From $\forall x x = x$ (in axiom group 5) we deduce

$$fx_1x_2 = fx_1x_2.$$  

The three displayed formulas tautologically imply

$$x_1 = y_1 \rightarrow x_2 = y_2 \rightarrow fx_1x_2 = fy_1y_2.$$  

**EXAMPLE**

(a) $\{\forall x (P_x \rightarrow Q_x), \forall z P z\} \vdash Qc$. It is not hard to show that such a deduction exists. The deduction itself consists of seven formulas.

(b) $\{\forall x (P_x \rightarrow Q_x), \forall z P z\} \vdash Qy$. This is just like (a). The point we are interested in here is that we can use the *same* seven-element deduction, with $c$ replaced throughout by $y$.

(c) $\{\forall x (P_x \rightarrow Q_x), \forall z P z\} \vdash \forall y Qy$. This follows from (b) by the generalization theorem.

(d) $\{\forall x (P_x \rightarrow Q_x), \forall z P z\} \vdash \forall x Qx$. This follows from (c) by use of the fact that $\forall y Qy \vdash \forall x Qx$.

Parts (a) and (b) of the foregoing example illustrate a sort of interchangeability of constant symbols with free variables. This interchangeability is the basis for the following variation on the generalization theorem, for which part (c) is an example. Part (d) is covered by Corollary 24G. $\varphi$ is, of course, the result of replacing $c$ by $y$ in $\varphi$.

**THEOREM 24F (GENERALIZATION ON CONSTANTS)** Assume that $\Gamma \vdash \varphi$ and that $c$ is a constant symbol that does not occur in $\Gamma$. Then there is a variable $y$ (which does not occur in $\varphi$) such that $\Gamma \vdash \forall y \varphi^y_c$. Furthermore, there is a deduction of $\forall y \varphi^y_c$ from $\Gamma$ in which $c$ does not occur.

**PROOF.** Let $(\alpha_0, \ldots, \alpha_n)$ be a deduction of $\varphi$ from $\Gamma$. (Thus $\alpha_n = \varphi$.) Let $y$ be the first variable that does not occur in any of the $\alpha_i$'s.

We claim that

$$((\alpha_0)^y_c, \ldots, (\alpha_n)^y_c)$$  

is a deduction from $\Gamma$ of $\varphi^y_c$. So we must check that each $(\alpha_k)^y_c$ is in $\Gamma \cup \boxed{\Lambda}$ or is obtained from earlier formulas by modus ponens.

Case 1: $\alpha_k \in \Gamma$. Then $c$ does not occur in $\alpha_k$. So $(\alpha_k)^y_c = \alpha_k$, which is in $\Gamma$.

Case 2: $\alpha_k$ is a logical axiom. Then $(\alpha_k)^y_c$ is also a logical axiom. (Read the list of logical axioms and note that introducing a new variable will transform a logical axiom into another one.)

Case 3: $\alpha_k$ is obtained by modus ponens from $\alpha_i$ and $\alpha_j$ (which is $(\alpha_i \rightarrow \alpha_j)$ for $i, j$ less than $k$. Then $(\alpha_k)^y_c = ((\alpha_i)^y_c \rightarrow (\alpha_j)^y_c)$. So $(\alpha_k)^y_c$ is obtained by modus ponens from $(\alpha_i)^y_c$ and $(\alpha_j)^y_c$. 


This completes the proof that (§) above is a deduction of \( \varphi^c \).
Let \( \Phi \) be the finite subset of \( \Gamma \) that is used in (§). Thus (§) is a deduction of \( \varphi^c \) from \( \Phi \), and \( y \) does not occur in \( \Phi \). So by the generalization theorem, \( \Phi \vdash \forall y \varphi^c_y \). Furthermore, there is a deduction of \( \forall y \varphi^c_y \) from \( \Phi \) in which \( c \) does not appear. (For the proof to the generalization theorem did not add any new symbols to a deduction.) This is also a deduction from \( \Gamma \) of \( \forall y \varphi^c_y \).  

We will sometimes want to apply this theorem in circumstances in which not just any variable will do. In the following version, there is a variable \( x \) selected in advance.

**Corollary 24G** Assume that \( \Gamma \vdash \varphi^x_c \), where the constant symbol \( c \) does not occur in \( \Gamma \) or in \( \varphi \). Then \( \Gamma^c \vdash \forall x \varphi \), and there is a deduction of \( \forall x \varphi \) from \( \Gamma \) in which \( c \) does not occur.

**Proof.** By the above theorem we have a deduction (without \( c \)) from \( \Gamma \) of \( \forall y ((\varphi^x_c)^y) \), where \( y \) does not occur in \( \varphi^x_c \). But since \( c \) does not occur in \( \varphi \),

\[
(\varphi^x_c)^y = \varphi^x.
\]

It remains to show that \( \forall y \varphi^x \vdash \forall x \varphi \). We can easily do this if we know that

\[
(\forall y \varphi^x) \rightarrow \varphi
\]

is an axiom. That is, \( x \) must be substitutable for \( y \) in \( \varphi^x \), and \( (\varphi^x)^y \) must be \( \varphi \). This is reasonably clear; for details see the re-replacement lemma (Exercise 9).  

**Corollary 24H (Rule EI)** Assume that the constant symbol \( c \) does not occur in \( \varphi, \psi \), or \( \Gamma \), and that

\[
\Gamma; \varphi^c \vdash \psi.
\]

Then

\[
\Gamma; \exists x \varphi \vdash \psi
\]

and there is a deduction of \( \psi \) from \( \Gamma; \exists x \varphi \) in which \( c \) does not occur.

**Proof.** By contraposition we have

\[
\Gamma; \neg \psi \vdash \neg \varphi^c
\]

So by the preceding corollary we obtain

\[
\Gamma; \neg \psi \vdash \forall x \neg \varphi.
\]

Applying contraposition again, we have the desired result.