Unless otherwise specified, \( p \) is a fixed prime number \( \geq 2 \), and the setting is the \( p \)-adics \( \mathbb{Q}_p \) with the \( p \)-adic absolute value \( | \cdot |_p \) (p. 176) and the associated metric, defined in Problem 1.

(1) Exercise 5.4.2 page 176.

(2) Show that \( d_p(r, s) = |r - s|_p \) gives a metric on \( \mathbb{Q} \), but that \( \mathbb{Q} \) is not complete with respect to this metric.

(3) (“All triangles are isosceles”) Suppose \( x, y, z \in \mathbb{Q}_p \). Show that if \( |x - y|_p \neq |y - z|_p \) then \( |x - z|_p = \max(|x - y|_p, |y - z|_p) \). In other words, two sides of any triangle are equal.

(4) (“Every point of an open ball is a center”) For \( a \in \mathbb{Q}_p \) and \( \epsilon \) a positive real number, define as usual \( B_\epsilon(a) = \{ b : |a - b|_p < \epsilon \} \subseteq \mathbb{Q}_p \). Show that if \( b \in B_\epsilon(a) \), then \( B_\epsilon(b) = B_\epsilon(a) \).

(5) Show that for any \( a, \epsilon \) as in the previous problem, any open ball \( B_\epsilon(a) \) is both open and closed.

(6) Show that \( \{ x : |x|_p = 1 \} \) is not the boundary of \( B_1(0) \). What is the boundary?

(7) Working in \( \mathbb{Q}_p \), show that the ordinary integers \( \mathbb{Z} \) are dense in the \( p \)-adic integers.

(8) Consider the sequence \( \{a_n\}_{n \in \mathbb{N}} \) given by:

\[
a_1 = 4, \ a_2 = 34, \ a_3 = 334, \ a_4 = 3334, \ldots
\]

Show that in \( \mathbb{Q}_5 \), this sequence is Cauchy and converges to \( \frac{2}{7} \). (Suggestion: For the last part, show that for each \( n \), \( |3a_n - 2|_5 = 5^{-n} \).)

(9) (a) Show that the series \( \sum_{n=1}^{\infty} p^n \) converges in \( \mathbb{Q}_p \).

(b) Does the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) converge in \( \mathbb{Q}_p \)? Why or why not?

(10) Prove that the \( p \)-adic integers are a compact subset of \( \mathbb{Q}_p \).

CP: Assigned in class.