MATH 277: HOMEWORK 8

DUE WEDNESDAY, DECEMBER 1 AT 3PM

All problems take place in first-order logic.

This week's assignment goes a little deeper into compactness. Each of these problems, excepting the example 4, can be solved by a simple application of compactness. The point is to think through each definition until you see precisely how and why this should work.

(1) Let T be a complete theory and let $p(x) \in S_1(T)$ be a complete 1-type consistent with T. Prove that the following are equivalent:
   (a) In every model $M \models T$, there are only finitely many distinct elements which realize p. (Recall that a realizes p in M if $M \models \varphi(a)$ for every $\varphi(x) \in p(x)$.)
   (b) There is a formula $\varphi(x) \in p(x)$ and a natural number k such that in every $M \models T$, $\{a \in \mathrm{dom}(M) : M \models \varphi(a)\}$ has cardinality $\leq k$.

(2) A class $\mathcal{K}$ of $\mathcal{L}$-structures is said to be an *elementary class* if and only if there exists an $\mathcal{L}$-theory T such that $\mathcal{K}$ is exactly the class of models of T. Let $\mathcal{L} = \{<\}$ and let $\mathcal{K}$ be the class of models in which $<$ is a well-ordering (i.e. it is a linear order on the domain and every nonempty subset of the domain has a $<$-least element). Show that $\mathcal{K}$ is not an elementary class.

(3) Let T be a complete theory and $p(\overline{x}) = p(x_1, \ldots x_n)$ a type consistent with T. The type $p(\overline{x})$ is said to be *principal* if there is a formula $\varphi(\overline{x}) \in p(\overline{x})$ such that for all formulas $\psi(\overline{x}) \in p$, $T \vdash \forall \overline{x}(\varphi(\overline{x}) \to \psi(\overline{x}))$.
   Prove that any principal type consistent with T is realized in every model of T.

(4) Give an example of a complete theory T, a nonprincipal type consistent p with T, and a model $M \models T$ in which p is not realized. Justify your answer.

(5) Let T be complete and $p(x_1, \ldots x_n)$ a type consistent with T. Prove that the following are equivalent:
   (a) p is not principal.
   (b) For every formula $\psi(x_1, \ldots x_n)$ consistent with T, there exists $\varphi \in p$ such that $\psi \wedge \neg\varphi$ is consistent with T.