MATH 203: HOMEWORK 1

DUE WEDNESDAY, APRIL 7 AT THE BEGINNING OF CLASS

(1) Prove that the scalar product is a positive definite symmetric bilinear form on \( \mathbb{E}^n \).

(2) If \( v, w \in \mathbb{E}^n \), prove that:
   (a) \( ||v + w|| \leq ||v|| + ||w|| \) (the triangle inequality)
   (b) \( ||v + w||^2 + ||v - w||^2 = 2(||v||^2 + ||w||^2) \) (the parallelogram law)

(3) Let \( v, w \in \mathbb{E}^n \). Prove that \( |\langle v, w \rangle| = ||v||||w|| \) iff they are linearly dependent.

(4) (a) Show that the 0 vector in \( \mathbb{E}^n \) is orthogonal to every vector in \( \mathbb{E}^n \).
   (b) If \( v \) is a nonzero vector in \( \mathbb{E}^n \), show that the collection \( W = \{ w \in \mathbb{E}^n | \langle w, v \rangle = 0 \} \) is an \((n - 1)\)-dimensional subspace of \( \mathbb{E}^n \).

(5) If \( v_1, \ldots, v_k \) are pairwise orthogonal nonzero vectors in \( \mathbb{E}^n \), show that they form a linearly independent set in \( \mathbb{E}^n \).

(6) Let \( v, w \in \mathbb{E}^3 \) be linearly independent and nonzero vectors and let \( \theta \) be the angle between \( v \) and \( w \). Show that
\[
\langle v, v \times w \rangle = \langle w, v \times w \rangle = 0
\]
\[
||v \times w|| = ||v||||w|| \sin(\theta)
\]

(7) Let \( v_1 = (2, -1, 1), \ v_2 = (1, 2, -1), \ v_3 = (1, 1, -2) \) be vectors in \( \mathbb{E}^3 \). Find all vectors \( v \in \mathbb{E}^3 \) of the form \( \alpha v_2 + \beta v_3 \) for \( \alpha, \beta \in \mathbb{R} \) which are orthogonal to \( v_1 \) and have length 1.

(8) Exercise 2.5.21 p. 74.

(9) Prove that three distinct points \( p_1, p_2, p_3 \in \mathbb{R}^3 \) lie on a line iff \( (p_2 - p_1) \times (p_3 - p_1) = 0 \). [In particular, this implies that the line through two distinct points \( p_2, p_3 \in \mathbb{R}^3 \) is the set of all points \( p \in \mathbb{R}^3 \) such that \( (p_2 - p) \times (p_3 - p) = 0 \).]

(10) A Hamel basis is a basis of the vector space of \( \mathbb{R} \) over the field of rationals \( \mathbb{Q} \), i.e. a set \( B = \{ b_i : i \in I \} \) such that every real \( r \) can be uniquely written in the form \( r = \alpha_0 b_0 + \cdots + \alpha_n b_n \), where \( n \) is finite, the \( \alpha_i \) are nonzero rationals and the \( b_i \) are distinct elements of \( B \). Prove that for any Hamel basis \( B \), \( \text{card}(B) > \aleph_0 \), i.e. a Hamel basis cannot be countable.

Week 1 Challenge Problems:

CP1: The rabbit problem, given in class 3/29.

CP2: (Proof or Counterexample) Let \((A, <_A)\) and \((B, <_B)\) be two linearly ordered sets. Suppose that there exist order-preserving injective maps \( f : A \rightarrow B \) and \( g : B \rightarrow A \). (In other words, \( f \) is an injection with the property that \( f(a_1) <_B f(a_2) \) iff \( a_1 <_A a_2 \), and similarly for \( g \).) Must there exist an order-preserving bijection \( h : A \rightarrow B? \)

CP3: Expression for cardinality of vector space in terms of that of basis and field, given in class 3/31.