Definition 1. A set is called transitive if $y \in z$ and $z \in y$ imply $z \in x$ (or equivalently, if $y \in x \implies y \subseteq x$).

Call a set an $N$-set if it is transitive and well-ordered by $\epsilon$.

(1) (a) Give an example of a transitive set and a non-transitive set.
    (b) Show that every element of an $N$-set is an $N$-set.

(2) (a) If $X$ is an $N$-set and $Y \subseteq X$ is one of its initial segments, show that $Y$ is an $N$-set and either $Y = X$ or $Y \in X$.
    (b) If $X, Y$ are $N$-sets, show that: either $X = Y$ or $X \in Y$ or $Y \in X$.

(3) Given any two $N$-sets $x, y$, define $x < y$ iff $x \in y$. Show that the relation $<$ is irreflexive, transitive (in the usual sense: $(x < y) \land (y < z) \implies (x < z)$), and trichotomous ($\forall x, y(x \leq y \lor y \leq x)$ and if both hold then $x = y$). Show that moreover, if $B$ is a nonempty set of $N$-sets, then there is a smallest element of $B$ with respect to $<$ (“well order”).

(4) Show that the following are true:
    - $1 + \omega = \omega$, $\omega + 1 \neq \omega$
    - $2 \cdot \omega = \omega$, $\omega \cdot 2 \neq \omega$

(5) For each of the following, which one is bigger? (Briefly justify your answer.)
    (a) $\omega + k$ or $k + \omega$ (for $k$ a positive integer)
    (b) $k \cdot \omega$ or $\omega \cdot k$ (for $k$ a positive integer)
    (c) $\omega + \omega_1$ or $\omega_1 + \omega$
    (d) $P(\omega) = \omega^n \cdot a_n + \cdots + \omega \cdot a_1 + a_0$ or $\omega^{n+1}$, where $n \geq 1$ and $a_0, \ldots, a_n$ are natural numbers
    (e) $P(\omega) = \omega^n \cdot a_n + \cdots + \omega \cdot a_1 + a_0$ or $Q(\omega) = \omega^m \cdot b_m + \cdots + \omega \cdot b_1 + b_0$, where $n, m, a_0, \ldots, a_n, b_0, \ldots, b_m$ are natural numbers.