(1) (a) Suppose $\mathcal{L} = \{ f \}$, a binary function symbol. Write down sentences of first order logic expressing “$f$ is surjective” and “$f$ is injective.”

(b) Suppose $\mathcal{L} = \{ +, \times, 0, 1 \}$. Write down infinitely many sentences of first order logic which together express “every polynomial has a root” and “the characteristic is zero.”

(2) Let $T$ be a consistent set of sentences in some language $\mathcal{L}$. Prove that it can be extended to a maximal consistent set of sentences.

(3) Let $M$ be a model in some language $\mathcal{L}$ and suppose that every element of $M$ is the interpretation of some constant symbol. Verify that $T = Th(M)$ has witnesses in $\mathcal{L}$.

(4) Prove that if $\langle M_n : n < \omega \rangle$ is an increasing chain of $\mathcal{L}$-models and $N$ is their union, then any sentence $\varphi$ which is “$\forall \cdots \exists \cdots$” and which is true in each $M_n$ is also true in $N$.

(5) Let $\mathbb{F}_p$ denote the algebraic closure of the finite field with $p$ elements (so, a countable algebraically closed field of characteristic $p$). Let $\mathcal{D}$ be a nonprincipal ultrafilter on $P$, the primes. Prove\footnote{You are welcome to use the fact that if $\mathcal{D}$ is a nonprincipal ultrafilter on a countable set $I$, and each $M_i (i \in I)$ is countable, then the ultraproduct $\prod_{i \in I} M_i / \mathcal{D}$ has size continuum.} that the ultraproduct $\prod_{p \in P} \mathbb{F}_p / \mathcal{D}$ is an algebraically closed field of characteristic zero and size $\mathfrak{c}$. (Briefly check the axioms directly.)

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Turn over for the Challenge Problem.
This problem is about thinking through some interesting definitions. It continues ideas from HW2.5.

A family $\mathcal{F}$ of infinite subsets of $\omega$ has a pseudointersection, sometimes called an infinite pseudointersection for emphasis, if there is an infinite $A \subseteq \omega$ such that $A \subseteq^* B$ for every $B \in \mathcal{F}$, where $\subseteq^*$ means that there are at most finitely many elements of $A$ which do not belong to $B$. Prove that the family $\mathcal{F}$ of co-finite subsets of $\omega$ has the strong finite intersection property, meaning that the intersection of any finitely many (but at least one) elements of the family is infinite, and that it has a pseudo-intersection. Prove that the family $\mathcal{G} = \{\omega \setminus \{0, \ldots, n\} : n < \omega\}$ is well ordered by $\subseteq^*$, and that it has a pseudo-intersection.

Let $p$ be the smallest size of a family of infinite subsets of $\omega$ with the strong finite intersection property and no pseudointersection. Let $t$ be the smallest size of a family of infinite subsets of $\omega$ which is well ordered by $\subseteq^*$ and has no pseudointersection. Verify that $p \leq t$. 