REMARKS ON OPEN SETS

Observation 1. There are countably many rational numbers.

Corollary 1. Any disjoint collection of open intervals of \( \mathbb{R} \) is countable.

Proof. Select a rational number from every interval. Since the intervals are disjoint, the number of intervals cannot be greater than the number of rational numbers. □

Remark 1. This argument generalizes to \( \mathbb{R}^n \), since the points with rational coordinates are dense and there are countably many of them.

Remark 2. The corollary remains true when the word “intervals” is replaced by “sets”.

Lemma 1. Any open subset \( A \subseteq \mathbb{R} \) can be written as the disjoint union of countably many open intervals.

Hint. There are many open intervals around. How to combine these to cover \( A \) in the desired way?

Proof. For any \( a \in A \), define \( U_a \) to be the union of all open intervals \((c, d) \subseteq A\) which contain the point \( a \).

First, for each \( a \in A \), \( U_a \subseteq A \) and \( U_a \) is open because it is a union of open subsets of \( A \).

Second, consider \( a, b \in A \). If there exists an interval \((c, d) \subseteq A\) with \( a, b \in (c, d) \), then for any \( a' \in U_a \), let us show \( a' \in U_b \). By construction there must be some interval \((c', d') \subseteq A\) containing both \( a \) and \( a' \). Now \((c', d') \cap (c, d) \neq \emptyset\) and so the union of these two intervals is an open interval containing both \( a' \) and \( b \), as desired. This argument shows that \( U_a \cap U_b \neq \emptyset \) iff there is an interval in \( A \) containing both \( a \) and \( b \) iff \( U_a = U_b \). If \( U_a \neq U_b \) then they are disjoint. By a similar argument, given any \( a', b \) both in \( U_a \), all elements \( x \) such that \( a' \leq x \leq b \) must belong to \( U_a \), so \( U_a \) is an interval. Thus \( U_a \) is the largest open interval in \( A \) which contains \( a \).

Choose any subset \( A' \subseteq A \) to be maximal such that \( a, b \in A' \) implies \( a \notin U_b \) (i.e. choose a maximal set of points in distinct intervals.) The union of the sets \( U_c \ (c \in A') \) is precisely our original set \( A \), because on one hand each \( U_a \subseteq A \), and on the other each \( a \in A \) belongs to \( U_c \) for some \( c \in A' \); if not, we contradict maximality of \( A' \). By choice of \( A' \), these sets \( U_c \) are disjoint. By the Corollary above there are at most countably many, and so we finish. □

Second proof (sketch). Define an equivalence relation on elements of \( X \) as follows. For any two elements \( a, b \in X \), say \( a \sim b \) iff \([a, b] \subseteq S\). Check that this is an equivalence relation on \( X \) (text, p. 15). The equivalence classes will be open intervals, and different equivalence classes will be disjoint (why?). Therefore by the Corollary above, there are at most countably many of them. □