MATH 277: HINTS FOR HOMEWORK 8 PROBLEMS

Please note: These are informal suggestions intended to explain how to do the problems. On exams or homework, you should almost certainly give more details.

(1) Let T be a complete theory and let $p(x) \in S_1(T)$ be a complete 1-type consistent with T. Prove that the following are equivalent:
   (a) In every model $M \models T$, there are only finitely many distinct elements which realize p. (Recall that a realizes p in M if $M \models \varphi(a)$ for every $\varphi(x) \in p(x)$.)
   (b) There is a formula $\varphi(x) \in p(x)$ and a natural number k such that in every $M \models T$, $\{a \in \mathrm{dom}(M) : M \models \varphi(a)\}$ has cardinality $\leq k$.

   Suggestion. (1) implies (2): Let $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \mathbb{N}\}$ be a language with countably many new constants. Let $T' = T \cup \{\psi(c_n) : \psi \in p, n \in \mathbb{N}\} \cup \{c_n \neq c_m : n \neq m \in \mathbb{N}\}$. In other words, this theory says that there are countably many distinct elements (constants) all of which realize the type p. By (1), we know that T' must be inconsistent. Use compactness to show that there must be some finite subset $\{\psi_1(x), \ldots \psi_k(x)\}$ and some finite m such that T proves that there do not exist m distinct elements satisfying $\bigwedge_{i \leq k} \psi_i(x)$. Note that this conjunction in fact a formula of the type because p is complete, i.e. maximal consistent.

   (2) implies (1): Actually, to step back for a moment, notice that (2) tells us that in no model of T can there be infinitely many distinct elements satisfying $\varphi(x)$. Formally,
   $T \cup \{\exists x_1 \ldots \exists x_n \left(\bigwedge_{i<j\leq n} x_i \neq x_j \wedge \bigwedge_{i \leq n} \varphi(x_i) \right) : n \in \mathbb{N}\}$ is not consistent. By compactness, there is a finite subset which is inconsistent. [You could have begun the problem here:] That is, there is $k \in \mathbb{N}$ such that $T \models$ "there are no more than k distinct elements satisfying φ". (Note that we can in fact replace $\models$ with $\vdash$.) In particular, since only k elements can realize φ, there can be at most k elements realizing the whole type in any model of T.

(2) A class $\mathcal{K}$ of $\mathcal{L}$-structures is said to be an *elementary class* if and only if there exists an $\mathcal{L}$-theory T such that $\mathcal{K}$ is exactly the class of models of T. Let $\mathcal{L} = \{<\}$ and let $\mathcal{K}$ be the class of models in which $<$ is a well-ordering (i.e. it is a linear order on the domain and every nonempty subset of the domain has a $<$-least element). Show that $\mathcal{K}$ is not an elementary class.

   Suggestion. Suppose for a contradiction that $\mathcal{K}$ is the class of models of some theory T. Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$ and for each n, let $\psi_n(x)$ say that there exist at least n distinct elements less than x. Use compactness to show that $T \cup \{\psi_n(c) : n \in \mathbb{N}\}$ is consistent and therefore satisfiable. Explain why the reduct to $\mathcal{L}$ of any model of this theory will not be well ordered.

(3) Let T be a complete theory and $p(\overline{x}) = p(x_1, \ldots x_n)$ a type consistent with T. The type $p(\overline{x})$ is said to be *principal* if there is a formula $\varphi(\overline{x}) \in p(\overline{x})$ such that for all formulas $\psi(\overline{x}) \in p$, $T \vdash \forall \overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$.

Prove that any principal type consistent with T is realized in every model of T.

Suggestion. By definition of "consistent with T" and the fact that T is complete, $T \vdash \exists x_1 \ldots \exists x_n \varphi(x_1, \ldots x_n)$. So this will be true in any model of T. Use the definition of "principal" to show that any $a_1, \ldots a_n$ in the domain of a model $M \models T$ satisfying $\varphi(a_1, \ldots a_n)$ will realize every other formula in the type too.

(4) Give an example of a complete theory T, a nonprincipal type consistent p with T, and a model $M \models T$ in which p is not realized. Justify your answer.

Suggestion. Let $T = Th(M)$ for $M = \langle \mathbb{N}; S \rangle$ be the theory of the natural numbers with successor and let $p(x)$ be the type in one free variable given by

$$\{\neg(x = S^n(0)) : n \in \mathbb{N}\}$$

(recall that 0 is definable). This type is consistent with T by compactness, but it is not realized in M. So by the previous problem, since T is complete, it cannot be principal.

(5) Let T be complete and $p(x_1, \ldots x_n)$ a type consistent with T. Prove that the following are equivalent:

(a) p is not principal.

(b) For every formula $\psi(x_1, \ldots x_n)$ consistent with T, there exists $\varphi \in p$ such that $\psi \wedge \neg\varphi$ is consistent with T.

Suggestion. (2) implies (1): Use the contrapositive and the definition of "principal."

(1) implies (2): Use the contrapositive and the definition of "principal." (There is a formula ψ such that for all φ in p, $\psi \wedge \neg\varphi$ is not consistent with T...)