1. Logic

1.1. Logical Systems

In a logical system, the symbols used are elements of a set \( L \), and the interpretation of these symbols is given through a function \( I \). A logical system consists of a set of symbols and a binary relation \( \models \) that defines the truth conditions for the symbols. The set of formulas is a subset of the set of sequences of symbols, and the logical system is a set of rules for manipulating these formulas.

1.2. Theorem

Theorem: For a given logical system \( \mathcal{L} \) and a set of formulas \( \mathcal{F} \), the following property holds:

\[ \text{For all } \phi, \psi \in \mathcal{F}, \text{ if } \phi \models \psi \text{ then } \forall x. (\phi(x) \Rightarrow \psi(x)) \]

Proof: By the semantics of \( \mathcal{L} \), if \( \phi \models \psi \), then \( \phi \Rightarrow \psi \) is true in all models of \( \mathcal{L} \). For any variable \( x \), by the semantics of \( \mathcal{L} \), \( \phi(x) \Rightarrow \psi(x) \) is also true in all models of \( \mathcal{L} \).

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Theorem: For a given logical system \( \mathcal{L} \) and a set of formulas \( \mathcal{F} \), the following property holds:

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does not contain a second-order quantifier.

\[ \phi \models [\forall \Pi] \iff \phi \models [\forall \Pi \eta] \]

I. A. Existence, Let be given.

1. 1. Properties of logical systems for which the compactness theorem holds.

\[ \Gamma \models \phi \iff \Gamma \models \psi \]

1. 2. Lemma, and Compactness Theorem (\( \Gamma \models \phi \iff \Gamma \models \psi \)).

1. 3. Let be a regular logical system such that

sat \( \phi \) is a set of sentences of which \( \phi \) is a member.

1. 4. The compactness theorem holds for \( \phi \) if and only if

sat \( \phi \) is a set of sentences of which \( \phi \) is a member.

1. 5. Let be a model of \( \phi \).

1. 6. Let be a set of sentences of which \( \phi \) is a member.

model \( \phi \) is a set of sentences of which \( \phi \) is a member.

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II. Characterizing First-Order Logic

[Continued on next page]
For all \( \phi \), we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Because the substitution in \( \mathcal{S} \) is defined as the empty substitution, we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

For all \( \phi \), we similarly have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Two substitutions are identical if and only if:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Therefore, since \( \mathcal{S} \) is a finite set of substitutions, there are finitely many substitutions in \( \mathcal{S} \).

So there is a finite set of substitutions in \( \mathcal{S} \).

By Lemma 2.2, there is a finite set of substitutions in \( \mathcal{S} \).

Since a finite set of substitutions is a finite set, we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Thus, the substitution in \( \mathcal{S} \) is identical to the substitution in \( \mathcal{S} \).

Therefore, we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

For all \( \phi \), we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Since \( \mathcal{S} \) is finite, we can define a finite set of substitutions in \( \mathcal{S} \).

Therefore, we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

Finally, we have:

\[
\phi_{\mathcal{S}} = \phi_{\mathcal{S}}
\]

In this section, we will prove that a regular logical system is sound.
We now have all tools available to obtain the following characterization of

first-order logic.

Theorem 3.5. Let $S$ be a regular logical system such that

- $S$ is a first-order theory.
- $L(S)$ is a consistent, first-order language.
- $T$ is a set of sentences in $L(S).

Then $T$ is sound and complete in $S$ if and only if $T$ is satisfiable by a model in $S$.

Proof. Suppose $T$ is satisfiable by a model $M$ in $S$. Then $T$ is a set of sentences in $L(S)$.

We define on $S$ the following semantic consequence relation $\models_{S}$: $A \models_{S} B$ if and only if $B$ is true in all models of $S$.

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We define on $S$ the following semantic consequence relation $\models_{S}$: $A \models_{S} B$ if and only if $B$ is true in all models of $S$.
The diagram in the middle of the page illustrates the relationship between the sets A and B, where A is a subset of B, and the operations and relations involved in the context of the logical expressions presented.

The text on the page continues with the discussion of logical expressions, focusing on the subset inclusion and the operations involving these sets. The expressions and theorems are presented in a formal mathematical notation, characteristic of advanced mathematical logic.

The bottom of the page contains a section of text that appears to be a continuation of the discussion, possibly introducing new theorems or propositions related to the set theory and logical expressions presented earlier in the document.
For instance, if there is no consistent model of a non-discrete logical system, then there is no consistent model of a discrete logical system. In such cases, any model $\mathcal{M}$ of a discrete logical system contains a sub-model $\mathcal{N}$ of a non-discrete logical system.

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4.2 Definition Let $\mathcal{L}$ be a discrete logical system.

There is a finite subset $\mathcal{L}_0$ of such that $\mathcal{L} \subseteq \mathcal{L}_0$.

4.3 Definition Let $\mathcal{L}$ be a non-discrete logical system. $\mathcal{L}$ is called an effective logical system if there is a consistent model $\mathcal{M}$ of $\mathcal{L}$.

When speaking of a decidable set of words, we understand a set of words that is decidable relative to $\mathcal{L}_0$. The set of words is decidable relative to $\mathcal{L}_0$ if there is a finite subset $\mathcal{L}_0$ of such that $\mathcal{L} \subseteq \mathcal{L}_0$.

In our consideration of logical systems, we now pay special attention to

§ 4.4. Laidon's Second Theorem

But if there is no consistent model of a non-discrete logical system, then there are no consistent models of a non-discrete logical system. If there is a consistent model of a non-discrete logical system, then there is a consistent model of a discrete logical system.

In our consideration of logical systems, we now pay special attention to the properties of such systems. Whether or not such systems exist is one of the most interesting open questions in current research.