Assume each language has a binary relation symbol =, interpreted as equality.

(1) Let \( \mathcal{L} = \{ < \} \) and consider the model \( M_1 = (\mathbb{Q}; <) \), i.e. the rationals in which \( < \) is interpreted to mean the usual linear order. Prove that for any \( a, b \in \mathbb{Q} \), there is an automorphism of \( M_1 \) which sends \( a \) to \( b \).

(2) Give examples of a language \( \mathcal{L} \) and a model with infinite domain which does not have any automorphisms except for the identity: (a) one example where \( \mathcal{L} \) is allowed to be infinite, (b) a different example where \( \mathcal{L} \) is finite.

(3) Consider the model \( M_3 = (\mathbb{Z}; <) \) meaning: \( \text{Dom}(M_3) = \mathbb{Z}, \mathcal{L} = \{ < \} \), \( < \) is a binary relation interpreted as the usual linear order on \( \mathbb{Z} \).
   (a) Describe the submodels of \( M_3 \).
   (b) How does the answer change if we assume the language also has a constant symbol \( c \), interpreted as 0?
   (c) How does the answer change if we assume that, in addition to the constant from (b), the language also has a unary function \( S \) interpreted to mean “successor” (i.e. \( S_{M_3}(a) = a + 1 \) for each \( a \in \mathbb{Z} \))?  

(4) Consider a model \( M_4 \) in the language \( \mathcal{L} = \{ R \} \), \( R \) a binary relation symbol. Assume \( R \) is symmetric and irreflexive. Call \( M_4 \) special if it satisfies: there exist infinitely many elements, and for any \( n < \omega \), whenever \( \{ a_1, \ldots, a_n \} \), \( \{ b_1, \ldots, b_n \} \) are disjoint sets of distinct elements of \( \text{Dom}(M_4) \), there exists \( c \in \text{Dom}(M_4) \) such that \( i \leq n \implies R_{M_4}(c, a_i) \) and \( i \leq n \implies \neg R_{M_4}(c, b_i) \).
  Prove that any two countable (so, countably infinite) special graphs are isomorphic.

(5) Suppose the language has a binary relation symbol \( R \). Write down first order axioms to ensure
   (a) that \( R \) is a special graph.
   (b) that \( R \) is an equivalence relation with precisely one class of size \( n \) for each finite \( n \).

Please make your answers on this problem easy to read (also explain the formulas you have written in English).
Challenge problem (optional; turn in on a separate page):
Will any two special graphs of the same uncountable size also be isomorphic?