C_s^{21} & C_s^{22} \end{pmatrix}.$$

Then the compatibility condition (5.5) takes the form

$$\left( I + \frac{A_s}{\zeta - (s + 1)} \right) \cdot (\Lambda\zeta + C_s) = (\Lambda\zeta + C_{s+1}) \cdot \left( I + \frac{A_{s+1}}{\zeta - (s + 1)} \right), \quad (9.1)$$

where  $A_s = \begin{pmatrix} p_s & q_s \\ r_s & -p_s \end{pmatrix}$ . The following result allows us to find a convenient reparameterization of the matrices  $A_s$  and  $C_s$  which leads to an explicit solution of (9.1).

**Lemma 9.1.** *We have*

$$C_s^{11} + p_s = -k, \quad C_s^{22} - \xi p_s = \xi k + \tau, \quad C_s^{12} C_s^{21} = C_s^{11} C_s^{22} \quad (9.2)$$

for all  $s \in \mathbb{Z}_{\geq k}, s \leq N$ .

*Proof.* Taking the asymptotics of both sides of (9.1) as  $\zeta \rightarrow \infty$  gives

$$C_s + A_s \Lambda = C_{s+1} + \Lambda A_{s+1}, \quad (9.3)$$

which implies that

$$C_s^{11} + p_s = C_{s+1}^{11} + p_{s+1} \quad \text{and} \quad C_s^{22} - \xi p_s = C_{s+1}^{22} - \xi p_{s+1}$$

for all  $s \in \mathbb{Z}_{\geq k}, s < N$ . This means that the expressions  $C_s^{11} + p_s$  and  $C_s^{22} - \xi p_s$  do not depend on  $s$ . But we know from Proposition 6.5 that

$$C_k = \begin{pmatrix} -k - p_k & -q_k \\ -\xi r_k + (1 - \xi) \sum_{m=0}^{k-1} \rho_m \tau + \xi(p_k + k) \end{pmatrix}, \quad (9.4)$$

whence  $C_k^{11} + p_k = -k$  and  $C_k^{22} - \xi p_k = \xi k + \tau$ , proving the first two equalities in (9.2). To prove the third equality, note that  $\det M_s(\zeta) = \det D(\zeta)$  for all  $\zeta$ , by (3.4) and Lemma 2.1, and take  $\zeta = 0$ .  $\square$

9.2. It follows from Lemma 9.1 that for all  $s \in \mathbb{Z}_{\geq k}$ ,  $s \leq N$ , the matrices  $A_s$  and  $C_s$  can be naturally parameterized as follows:

$$A_s = (k + b_s) \begin{pmatrix} -1 & -\alpha_s \beta_s \\ 1/(\alpha_s \beta_s) & 1 \end{pmatrix}, \quad C_s = \begin{pmatrix} b_s & b_s \beta_s \\ (\tau - \xi b_s)/\beta_s & \tau - \xi b_s \end{pmatrix}. \tag{9.5}$$

*Remark 9.2.* This parameterization is taken from [4] (see Eq. (6.8) *ibid.*). The proof of the theorem below is based on the same idea as the proof of Proposition 6.3 in [4]. In fact, the situation considered in Sect. 6 of [4] corresponds, with minor modifications, to the weight function for Meixner polynomials (see Sect. 7 above). The only essential difference with the present paper is that we consider a slightly more general situation by letting  $d_2(\zeta)$  be an arbitrary linear function, which allows us to treat the cases of the Krawtchouk and Charlier polynomials, as well as the case of the Meixner polynomials.

Note also that the parameterization (9.5) is only valid if the matrices  $A_s$  and  $C_s$  are sufficiently generic. If, for instance,  $C_s^{12} = 0$ , but  $C_s^{11} \neq 0$ , then (9.5) does not make sense. When we try to solve (9.1) in terms of the parameterization (9.5), we will encounter a similar difficulty: the formulas will involve rational functions of the parameters, and it is not clear a priori that the denominators of all fractions do not vanish. We refer the reader to [4], §6, where this problem is discussed in detail. The argument of [4] can be easily adapted to our situation.

**Theorem 9.3.** (a) Assume that  $\xi \neq 0$ . Introduce new variables  $f_s, g_s$  by

$$f_s = -k - b_s + \frac{s}{1 - \alpha_s}, \quad g_s = -\alpha_s. \tag{9.6}$$

Then with the parameterization (9.5), the recurrence relation (9.1) has the following solution:

$$f_{s+1} + f_s = -(k + \tau/\xi) + \frac{s}{1 + g_s} + \frac{\tau/\xi + s + 1}{1 + \xi g_s}, \tag{9.7}$$

$$g_{s+1} g_s = \frac{(f_{s+1} - 1 - s)(f_{s+1} - 1 - s + k)}{\xi f_{s+1}(f_{s+1} + k + \tau/\xi)}, \tag{9.8}$$

$$\frac{\beta_{s+1}}{\beta_s} = -\frac{\xi g_s}{g_{s+1}} \cdot \frac{(1 + g_{s+1})f_{s+1} + (k + \tau/\xi)g_{s+1} - s - 1}{(1 + \xi g_s)f_{s+1} + k - s - 1}. \tag{9.9}$$

(b) Now let  $\xi = 0$ , and introduce new variables  $f_s, g_s$  by

$$f_s = \alpha_s^{-1}, \quad g_s = \tau \alpha_s + b_s + s + 1. \tag{9.10}$$

Then with the parameterization (9.5), the recurrence relation (9.1) has the following solution:

$$f_s f_{s+1} = \frac{\tau g_s}{(g_s - s - 1)(g_s + k - s - 1)}, \tag{9.11}$$

$$g_s + g_{s+1} = \frac{\tau}{f_{s+1}} - \frac{s + 1}{1 - f_{s+1}} - k + 2s + 3, \tag{9.12}$$

$$\frac{\beta_{s+1}}{\beta_s} = \frac{\tau}{f_s(g_s + k - s - 1)}. \tag{9.13}$$

*Remark 9.4.* (a) If we set  $f = f_s, \bar{f} = f_{s+1}, g = g_s, \bar{g} = g_{s+1}$ , then the relations (9.7), (9.8) form a special case of the difference Painlevé V equation ( $d - P_V$ ) of [17], §7. The parameters  $\lambda, a_0, a_1, a_2, a_3, a_4$  in our case are as follows:

$$a_0 = \tau/\xi + s + 1, \quad a_1 = s, \quad a_2 = -s,$$

$$a_3 = -(k + \tau/\xi), \quad a_4 = k, \quad \lambda = a_1 + 2a_2 + a_3 + a_4 + a_0 = 1.$$

(b) If we set  $f = f_{s+1}, \bar{f} = f_s, g = g_{s+1}, \bar{g} = g_s$ , then the relations (9.11), (9.12) form a special case of the difference Painlevé IV equation ( $d - P_{IV}$ ) of [17], §7. The parameters  $\lambda, a_0, a_1, a_2, a_3$  in our case are as follows:

$$a_0 = -s - 2, \quad a_1 = 1, \quad a_2 = k,$$

$$a_3 = s + 2 - k, \quad \lambda = a_1 + a_2 + a_3 + a_0 = 1.$$

*Proof of Theorem 9.3.* For the first part of the proof we do not need to distinguish between the cases where  $\xi \neq 0$  and  $\xi = 0$ . Taking the residues of both sides of (9.1) at  $\zeta = s + 1$  yields

$$A_s \cdot ((s + 1)\Lambda + C_s) = ((s + 1)\Lambda + C_{s+1}) \cdot A_{s+1},$$

i.e.,

$$\begin{aligned} & (k + b_s) \cdot \begin{pmatrix} -1 & -\alpha_s \beta_s \\ 1/(\alpha_s \beta_s) & 1 \end{pmatrix} \cdot \begin{pmatrix} s + 1 + b_s & b_s \beta_s \\ (\tau - \xi b_s)/\beta_s & \xi(s + 1) + \tau - \xi b_s \end{pmatrix} \\ &= (k + b_{s+1}) \cdot \begin{pmatrix} s + 1 + b_{s+1} & b_{s+1} \beta_{s+1} \\ (\tau - \xi b_{s+1})/\beta_{s+1} & \xi(s + 1) + \tau - \xi b_{s+1} \end{pmatrix} \cdot \begin{pmatrix} -1 & -\alpha_{s+1} \beta_{s+1} \\ 1/(\alpha_{s+1} \beta_{s+1}) & 1 \end{pmatrix}. \end{aligned}$$

Comparing the diagonal terms on both sides, we get a system of two equations:

$$\begin{aligned} & (k + b_s) \cdot [-(\tau - \xi b_s)\alpha_s - (s + 1) - b_s] \\ &= (k + b_{s+1}) \cdot [b_{s+1}/\alpha_{s+1} - (s + 1) - b_{s+1}], \end{aligned} \tag{9.14}$$

$$\begin{aligned} & (k + b_s) \cdot [b_s/\alpha_s + \xi(s + 1) + \tau - \xi b_s] \\ &= (k + b_{s+1}) \cdot [-(\tau - \xi b_{s+1})\alpha_{s+1} + \xi(s + 1) + \tau - \xi b_{s+1}]. \end{aligned} \tag{9.15}$$

If we multiply (9.14) by  $\alpha_{s+1}$ , (9.15) by  $\alpha_s$ , and add the results, we obtain an equation which can be written as follows:

$$\begin{aligned} & (k + b_s) \cdot [(1 - \xi\alpha_s)(1 - \alpha_{s+1})b_s + \tau\alpha_s(1 - \alpha_{s+1}) + \xi(s + 1)\alpha_s - (s + 1)\alpha_s] \\ &= (k + b_{s+1}) \cdot [(1 - \xi\alpha_s)(1 - \alpha_{s+1})b_{s+1} + \tau\alpha_s(1 - \alpha_{s+1}) + \xi(s + 1)\alpha_s - (s + 1)\alpha_s]. \end{aligned} \tag{9.16}$$

Now we subtract the LHS of the last equation from its RHS and divide the result by  $(1 - \xi\alpha_s)(1 - \alpha_{s+1})(b_{s+1} - b_s)$ ; noting that  $(k + b_{s+1})b_{s+1} - (k + b_s)b_s = (k + b_{s+1} + b_s)(b_{s+1} - b_s)$ , we get

$$k + b_s + b_{s+1} + \frac{\tau\alpha_s + s + 1}{1 - \xi\alpha_s} - \frac{s + 1}{1 - \alpha_{s+1}} = 0. \tag{9.17}$$

Let us now assume that  $\xi \neq 0$ . In this case, it is easy to see that with the notation (9.6), the last equation is equivalent to (9.7). To obtain (9.8), we divide (9.15) by (9.14), which yields

$$\frac{b_s/\alpha_s + \xi(s+1) + \tau - \xi b_s}{-(\tau - \xi b_s)\alpha_s - (s+1) - b_s} = \frac{-(\tau - \xi b_{s+1})\alpha_{s+1} + \xi(s+1) + \tau - \xi b_{s+1}}{b_{s+1}/\alpha_{s+1} - (s+1) - b_{s+1}}. \quad (9.18)$$

From (9.6) and (9.17), we have

$$\begin{aligned} & b_s/\alpha_s + \xi(s+1) + \tau - \xi b_s \\ &= \frac{1}{\alpha_s}(1 - \xi\alpha_s)b_s + \xi(s+1) + \tau = \frac{1 - \xi\alpha_s}{\alpha_s}(f_{s+1} - 1 - s), \end{aligned}$$

$$-(\tau - \xi b_s)\alpha_s - (s+1) - b_s = -[(1 - \xi\alpha_s)b_s + \tau\alpha_s + s + 1] = -(1 - \xi\alpha_s)f_{s+1},$$

$$\begin{aligned} & -(\tau - \xi b_{s+1})\alpha_{s+1} + \xi(s+1) + \tau - \xi b_{s+1} \\ &= \tau(1 - \alpha_{s+1}) - \xi \cdot [(1 - \alpha_{s+1})b_{s+1} - (s+1)] \\ &= (1 - \alpha_{s+1})(\tau + \xi f_{s+1} + \xi k), \end{aligned}$$

$$\begin{aligned} & b_{s+1}/\alpha_{s+1} - (s+1) - b_{s+1} \\ &= \frac{1}{\alpha_{s+1}} \cdot [(1 - \alpha_{s+1})b_{s+1} - (s+1)\alpha_{s+1}] = \frac{1 - \alpha_{s+1}}{\alpha_{s+1}}(s+1 - f_{s+1} - k). \end{aligned}$$

This computation immediately implies that (9.18) is equivalent to (9.8). To complete the proof of part (a), we equate the (2, 1) elements of both sides of (9.3), which gives

$$(\tau - \xi b_s)/\beta_s + (k + b_s)/(\alpha_s\beta_s) = (\tau - \xi b_{s+1})/\beta_{s+1} + \xi(k + b_{s+1})/(\alpha_{s+1}\beta_{s+1}).$$

We can rewrite the last equation as

$$\frac{1}{\beta_s}[(\tau - \xi b_s) + (k + b_s)/\alpha_s] = \frac{1}{\beta_{s+1}}[(\tau - \xi b_{s+1}) + \xi(k + b_{s+1})/\alpha_{s+1}].$$

It is easily seen to be equivalent to (9.9), by (9.6).

Now we assume that  $\xi = 0$ . Then (9.17) becomes

$$k + b_s + b_{s+1} + \tau\alpha_s + s + 1 - \frac{s+1}{1 - \alpha_{s+1}} = 0. \quad (9.19)$$

It is clear that with the notation (9.10), the last equation is equivalent to (9.11). To obtain (9.12), we divide (9.15) by (9.14), which gives

$$\frac{b_s/\alpha_s + \tau}{-\tau\alpha_s - (s+1) - b_s} = \frac{-\tau\alpha_{s+1} + \tau}{b_{s+1}/\alpha_{s+1} - (s+1) - b_{s+1}}. \quad (9.20)$$

From (9.10) and (9.19), we have

$$b_s/\alpha_s + \tau = \frac{1}{\alpha_s}(\tau\alpha_s + b_s) = \frac{1}{\alpha_s}(g_s - s - 1),$$

$$-\tau\alpha_s - b_s - (s + 1) = -g_s,$$

$$-\tau\alpha_{s+1} + \tau = (1 - \alpha_{s+1})\tau,$$

$$\begin{aligned} b_{s+1}/\alpha_{s+1} - (s + 1) - b_s &= \frac{1}{\alpha_{s+1}} \cdot [(1 - \alpha_{s+1})b_{s+1} - (s + 1)\alpha_{s+1}] \\ &= \frac{1 - \alpha_{s+1}}{\alpha_{s+1}}(s + 1 - k - g_s). \end{aligned}$$

This computation immediately implies that (9.20) is equivalent to (9.12). Finally, we compare the (2, 1) elements of both sides of (9.3). This yields

$$\frac{\tau}{\beta_s} + \frac{k + b_s}{\alpha_s\beta_s} = \frac{\tau}{\beta_{s+1}}, \quad \text{i.e.,} \quad \frac{\beta_{s+1}}{\beta_s} = \frac{\tau}{\alpha_s^{-1}(\tau\alpha_s + k + b_s)},$$

which gives (9.13), completing the proof of part (b).  $\square$

### 10. A Connection with the $q - P_{VI}$ Equation of M. Jimbo and H. Sakai

*10.1 Reduction to the  $q - P_{VI}$  system.* In this section we show that the compatibility conditions for the Lax pairs corresponding to some of the families of polynomials orthogonal on  $q$ -lattices are equivalent to the  $q - P_{VI}$  system of M. Jimbo and H. Sakai (Eqs. (19)–(20) in [10]) for an appropriate choice of the parameters, or to a certain degeneration of this system. Thus, we now assume that the orthogonality set is of the form  $\mathfrak{X} = \{q^{-s}\}$ , where  $s$  runs either over  $\mathbb{Z}_{\geq 0}$  or over  $\{0, \dots, N\}$  ( $N \in \mathbb{Z}_{\geq 0}$ ), and  $|q| \neq 0, 1$ . Hence we have  $\sigma\zeta = q\zeta$  and  $\eta = q$ . Then the corresponding Lax pair has the following form:

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - q^{-s}} \right) m_s(\zeta), \quad m_s(q\zeta) = M_s(\zeta)m_{s+1}(\zeta)D(\zeta)^{-1}, \quad (10.1)$$

and the compatibility condition is (cf. Eq. (5.5)):

$$\left( I + \frac{A_s}{q\zeta - q^{-s}} \right) M_s(\zeta) = M_{s+1}(\zeta) \left( I + \frac{A_{s+1}}{\zeta - q^{-s-1}} \right). \quad (10.2)$$

To relate our situation to the one considered in [10], we make the following change of notation:  $x := \zeta, t := q^{-s}$ . Then we define

$$A(x, t) = M_s(x)((x - t)I + A_s), \quad (10.3)$$

$$B_0(t) = -qtI - A_{s-1}, \quad (10.4)$$

$$B(x, t) = \frac{x(xI + B_0(t))}{(x - qt)^2}. \quad (10.5)$$

**Theorem 10.1.** (a) *The compatibility condition (10.2) with  $s$  replaced by  $s - 1$  is equivalent to the following equation:*

$$A(x, qt)B(x, t) = B(qx, t)A(x, t). \quad (10.6)$$

(b) If the matrix  $D(\zeta) = \text{diag}(d_1(\zeta), d_2(\zeta))$  is linear in  $\zeta$ , so that  $d_1(\zeta) = \lambda_1(\zeta - a_3)$ ,  $d_2(\zeta) = \lambda_2(\zeta - a_4)$  with  $\lambda_1, \lambda_2 \neq 0$ , then the matrix  $A(x, t)$  is quadratic in  $x$ , and if we write

$$A(x, t) = A_0(t) + A_1(t)x + A_2x^2, \tag{10.7}$$

we have:

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad \kappa_1 = q^k \lambda_1, \quad \kappa_2 = q^{-k} \lambda_2, \tag{10.8}$$

$$A_0(t) \text{ has eigenvalues } t\theta_1, t\theta_2, \tag{10.9}$$

$$\det A(x, t) = \kappa_1 \kappa_2 (x - t)^2 (x - a_3)(x - a_4), \tag{10.10}$$

and the parameters  $\kappa_j, a_j, \theta_j$  are independent of  $t$ .

Recall that the natural number  $k$  in (10.8) defines the asymptotics of the solutions  $m_s(\zeta)$  as  $\zeta \rightarrow \infty$  (see Theorem 3.1).

*Remark 10.2.* Equation (10.6) can be viewed as the compatibility condition for the following pair of  $q$ -difference matrix equations, cf. [10]:

$$n(x, qt) = B(x, t)n(x, t), \quad n(qx, t) = A(x, t)n(x, t). \tag{10.11}$$

One way to construct a matrix  $n(x, t)$  which solves the system (10.11) is as follows. Let  $V(\zeta) = \text{diag}(v_1(\zeta), v_2(\zeta))$  be a diagonal matrix such that  $V(q\zeta) = D(\zeta)V(\zeta)$  for all  $\zeta$ . Then for all  $s$ , we define a new matrix  $n_s(\zeta)$  by

$$m_s(\zeta) = \begin{cases} \zeta^s q^{\binom{s}{2}} \prod_{i=-\infty}^{s-1} (q^i \zeta - 1)^{-1} \cdot n_s(\zeta) \cdot V(\zeta)^{-1} & \text{if } |q| > 1, \\ \zeta^s q^{\binom{s}{2}} \prod_{i=s}^{+\infty} (q^i \zeta - 1) \cdot n_s(\zeta) \cdot V(\zeta)^{-1} & \text{if } 0 < |q| < 1. \end{cases} \tag{10.12}$$

Substituting this into the first equation of the Lax pair (10.1), replacing  $s$  by  $s - 1$  and simplifying yields

$$\zeta \cdot n_s(\zeta) = ((\zeta - q^{-s+1})I + A_{s-1}) \cdot n_{s-1}(\zeta).$$

Since  $A_{s-1}^2 = 0$ , this is equivalent to

$$n_{s-1}(\zeta) = \frac{\zeta(\zeta I + (-q^{-s+1}I - A_{s-1}))}{(\zeta - q^{-s+1})^2} n_s(\zeta). \tag{10.13}$$

On the other hand, if we substitute the first equation of the Lax pair (10.1) into the second one, use (10.12) and simplify the result, we obtain

$$n_s(q\zeta) = M_s(\zeta) \cdot ((\zeta - q^{-s}) + A_s)n_s(\zeta). \tag{10.14}$$

Now if we let  $x = \zeta, t = q^{-s}$  and  $n(x, t) = n_s(\zeta)$ , then with the notation (10.3), (10.4), (10.5) of Theorem 10.1, the system (10.13), (10.14) leads to (10.11).

*Proof of Theorem 10.1.* (a) Since  $A_{s-1}^2 = 0$ , we have

$$A(x, qt)B(x, t) = M_{s-1}(\zeta) \cdot ((\zeta - q^{-s+1})I + A_{s-1}) \cdot \frac{\zeta((\zeta - q^{-s+1})I - A_{s-1})}{(\zeta - q^{-s+1})^2} = \zeta M_{s-1}(\zeta)$$

and

$$B(qx, t)A(x, t) = \frac{q\zeta(q(\zeta - q^{-s})I - A_{s-1})}{q^2(\zeta - q^{-s})^2} \cdot M_s(\zeta) \cdot ((\zeta - q^{-s})I + A_s) = \zeta \cdot \left( I - \frac{q^{-1}A_{s-1}}{\zeta - q^{-s}} \right) \cdot M_s(\zeta) \cdot \left( I + \frac{A_s}{\zeta - q^{-s}} \right).$$

Hence, if we multiply both sides of (10.6) by  $(I - (\zeta - q^{-s})^{-1}q^{-1}A_{s-1})^{-1} = I + (\zeta - q^{-s})^{-1}q^{-1}A_{s-1}$  and divide by  $\zeta$ , we obtain

$$\left( I + \frac{q^{-1}A_{s-1}}{\zeta - q^{-s}} \right) \cdot M_{s-1}(\zeta) = M_s(\zeta) \cdot \left( I + \frac{A_s}{\zeta - q^{-s}} \right).$$

Replacing  $s$  by  $s + 1$  yields (10.2).

(b) Since the matrix  $D(\zeta) = \text{diag}(\lambda_1(\zeta - a_3), \lambda_2(\zeta - a_4))$  is linear in  $\zeta$ , it follows from Theorem 3.1(c) that we can write

$$M_s(\zeta) = \begin{pmatrix} \kappa_1 \zeta & 0 \\ 0 & \kappa_2 \zeta \end{pmatrix} + C_s$$

for a constant matrix  $C_s$ , where  $\kappa_1, \kappa_2$  are given by (10.8). This immediately implies that the matrix  $A(x, t)$  is quadratic in  $x$  with the leading coefficient  $A_2 = \text{diag}(\kappa_1, \kappa_2)$ . Since  $\det M_s(\zeta) = \det D(\zeta)$  for all  $\zeta$  by Lemma 2.1, and the matrix  $A_s$  is nilpotent, we see from (10.3) that  $\det A(x, t) = \kappa_1 \kappa_2 (x - t)^2 (x - a_3)(x - a_4)$ . In particular, taking  $x = 0$ , we see that  $\det A_0(t) = t^2 \delta$ , where  $\delta$  is a constant independent of  $t$ . To complete the proof, it therefore suffices to show that  $\text{Tr } A_0(t) = t\tau$ , where  $\tau$  is a constant independent of  $t$ . To this end, we study the compatibility condition (10.2). Taking residues of both sides of (10.2) at  $\zeta = q^{-s-1}$  yields

$$A_s \cdot \left\{ \begin{pmatrix} \kappa_1 q^{-s-1} & 0 \\ 0 & \kappa_2 q^{-s-1} \end{pmatrix} + C_s \right\} = q \cdot \left\{ \begin{pmatrix} \kappa_1 q^{-s-1} & 0 \\ 0 & \kappa_2 q^{-s-1} \end{pmatrix} + C_{s+1} \right\} \cdot A_{s+1}. \tag{10.15}$$

If we compare the asymptotics of both sides of (10.2) as  $\zeta \rightarrow \infty$ , we get

$$C_s + A_s \cdot \begin{pmatrix} \kappa_1 q^{-1} & 0 \\ 0 & \kappa_2 q^{-1} \end{pmatrix} = C_{s+1} + \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \cdot A_{s+1}. \tag{10.16}$$

Multiplying (10.16) by  $q^{-s}$  and subtracting it from (10.15) gives

$$(A_s - q^{-s}I)C_s = qC_{s+1}(A_{s+1} - q^{-s-1}I).$$

Since  $A_0(t) = C_s(A_s - q^{-s}I)$ , this shows that  $\text{Tr } A_0(qt) = q \cdot \text{Tr } A_0(t)$  for all  $t$ .  $\square$

*10.2 Solution of the  $q - P_{VI}$  system.* We are now in a position to quote a result of M. Jimbo and H. Sakai [10]. The situation considered in their work is more general: it is assumed that  $B(x, t)$  has the form

$$B(x, t) = \frac{x(xI + B_0(t))}{(x - qta_1)(x - qta_2)}, \quad (10.17)$$

and instead of (10.10), it is assumed that

$$\det A(x, t) = \kappa_1 \kappa_2 (x - ta_1)(x - ta_2)(x - a_3)(x - a_4). \quad (10.18)$$

Thus, the situation of Theorem 10.1 corresponds to the specialization  $a_1 = a_2 = 1$ . Now we have

**Theorem 10.3 (Jimbo–Sakai, [10]).** *Assume (10.17), (10.7), (10.8), (10.9) and (10.18). Also, suppose that  $\kappa_j, \theta_j \neq 0$  ( $j = 1, 2$ ),  $a_j \neq 0$  ( $j = 1, 2, 3, 4$ ) and  $\kappa_1 \neq \kappa_2$ .*

(a) *Define  $y = y(t)$  and  $z_i = z_i(t)$  ( $i = 1, 2$ ) by*

$$A_{12}(y, t) = 0, \quad A_{11}(y, t) = \kappa_1 z_1, \quad A_{22}(y, t) = \kappa_2 z_2,$$

where  $A_{ij}(x, t)$  are the elements of the matrix  $A(x, t)$ , so that  $z_1 z_2 = (y - ta_1)(y - ta_2)(y - a_3)(y - a_4)$ . In terms of  $y, z_1, z_2$ , the matrix  $A(x, t)$  can be parameterized as follows:

$$A(x, t) = \begin{pmatrix} \kappa_1((x - y)(x - \alpha) + z_1) & \kappa_2 w(x - y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2((x - y)(x - \beta) + z_2) \end{pmatrix},$$

where

$$\alpha = \frac{1}{\kappa_1 - \kappa_2} \left[ y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) - \kappa_2((a_1 + a_2)t + a_3 + a_4 - 2y) \right],$$

$$\beta = \frac{1}{\kappa_1 - \kappa_2} \left[ -y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - \kappa_2 z_2) + \kappa_1((a_1 + a_2)t + a_3 + a_4 - 2y) \right],$$

$$\gamma = z_1 + z_2 + (y + \alpha)(y + \beta) + (\alpha + \beta)y - a_1 a_2 t^2 - (a_1 + a_2)(a_3 + a_4)t - a_3 a_4,$$

$$\delta = y^{-1}(a_1 a_2 a_3 a_4 t^2 - (\alpha y + z_1)(\beta y + z_2)).$$

(b) *Define  $z = z(t)$  by*

$$z_1 = \frac{(y - ta_1)(y - ta_2)}{q\kappa_1 z}, \quad z_2 = q\kappa_1(y - a_3)(y - a_4)z.$$

Introduce the notation  $\bar{y} = y(qt)$ ,  $\bar{z} = z(qt)$ ,  $\bar{w} = w(qt)$ , and set

$$b_1 = \frac{a_1 a_2}{\theta_1}, \quad b_2 = \frac{a_1 a_2}{\theta_2}, \quad b_3 = \frac{1}{q\kappa_1}, \quad b_4 = \frac{1}{\kappa_2}.$$

Then the compatibility condition (10.6) is equivalent to the following system of equations:

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z} - tb_1)(\bar{z} - tb_2)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \tag{10.19}$$

$$\frac{z\bar{z}}{b_3b_4} = \frac{(y - ta_1)(y - ta_2)}{(y - a_3)(y - a_4)}, \tag{10.20}$$

$$\frac{\bar{w}}{w} = \frac{b_4}{b_3} \cdot \frac{\bar{z} - b_3}{\bar{z} - b_4}. \tag{10.21}$$

*Remark 10.4.* Equations (10.19)–(10.21) allow us to compute  $(\bar{y}, \bar{z}, \bar{w})$  if we know  $(y, z, w)$ , and vice versa.

*Remark 10.5.* Unfortunately, it was beyond our technical abilities to follow the proof of Theorem 10.3 in [10]. However, we were able to verify the statement of the theorem using computer simulations with random values of the parameters  $\kappa_j, a_j, \theta_j$ .

*10.3 Degeneration of  $q - P_{VI}$ .* We have mentioned in Subsect. 7.5 that the compatibility conditions for Lax pairs corresponding to certain families of orthogonal polynomials are equivalent not to the  $q - P_{VI}$  system but to a degeneration of it. We now describe this degeneration.

**Theorem 10.6.** *With the notation of Theorem 10.1, suppose that the matrix  $D(\zeta) = \text{diag}(d_1(\zeta), d_2(\zeta))$  is such that  $d_1(\zeta) = \lambda_1(\zeta - a_3^\circ)$ ,  $d_2(\zeta) = \lambda_2$ , where  $\lambda_j, a_3^\circ \neq 0$ . Then*

(a) *The matrix  $A(x, t)$  is quadratic in  $x$ , and if we write*

$$A(x, t) = A_0(t) + A_1(t)x + A_2x^2, \tag{10.22}$$

*we have:*

$$A_2 = \begin{pmatrix} \kappa_1^\circ & 0 \\ 0 & 0 \end{pmatrix}, \quad \kappa_1^\circ = q^k \lambda_1, \tag{10.23}$$

$$A_0(t) \text{ has eigenvalues } t\theta_1^\circ, t\theta_2^\circ, \tag{10.24}$$

$$\det A(x, t) = \kappa_1^\circ \kappa_2^\circ (x - t)^2 (x - a_3^\circ), \quad \kappa_2^\circ = q^{-k} \lambda_2, \tag{10.25}$$

*and the parameters  $\kappa_j^\circ, \theta_j^\circ, a_3^\circ$  are independent of  $t$ .*

(b) *We can parameterize the matrix  $A(x, t)$  as follows:*

$$A(x, t) = \begin{pmatrix} \kappa_1^\circ((x - y^\circ)(x - \alpha^\circ) + z_1^\circ) & \kappa_2^\circ w^\circ (x - y^\circ) \\ \kappa_1^\circ (w^\circ)^{-1} (\gamma^\circ x + \delta^\circ) & \kappa_2^\circ (x - y^\circ + z_2^\circ) \end{pmatrix}, \tag{10.26}$$

*where*

$$\alpha^\circ = \frac{1}{\kappa_1^\circ} \left[ (y^\circ)^{-1} ((\theta_1^\circ + \theta_2^\circ)t - \kappa_1^\circ z_1^\circ - \kappa_2^\circ z_2^\circ) + \kappa_2^\circ \right],$$

$$\gamma^\circ = z_2^\circ - (y^\circ + \alpha^\circ) - y^\circ + 2t + a_3^\circ,$$

$$\delta^\circ = (y^\circ)^{-1}(-a_3^\circ t^2 + (\alpha^\circ y^\circ + z_1^\circ)(y^\circ - z_2^\circ)).$$

Define  $z^\circ = z^\circ(t)$  by

$$z_1^\circ = \frac{(y^\circ - t)^2}{q\kappa_1^\circ z^\circ}, \quad z_2^\circ = q\kappa_1^\circ(y - a_3^\circ)z^\circ.$$

Introduce the notation  $\bar{y}^\circ = y^\circ(qt)$ ,  $\bar{z}^\circ = z^\circ(qt)$ ,  $\bar{w}^\circ = w^\circ(qt)$ , and set

$$b_1^\circ = \frac{1}{\theta_1^\circ}, \quad b_2^\circ = \frac{1}{\theta_2^\circ}, \quad b_3^\circ = \frac{1}{q\kappa_1^\circ}.$$

Then the compatibility condition (10.6) is equivalent to the following system of equations:

$$\frac{y^\circ \bar{y}^\circ}{\kappa_2^\circ a_3^\circ} = \frac{(\bar{z}^\circ - tb_1^\circ)(\bar{z}^\circ - tb_2^\circ)}{\bar{z}^\circ - b_3^\circ}, \tag{10.27}$$

$$\frac{z^\circ \bar{z}^\circ}{b_3^\circ} = \frac{(y^\circ - t)^2}{\kappa_2^\circ (y^\circ - a_3^\circ)}, \tag{10.28}$$

$$\frac{\bar{w}^\circ}{w^\circ} = \frac{b_3^\circ - \bar{z}^\circ}{b_3^\circ}. \tag{10.29}$$

*Proof.* (a) The proof of this part is almost identical to that of Theorem 10.1(b) and will therefore be omitted.

(b) One checks directly that all formulas of [10] are compatible with the following limit transition:

$$\begin{aligned} \kappa_1 &\rightarrow \kappa_1^\circ, & a_1, a_2 &\rightarrow 1, & \theta_1 &\rightarrow \theta_1^\circ, & \theta_2 &\rightarrow \theta_2^\circ, \\ & & a_3 &\rightarrow a_3^\circ, & \alpha &\rightarrow \alpha^\circ, & y &\rightarrow y^\circ, & z_1 &\rightarrow z_1^\circ, & z &\rightarrow z^\circ, \\ \kappa_2 &\rightarrow 0, & \kappa_2 a_4 &\rightarrow -\kappa_2^\circ, & \kappa_2 \beta &\rightarrow -\kappa_2^\circ, & \kappa_2 \gamma &\rightarrow \kappa_2^\circ \gamma^\circ, \\ & & \kappa_2 \delta &\rightarrow \kappa_2^\circ \delta^\circ, & \kappa_2 z_2 &\rightarrow \kappa_2^\circ z_2^\circ, & \kappa_2 w &\rightarrow \kappa_2^\circ w^\circ. \end{aligned}$$

This limit transition takes the parameterization of  $A(x, t)$  given in Theorem 10.3(a) to the parameterization (10.26), and it takes the system (10.19)–(10.21) to the system (10.27)–(10.29).  $\square$

### 11. Applications: Recurrence Relations for Some Polynomials of the Askey Scheme

*11.1 Notation.* In this section we illustrate the results of Sect. 3–10 by considering several specific examples: Charlier polynomials, Meixner polynomials, Krawtchouk polynomials,  $q$ -Charlier polynomials, little  $q$ -Laguerre/Wall polynomials, alternative  $q$ -Charlier polynomials, little  $q$ -Jacobi polynomials and  $q$ -Krawtchouk polynomials. In the first four cases we solve the compatibility condition explicitly, we write down a recurrence relation for the corresponding Fredholm determinants in terms of the solution, and we provide the initial conditions for all of our recurrence relations. This is done in Subsects. 11.2, 11.4, 11.5 and 11.6. For the other four families, we content ourselves with making some general remarks in Subsect. 11.7.

The general notation of this section is that of Sects. 4 and 5. Recall that we are considering a family  $\{P_n(\zeta)\}_{n=0}^N$  of monic polynomials orthogonal on a discrete, but not

necessarily locally finite subset  $\mathfrak{X} = \{\pi_x\}_{x=0}^N$  of  $\mathbb{R}$  (where  $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ ), with respect to a strictly positive weight function  $\omega : \mathfrak{X} \rightarrow \mathbb{R}$ . If  $k$  is a natural number,  $k \leq N$ , we can consider a kernel  $K$  on  $\mathfrak{X} \times \mathfrak{X}$  defined by the formula

$$K(\pi_x, \pi_y) = \begin{cases} \frac{\|P_{k-1}\|_{\omega}^{-2} \sqrt{\omega(x)\omega(y)} (P_k(\pi_x)P_{k-1}(\pi_y) - P_{k-1}(\pi_x)P_k(\pi_y))}{(\pi_x - \pi_y)}, & x \neq y, \\ \|P_{k-1}\|_{\omega}^{-2} \omega(x) (P'_k(\pi_x)P_{k-1}(\pi_x) - P'_{k-1}(\pi_x)P_k(\pi_x)), & x = y, \end{cases} \tag{11.1}$$

where  $\|P_{k-1}\|_{\omega} = (P_{k-1}, P_{k-1})_{\omega}^{1/2}$  denotes the norm of  $P_{k-1}(\zeta)$  with respect to the inner product defined by  $\omega$ . Up to conjugation, this coincides with the kernel introduced in the beginning of Sect. 4 (see Eq. (4.1)). For all  $k \leq s \leq N$ , we define a subset  $\mathfrak{Y}_s = \{\pi_x\}_{x=s}^N \subseteq \mathfrak{X}$ , and we are interested in the Fredholm determinants

$$D_s = \det(1 - K|_{\mathfrak{Y}_s \times \mathfrak{Y}_s}). \tag{11.2}$$

11.2 Charlier polynomials ([13], §1.12). The  $n^{th}$  Charlier polynomial is defined by

$$C_n(x; a) = {}_2F_0\left(-n, -x \mid -\frac{1}{a}\right). \tag{11.3}$$

These polynomials satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} C_m(x; a) C_n(x; a) = a^{-n} e^a n! \delta_{mn}, \tag{11.4}$$

where  $a > 0$ . Thus the orthogonality set for Charlier polynomials is  $\mathfrak{X} = \mathbb{Z}_{\geq 0}$ , and the weight function is  $\omega(x) = \frac{a^x}{x!}$ . The polynomial  $C_n(x; a)$  is not monic in general; in fact, its leading coefficient is  $(-a)^{-n}$ . Hence the corresponding family of orthogonal polynomials (recall that the orthogonal polynomials that we use are monic, see Sect. 2) is  $\{P_n(\zeta) = (-a)^n C_n(\zeta; a)\}_{n=0}^{\infty}$ . We call  $P_n(\zeta)$  the  $n^{th}$  normalized Charlier polynomial. Now from (11.4), we find that  $(P_n, P_n)_{\omega} = a^n e^a n!$  for all  $n \geq 0$ . After these preliminaries, we can state our main result for Charlier polynomials.

**Theorem 11.1.** *If  $K$  is the kernel (11.1) corresponding to the family  $\{P_n(\zeta)\}_{n=0}^{\infty}$  of normalized Charlier polynomials, then the Fredholm determinants  $D_s$  defined by (11.2) can be computed from the following recurrence relation:*

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} = \frac{a^{s-1}}{(s+1)!} \cdot \frac{f_s^2}{e_s} \cdot (g_s - s - 1) \cdot h_s^2. \tag{11.5}$$

Here, the scalar sequences  $\{e_s\}_{s \geq k}$ ,  $\{f_s\}_{s \geq k}$ ,  $\{g_s\}_{s \geq k}$  and  $\{h_s\}_{s \geq k}$  satisfy the following recurrence relations:

$$e_{s+1} = \frac{ae_s}{f_s(g_s + k - s - 1)}, \tag{11.6}$$

$$f_{s+1} = \frac{ag_s}{f_s(g_s - s - 1)(g_s + k - s - 1)}, \tag{11.7}$$

$$g_{s+1} = \frac{a}{f_{s+1}} - \frac{s+1}{1-f_{s+1}} - g_s - k + 2s + 3, \tag{11.8}$$

$$h_{s+1} = a^{-1} \cdot f_s \cdot (g_s - s - 1) \cdot h_s. \tag{11.9}$$

The initial conditions for the recurrence relations (11.5)–(11.9) are given by

$$D_k = e^{-ak}, \quad D_{k+1} = e^{-ak} \cdot \Phi(-k; 1; -a), \tag{11.10}$$

$$e_k = \frac{a^k \cdot (k-1)!}{\Phi(1-k; 1; -a)}, \tag{11.11}$$

$$f_k = -\frac{a \cdot \Phi(1-k; 2; -a)}{\Phi(1-k; 1; -a)}, \tag{11.12}$$

$$g_k = k + 1 - \frac{(k+1) \cdot \Phi(1-k; 1; -a) \cdot \Phi(-k; 2; -a)}{\Phi(-k; 1; -a) \cdot \Phi(1-k; 2; -a)}, \tag{11.13}$$

$$h_k = k!, \tag{11.14}$$

where

$$\Phi(u; w; z) = {}_1F_1\left(\begin{matrix} u \\ w \end{matrix} \middle| z\right).$$

*Remark 11.2.* As we have seen in Sect. 9 (cf. Remark 9.4(b)), the recurrence relations (11.7), (11.8) form a special case of the  $d - P_{IV}$  equation of [17].

*Proof.* The proof is a straightforward application of our previous results. For the reader’s convenience we provide some remarks; the same ones apply to Theorems 11.4 and 11.6, and hence the proofs of those results will be omitted.

To make the notation of the present section more uniform, we have been writing  $e_s$  for  $\beta_s$  and  $h_s$  for  $m_s^{11}$ , where  $\{\beta_s\}$  and  $\{m_s^{11}\}$  are the scalar sequences defined in Sects. 9 and 4, respectively. The symbols  $f_s$  and  $g_s$  have the same meaning as in Sect. 9. Then the recurrence relations (11.5)–(11.9) are obtained directly from Theorem 4.2(b) and Theorem 9.3(b). To find the initial conditions, one uses the definitions of  $b_k, \alpha_k, \beta_k, f_k, g_k$ , together with Propositions 6.1, 6.2, 6.6, and the obvious identities  $h_k = m_k^{11} = k!, b_k \beta_k = -q_k$ , which follow from (6.1), (9.4).  $\square$

11.3. We now illustrate the concluding remark of Sect. 3 by showing how one can use Proposition 3.3 to obtain difference equations satisfied by orthogonal polynomials.

**Proposition 11.3** (cf. [13], §1.12, Eq. (1.12.5)). *The  $k^{\text{th}}$  normalized Charlier polynomial  $P_k(\zeta)$  solves the following difference equation:*

$$-kP_k(\zeta) = aP_k(\zeta + 1) - (\zeta + a)P_k(\zeta) + \zeta P_k(\zeta - 1). \tag{11.15}$$

*Proof.* We use the notation of Proposition 3.3. Recall that in the case of Charlier polynomials, we have  $D(\zeta) = \text{diag}(\zeta, a)$ . We also observe that from the proof of Theorem 2.4 it follows that the matrix  $m_{\mathfrak{X}}(\zeta)$  has a full asymptotic expansion in  $\zeta$  as  $\zeta \rightarrow \infty$ ; in particular, we can write, by (2.4),

$$m_{\mathfrak{X}}(\zeta) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} = I + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \zeta^{-1} + O(\zeta^{-2}).$$

Therefore, as  $\zeta \rightarrow \infty$ , we have

$$\begin{aligned}
 M(\zeta) &= m_{\mathfrak{X}}(\zeta - 1) \cdot D(\zeta) \cdot m_{\mathfrak{X}}^{-1}(\zeta) \\
 &= m_{\mathfrak{X}}(\zeta - 1) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \cdot \begin{pmatrix} \zeta & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \cdot m_{\mathfrak{X}}^{-1}(\zeta) \\
 &= \left\{ m_{\mathfrak{X}}(\zeta - 1) \cdot \begin{pmatrix} (\zeta - 1)^{-k} & 0 \\ 0 & (\zeta - 1)^k \end{pmatrix} \right\} \cdot \begin{pmatrix} (1 - 1/\zeta)^k & 0 \\ 0 & (1 - 1/\zeta)^{-k} \end{pmatrix} \cdot \begin{pmatrix} \zeta & 0 \\ 0 & a \end{pmatrix} \\
 &\quad \times \left\{ m_{\mathfrak{X}}(\zeta) \cdot \begin{pmatrix} \zeta^{-k} & 0 \\ 0 & \zeta^k \end{pmatrix} \right\}^{-1} \\
 &= \begin{pmatrix} 1 + \alpha\zeta^{-1} & \beta\zeta^{-1} \\ \gamma\zeta^{-1} & 1 + \delta\zeta^{-1} \end{pmatrix} \cdot \begin{pmatrix} \zeta - k & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} 1 - \alpha\zeta^{-1} & -\beta\zeta^{-1} \\ -\gamma\zeta^{-1} & 1 - \delta\zeta^{-1} \end{pmatrix} + O(\zeta^{-1}) \\
 &= \begin{pmatrix} \zeta - k & -\beta \\ \gamma & a \end{pmatrix} + O(\zeta^{-1}).
 \end{aligned}$$

Since  $M(\zeta)$  is entire by Proposition 3.3, the last term  $O(\zeta^{-1})$  is identically zero by Liouville’s theorem. Hence the system of Eqs. (3.5), (3.6) takes the following form:

$$\zeta \cdot P_k(\zeta - 1) = (\zeta - k) \cdot P_k(\zeta) - \beta \cdot cP_{k-1}(\zeta), \tag{11.16}$$

$$\zeta \cdot cP_{k-1}(\zeta - 1) = \gamma \cdot P_k(\zeta) + a \cdot cP_{k-1}(\zeta). \tag{11.17}$$

Note that since  $\det M(\zeta) = \det D(\zeta)$  for all  $\zeta$  by Lemma 2.1, we have  $\beta\gamma = ak$ ; in particular,  $\beta \neq 0$ . Now from (11.16), we find that

$$cP_{k-1}(\zeta) = \frac{1}{\beta} \cdot [(\zeta - k)P_k(\zeta) - \zeta P_k(\zeta - 1)].$$

Substituting this into (11.17), multiplying the result by  $\beta$  and using  $\beta\gamma = ak$ , we obtain

$$\begin{aligned}
 &\zeta \cdot [(\zeta - 1 - k)P_k(\zeta - 1) - (\zeta - 1)P_k(\zeta - 2)] \\
 &= ak \cdot P_k(\zeta) + a \cdot [(\zeta - k)P_k(\zeta) - \zeta P_k(\zeta - 1)].
 \end{aligned}$$

Dividing the last equation by  $\zeta$  and replacing  $\zeta$  by  $\zeta + 1$ , we find that it is equivalent to (11.15).  $\square$

*11.4 Meixner polynomials ([13], §1.9).* The  $n^{\text{th}}$  Meixner polynomial is defined by

$$M_n(x; \beta, c) = {}_2F_1 \left( \begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - \frac{1}{c} \right). \tag{11.18}$$

These polynomials satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_m(x; \beta, c) M_n(x; \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1 - c)^\beta} \delta_{mn}, \tag{11.19}$$

where  $\beta > 0$  and  $0 < c < 1$ . Thus the orthogonality set for Meixner polynomials is  $\mathfrak{X} = \mathbb{Z}_{\geq 0}$ , and the weight function is  $\omega(x) = \frac{(\beta)_x}{x!} c^x$ . The leading coefficient of the

polynomial  $M_n(x; \beta, c)$  is  $\frac{(1-1/c)^n}{(\beta)_n}$ , so the corresponding family of monic orthogonal polynomials is  $\{P_n(\zeta) = (\beta)_n(1-1/c)^{-n}M_n(\zeta; \beta, c)\}_{n=0}^\infty$ . We call  $P_n(\zeta)$  the  $n^{\text{th}}$  *normalized Meixner polynomial*. Now from (11.19), we find that  $(P_n, P_n)_\omega = \frac{(\beta)_n c^n n!}{(1-c)^{\beta+2n}}$  for all  $n \geq 0$ . Then we have

**Theorem 11.4.** *If  $K$  is the kernel (11.1) corresponding to the family  $\{P_n(\zeta)\}_{n=0}^\infty$  of normalized Meixner polynomials, then the Fredholm determinants  $D_s$  defined by (11.2) can be computed from the following recurrence relation:*

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} = \frac{(\beta)_s}{\beta + s} \cdot \frac{c^{s-1}}{(s+1)!} \cdot \frac{1 + cg_s}{e_s g_s^2} \cdot [(1 + cg_s)f_{s+1} - s - 1] \cdot h_s^2. \tag{11.20}$$

Here, the scalar sequences  $\{e_s\}_{s \geq k}$ ,  $\{f_s\}_{s \geq k}$ ,  $\{g_s\}_{s \geq k}$  and  $\{h_s\}_{s \geq k}$  satisfy the following recurrence relations:

$$e_{s+1} = -\frac{ce_s g_s}{g_{s+1}} \cdot \frac{(1 + g_{s+1})f_{s+1} + (\beta + k - 1)g_{s+1} - s - 1}{(1 + cg_s)f_{s+1} + k - s - 1}, \tag{11.21}$$

$$f_{s+1} = 1 - \beta - k - f_s + \frac{s}{1 + g_s} + \frac{\beta + s}{1 + cg_s}, \tag{11.22}$$

$$g_{s+1} = \frac{(f_{s+1} - 1 - s)(f_{s+1} - 1 - s + k)}{cg_s f_{s+1}(f_{s+1} + \beta + k - 1)}, \tag{11.23}$$

$$h_{s+1} = \frac{(1 + cg_s)(s + 1 - f_{s+1})}{c(\beta + s)g_s} \cdot h_s. \tag{11.24}$$

The initial conditions for the recurrence relations (11.20)–(11.24) are given by

$$D_k = (1 - c)^{k(\beta+k-1)}, \quad D_{k+1} = \frac{(\beta)_k}{k!} \cdot c^k \cdot (1 - c)^{k(\beta+k-1)} \cdot F(-k, -k; \beta; 1/c), \tag{11.25}$$

$$e_k = \frac{\beta c \cdot (k - 1)!^2}{F(1 - k, 1 - k; 1 + \beta; 1/c)}, \tag{11.26}$$

$$f_k = 0, \tag{11.27}$$

$$g_k = \frac{k}{\beta c} \cdot \frac{F(1 - k, 1 - k; 1 + \beta; 1/c)}{F(-k, 1 - k; \beta; 1/c)}, \tag{11.28}$$

$$h_k = k!, \tag{11.29}$$

where

$$F(u, v; w; z) = {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| z\right).$$

*Proof.* See the proof of Theorem 11.1.  $\square$

*Remark 11.5.* As we have seen in Sect. 9 (cf. Remark 9.4(a)), the recurrence relations (11.22), (11.23) form a special case of the  $d - P_V$  equation of [17].

11.5 *Krawtchouk polynomials* ([13], §1.10). The  $n^{\text{th}}$  *Krawtchouk polynomial* is defined by

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p}\right). \tag{11.30}$$

These polynomials satisfy the orthogonality relation

$$\sum_{x=0}^N \binom{N}{x} p^x (1-p)^{N-x} K_m(x; p, N) K_n(x; p, N) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n \delta_{mn}, \tag{11.31}$$

where  $N \in \mathbb{Z}_{\geq 0}$  and  $0 < p < 1$ . Thus the orthogonality set for Krawtchouk polynomials is  $\mathfrak{X} = \{0, \dots, N\}$ , and the weight function is  $\omega(x) = \binom{N}{x} p^x (1-p)^{N-x}$ . The leading coefficient of the polynomial  $K_n(x; p, N)$  is  $(-N)_n^{-1} p^{-n}$ , so the corresponding family of monic orthogonal polynomials is  $\{P_n(\zeta) = (-N)_n p^n K_n(\zeta; p, N)\}_{n=0}^N$ . We call  $P_n(\zeta)$  the  $n^{\text{th}}$  *normalized Krawtchouk polynomial*. Now from (11.31), we find that  $(P_n, P_n)_\omega = (-1)^n n! (-N)_n p^n (1-p)^n$  for all  $0 \leq n \leq N$ . Then we have

**Theorem 11.6.** *If  $K$  is the kernel (11.1) corresponding to the family  $\{P_n(\zeta)\}_{n=0}^N$  of normalized Krawtchouk polynomials, then the Fredholm determinants  $D_s$  defined by (11.2) can be computed from the following recurrence relation:*

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} = \binom{N}{s+1} \cdot \frac{p^{s-1} (1-p)^{N-s+1}}{(N-s)^2} \cdot \frac{1 + pg_s/(p-1)}{e_s g_s^2} \cdot \left[ (1 + pg_s/(p-1)) f_{s+1} - s - 1 \right] \cdot h_s^2. \tag{11.32}$$

Here, the scalar sequences  $\{e_s\}_{s \geq k}$ ,  $\{f_s\}_{s \geq k}$ ,  $\{g_s\}_{s \geq k}$  and  $\{h_s\}_{s \geq k}$  satisfy the following recurrence relations:

$$e_{s+1} = \frac{pe_s g_s}{(1-p)g_{s+1}} \cdot \frac{(1 + g_{s+1})f_{s+1} + (k - N - 1)g_{s+1} - s - 1}{(1 + pg_s/(p-1))f_{s+1} + k - s - 1}, \tag{11.33}$$

$$f_{s+1} = N + 1 - k - f_s + \frac{s}{1 + g_s} + \frac{(1-p)(N-s)}{p-1 + pg_s}, \tag{11.34}$$

$$g_{s+1} = \frac{(1-p)(f_{s+1} - 1 - s)(f_{s+1} - 1 - s + k)}{pg_s f_{s+1} (N + 1 - k - f_{s+1})}, \tag{11.35}$$

$$h_{s+1} = \frac{(p-1 + pg_s)(f_{s+1} - s - 1)}{p(N-s)g_s} \cdot h_s. \tag{11.36}$$

The initial conditions for the recurrence relations (11.32)–(11.36) are given by

$$D_k = (1-p)^{k(N+1-k)}, \quad D_{k+1} = \binom{N}{k} \cdot p^k \cdot (1-p)^{k(N-k)} \cdot F(-k, -k; -N; 1-1/p), \tag{11.37}$$

$$e_k = \frac{Np(1-p)^{N-1}(k-1)!^2}{F(1-k, 1-k; 1-N; 1-1/p)}, \tag{11.38}$$

$$f_k = 0, \tag{11.39}$$

$$g_k = \frac{k(1-p)}{Np} \cdot \frac{F(1-k, 1-k; 1-N; 1-1/p)}{F(-k, 1-k; -N; 1-1/p)}, \tag{11.40}$$

$$h_k = k!, \tag{11.41}$$

where

$$F(u, v; w; z) = {}_2F_1\left(\begin{matrix} u, v \\ w \end{matrix} \middle| z\right).$$

*Proof.* See the proof of Theorem 11.1.  $\square$

*Remark 11.7.* As we have seen in Sect. 9 (cf. Remark 9.4(a)), the recurrence relations (11.34), (11.35) form a special case of the  $d - P_V$  equation of [17].

*11.6  $q$ -Charlier polynomials ([13], §3.23).* In this subsection, we assume that  $q$  is a fixed real number,  $0 < q < 1$ . The  $n^{th}$   $q$ -Charlier polynomial is defined by

$$C_n(\zeta; a; q) = \mathfrak{z}\phi_1\left(\begin{matrix} q^{-n}, \zeta \\ 0 \end{matrix} \middle| q; -\frac{q^{n+1}}{a}\right). \tag{11.42}$$

These polynomials satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} \frac{a^x}{(q; q)_x} q^{\binom{x}{2}} C_m(q^{-x}; a; q) C_n(q^{-x}; a; q) = q^{-n} (-a; q)_{\infty} (-a^{-1}q, q; q)_n \delta_{mn}, \tag{11.43}$$

where  $a > 0$  and

$$(-a; q)_{\infty} = \prod_{l=0}^{\infty} (1 + aq^l).$$

Thus the orthogonality set for  $q$ -Charlier polynomials is  $\mathfrak{X} = \{q^{-x}\}_{x=0}^{\infty}$ , and the weight function is  $\omega(x) = \frac{a^x}{(q; q)_x} q^{\binom{x}{2}}$ . The leading coefficient of the polynomial  $C_n(\zeta; a; q)$  is  $(-1)^n q^{n^2} a^{-n}$ , so the corresponding family of orthogonal polynomials is  $\{P_n(\zeta) = (-1)^n a^n q^{-n^2} C_n(\zeta; a; q)\}_{n=0}^{\infty}$ . We call  $P_n(\zeta)$  the  $n^{th}$  normalized  $q$ -Charlier polynomial. Now from (11.43), we find that

$$(P_n, P_n)_{\omega} = a^{2n} q^{-2n^2-n} (-a; q)_{\infty} (-a^{-1}q, q; q)_n$$

for all  $n \geq 0$ . Then we have

**Theorem 11.8.** *If  $K$  is the kernel (11.1) corresponding to the family  $\{P_n(\zeta)\}_{n=0}^\infty$  of normalized  $q$ -Charlier polynomials, then the Fredholm determinants  $D_s$  defined by (11.2) satisfy the following recurrence relation:*

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+1}}{D_s} = \frac{a^{s-1}}{(q; q)_{s+1}} \cdot q^{\binom{s+1}{2}} \cdot u_s \cdot h_s^2, \tag{11.44}$$

where

$$u_s = q^k \cdot \frac{p_s}{q_s^2} \cdot (p_s \beta_s + a q^{-k} q_s) \tag{11.45}$$

for all  $s \geq 0$ . The scalar sequences  $\{p_s\}_{s \geq k}$ ,  $\{q_s\}_{s \geq k}$ ,  $\{\beta_s\}_{s \geq k}$  and  $\{h_s\}_{s \geq k}$  can be computed from the following recurrence relations (which involve additional sequences):

$$\epsilon_s = a(q^{-s-1} - 1) + q^{k-1}(a q^{-k} p_s - r_s \beta_s), \tag{11.46}$$

$$p_{s+1} = -q^{-1} p_s^{-1} \epsilon_s^{-1} \cdot (p_s \beta_s + a q^{-k} q_s) \cdot (r_s \alpha_s - p_s \gamma_s + q^{k-s-1} r_s), \tag{11.47}$$

$$q_{s+1} = q^{-1} q_s^{-1} \epsilon_s^{-1} \cdot (p_s \beta_s + a q^{-k} q_s)^2, \tag{11.48}$$

$$r_{s+1} = q^{-1} r_s^{-1} \epsilon_s^{-1} \cdot (r_s \alpha_s - p_s \gamma_s + q^{k-s-1} r_s)^2, \tag{11.49}$$

$$\alpha_{s+1} = \alpha_s + q^{k-1} p_s - q^k p_{s+1}, \tag{11.50}$$

$$\beta_{s+1} = \beta_s - q^k q_{s+1}, \tag{11.51}$$

$$\gamma_{s+1} = \gamma_s + q^{k-1} r_s, \tag{11.52}$$

$$h_{s+1} = \frac{p_s \beta_s + a q^{-k} q_s}{a q_s} \cdot h_s. \tag{11.53}$$

The initial conditions for the recurrence relations (11.44) and (11.47)–(11.53) are provided by

$$D_k = (-a; q)_\infty^{-k} \cdot a^{-\binom{k}{2}} \cdot \prod_{n=0}^{k-1} [(-a^{-1} q; q)_n^{-1} q^{\binom{n+1}{2}}], \tag{11.54}$$

$$D_{k+1} = (-a; q)_\infty^{-k} \cdot \frac{a^{k-\binom{k}{2}}}{(q; q)_k} \cdot q^{-\binom{k}{2}} \cdot G_q(q^{-k}, q^{-k}; -q^{2k}/a) \cdot \prod_{n=0}^{k-1} [(-a^{-1} q; q)_n^{-1} q^{\binom{n+1}{2}}], \tag{11.55}$$

$$p_k = (1 - q^{-k}) \cdot \frac{G_q(q^{-k}, q^{1-k}; -q^{2k-1}/a)}{G_q(q^{-k}, q^{-k}; -q^{2k}/a)}, \tag{11.56}$$

$$q_k = (q; q)_k^2 \cdot q^{-k(k+1)} \cdot G_q(q^{-k}, q^{-k}; -q^{2k}/a)^{-1}, \tag{11.57}$$

$$r_k = \frac{q^{k^2}(1 - q^{-k})}{(q; q)_k (q; q)_{k-1}} \cdot \frac{G_q(q^{-k}, q^{1-k}; -q^{2k-1}/a)^2}{G_q(q^{-k}, q^{-k}; -q^{2k}/a)}, \tag{11.58}$$

$$\alpha_k = -1 - q^k p_k, \tag{11.59}$$

$$\beta_k = -q^k q_k, \tag{11.60}$$

$$\gamma_k = \frac{q^{k^2-1}}{(q; q)_{k-1}^2} \cdot G_q(q^{1-k}, q^{1-k}; -q^{2k-2}/a), \tag{11.61}$$

$$h_k = q^{-k^2} \cdot (q; q)_k, \tag{11.62}$$

where

$$G_q(u, v; z) = {}_2\phi_0\left(\begin{matrix} u, v \\ - \end{matrix} \middle| q; z\right).$$

*Proof.* As before, the proof is quite straightforward. We have been writing  $h_s$  for  $m_s^{11}$  (defined in Sect. 4). The formulas (11.45) and (11.46)–(11.52) follow immediately from Theorem 8.2 (note that  $\delta_s = aq^{-k}$  for all  $s$ , since we have  $\delta_{s+1} = \delta_s$  from (8.11), and  $\delta_k = aq^{-k}$  from (6.10)). Then (11.44) and (11.53) are deduced from (4.7) and (4.6), respectively. Finally, the initial conditions (11.54)–(11.62) are easily obtained from Propositions 6.6, 6.2 and 6.5.  $\square$

*Remark 11.9.* As we have mentioned in Subsect. 7.5, the recurrence relation (11.47)–(11.52) for  $q$ -Charlier polynomials is in fact equivalent to a certain degeneration of the  $q - P_{VI}$  equation of [10]. This is a special case of Theorem 10.6.

*Remark 11.10.* In the case of  $q$ -Charlier polynomials, it is possible to solve the compatibility condition for the corresponding Lax pair by a method similar to the one used in Theorem 9.3(b). Namely, it is easy to see that the Lax pair can be parameterized as follows:

$$m_{s+1}(\zeta) = \left\{ I + (\zeta - q^{-s})^{-1} \begin{pmatrix} p_s & p_s a_s c_s \\ -p_s / (a_s c_s) & -p_s \end{pmatrix} \right\} \cdot m_s(\zeta),$$

$$m_s(q\zeta) = \begin{pmatrix} q^k(\zeta - 1) + b_s & b_s c_s \\ aq^{-k}/c_s & aq^{-k} \end{pmatrix} m_{s+1}(\zeta) \begin{pmatrix} (\zeta - 1)^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Then the compatibility condition gives the following recurrence relations for the parameters  $a_s, b_s, c_s, p_s$ :

$$p_{s+1} = \frac{p_s(b_s + aq^{-k}a_s)(q^{k-s-1} - q^k + b_s + aq^{-k}a_s)}{q^k p_s(b_s + aq^{-k}a_s) + q \cdot (q^{k-s-1} - q^k) \cdot aq^{-k}a_s},$$

$$b_{s+1} = b_s + q^{k-1}p_s - q^k p_{s+1},$$

$$a_{s+1} = \frac{b_s - q^k p_{s+1} + aq^{-k}a_s}{q^{k-s-1} - q^k + b_s - q^k p_{s+1} + aq^{-k}a_s},$$

$$c_{s+1} = \frac{aq^{-k}a_s c_s}{aq^{-k}a_s - q^{k-1}p_s}.$$

With this notation, the recurrence relation for the Fredholm determinants is (11.44), the same as before, but now we have

$$u_s = \frac{q^k \cdot (aq^{-k} + b_s/a_s)}{a_s c_s},$$

and the recurrence relation for  $h_s$  is given by

$$h_{s+1} = a^{-1} \cdot (aq^{-k} + b_s/a_s) \cdot h_s.$$

The initial values  $a_k, b_k, c_k$  can be easily found from (11.56)–(11.61).

The main difference between this situation and that of Sect. 9 is that we cannot further reduce the recurrence relations for  $a_s, b_s, p_s$  to relations involving only two sequences of parameters. So this method cannot be used to show that our recurrence relations in the case of  $q$ -Charlier polynomials are equivalent to one of H. Sakai’s  $q$ -difference

equations [17]. From the computational point of view, this method is slightly easier to use than the one presented in Theorem 11.8.

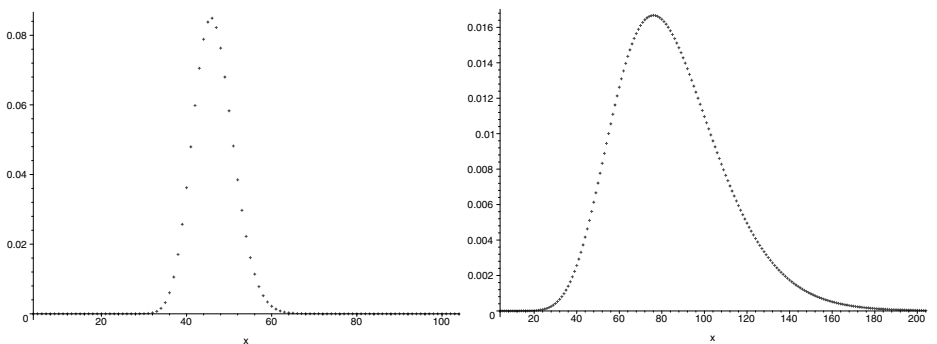
*11.7 Concluding remarks.* It follows from Theorem 10.1 that the recurrence relations corresponding to the little  $q$ -Jacobi polynomials and the  $q$ -Krawtchouk polynomials can be reduced to special cases of the  $q - P_{VI}$  system of [10]. Also, it follows from Theorem 10.6 that the recurrence relations corresponding to the  $q$ -Charlier polynomials and the little  $q$ -Laguerre/Wall polynomials can be reduced to special cases of a certain degeneration of the  $q - P_{VI}$  system described in Subsect. 10.3. However, it is more convenient to use the formulas of Sect. 8 for practical computations. In addition, the method of Sect. 8 covers the case of the alternative  $q$ -Charlier polynomials, whereas we do not know if the recurrence relation corresponding to these polynomials can be reduced to one of the equations of H. Sakai’s hierarchy.

As far as using the formulas of Sect. 8 is concerned, there is no essential difference between the  $q$ -Charlier polynomials and the other four families of basic hypergeometric orthogonal polynomials that we consider here. So we have decided not to write out explicitly the results we have obtained for these four families. On the other hand, we have carried out all the calculations for some specific values of parameters in Maple, and in Sect. 12 we present a few plots of the “density function” (difference or  $q$ -derivative of  $D_s$ ) for the eight families of orthogonal polynomials considered in this section.

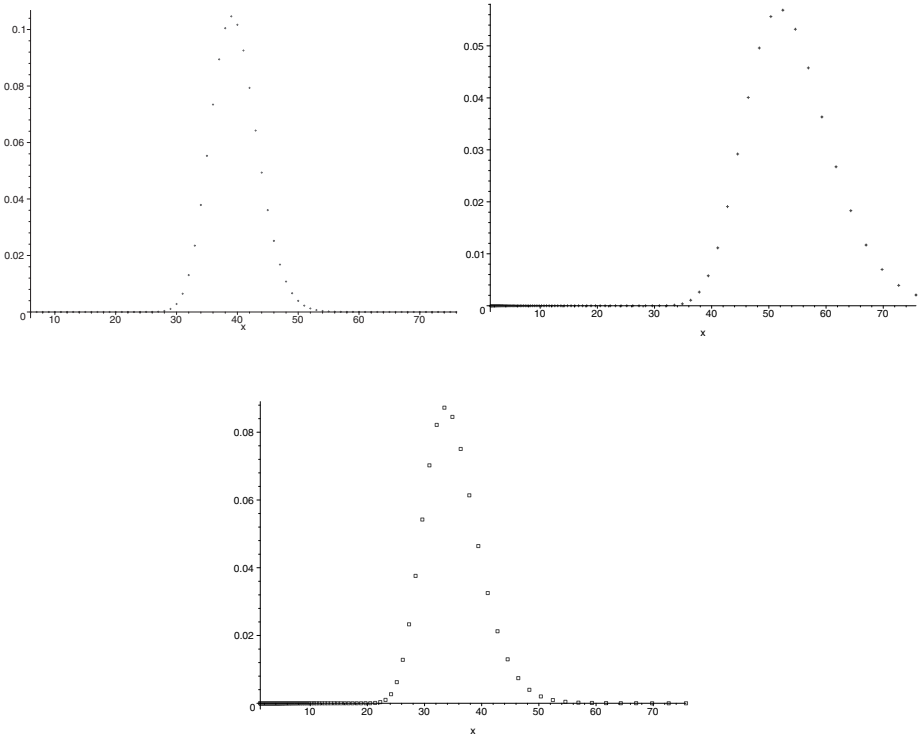
## 12. Numerical Computations

*12.1.* The plots in this section have been obtained in Maple by using the formulas of Sect. 8 and Subsects. 11.2, 11.4 and 11.5 for the specific values of parameters indicated below.

*12.2.* The following are two plots of the density function  $D_{s+1} - D_s$  for the family of Meixner polynomials. The parameters (cf. Subsect. 11.4 or Subsect. 7.2) are  $k = 4$ ,  $c = 0.01$ ,  $\beta = 3000$  for the first graph and  $k = 4$ ,  $c = 0.9$ ,  $\beta = 0.5$  for the second graph. The  $x$ -coordinate in each case is  $s$ .

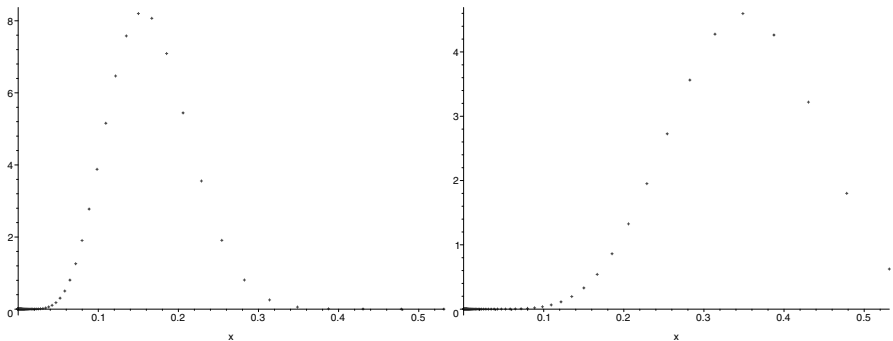


*12.3.* The following are the plots of the density function for the families of Charlier,  $q$ -Charlier and alternative  $q$ -Charlier polynomials (left to right). The parameters (cf. Subsect. 7.2) are  $k = 6$ ,  $a = 20$  for the Charlier polynomials (first graph) and  $k = 6$ ,

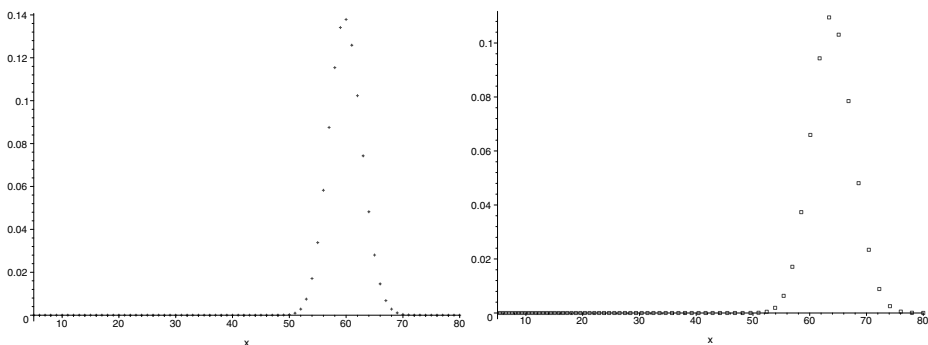


$a = 20, q = 0.96$  for the  $q$ -Charlier and the alternative  $q$ -Charlier polynomials (last two graphs). In the case of Charlier polynomials we plot the difference derivative,  $D_{s+1} - D_s$ , of  $D_s$ , and the  $x$ -coordinate is  $s$ , while in the other two cases we plot the  $q$ -derivative,  $q^s \cdot (D_{s+1} - D_s)/(1 - q)$ , of  $D_s$ , and the  $x$ -coordinate is  $q^{-s}$ .

12.4. The following are the plots of the density function  $q^{-s} \cdot (D_{s+1} - D_s)/(1 - q)$  for the families of little  $q$ -Laguerre/Wall polynomials (first graph) and little  $q$ -Jacobi polynomials (second graph). The parameters (cf. Subsect. 7.2) are  $k = 6, a = 0.5, q = 0.9$  for the little  $q$ -Laguerre polynomials and  $k = 6, a = 0.5, b = 1.5, q = 0.9$  for the little  $q$ -Jacobi polynomials. The  $x$ -coordinate in each case is  $q^s$ .



12.5. The following are the plots of the density function for the families of Krawtchouk polynomials (first graph) and  $q$ -Krawtchouk polynomials (second graph). The parameters (cf. Subsect. 7.2) are  $k = 5$ ,  $N = 80$ ,  $p = 1/(0.7 + 1)$  for the Krawtchouk polynomials and  $k = 5$ ,  $N = 80$ ,  $p = 0.7$ ,  $q = 0.98$  for the  $q$ -Krawtchouk polynomials. In the first case we plot the difference derivative,  $D_{s+1} - D_s$ , of  $D_s$ , and the  $x$ -coordinate is  $s$ , while in the second case we plot the normalized  $q$ -derivative,  $(q^{-N} - 1) \cdot q^s \cdot (D_{s+1} - D_s)/(1 - q)/N$ , of  $D_s$ , and the  $x$ -coordinate is  $N \cdot (q^{-s} - 1)/(q^{-N} - 1)$ .



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