COFINALITY SPECTRUM PROBLEMS: THE AXIOMATIC APPROACH

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Abstract. Our investigations are framed by two overlapping problems: finding the right axiomatic framework for so-called cofinality spectrum problems, and a 1985 question of Dow on the conjecturally nonempty (in ZFC) region of OK but not good ultrafilters. We define the lower-cofinality spectrum for a regular ultrafilter $\mathcal{D}$ on λ and show that this spectrum may consist of a strict initial segment of cardinals below λ and also that it may finitely alternate. We define so-called 'automorphic ultrafilters' and prove that the ultrafilters which are automorphic for some, equivalently every, unstable theory are precisely the good ultrafilters. We axiomatize a bare-bones framework called "lower cofinality spectrum problems", consisting essentially of a single tree projecting onto two linear orders. We prove existence of a lower cofinality function in this context and show by example that it holds of certain theories whose model theoretic complexity is bounded.

Dedicated to Alan Dow on the occasion of his birthday.

1. Introduction

Recall that two models M, N are elementarily equivalent, $M \equiv N$, if they satisfy the same sentences of first-order logic. A remarkable fact is that elementary equivalence may be characterized purely algebraically, without reference to logic:

Theorem A (Keisler 1964 under GCH; Shelah unconditionally)**.** $M \equiv N$ if and only if M, N have isomorphic ultrapowers, that is, if and only if there is a set I and an ultrafilter $\mathcal{D}$ on I such that $M^I/\mathcal{D} \cong N^I/\mathcal{D}$.

To prove this theorem, Keisler established that ultrafilters which are both regular and *good* exist on any infinite cardinal and that they have strong saturation properties. *For transparency in this introduction, all languages (e.g. vocabularies) are countable and all theories are (first-order) complete.* Regularity is an existential property of filters, showing a kind of strong incompleteness: a filter on I is regular if there exists a family $\mathcal{X} = \{X_i : i < |I|\} \subseteq \mathcal{D}$, called a regularizing family, such that the intersection of any infinitely many elements of $\mathcal{X}$ is empty. To motivate the definition of good, 1.1 below, we finish outlining the proof of Theorem A in the GCH case. If $\mathcal{D}$ is regular and good, and $|M| \leq |I|$, then $M^I/\mathcal{D}$ is of size $2^{|I|}$ (since $\mathcal{D}$ is regular) and is $|I|^+$-saturated (since $\mathcal{D}$ is in addition good). So choosing $|I| \geq \max\{|M|, |N|\}$ and assuming the relevant instance of GCH, the ultrapowers

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$M^I/\mathcal{D}$, $N^I/\mathcal{D}$ are elementarily equivalent, of the same cardinality, and saturated in that cardinality, therefore isomorphic.

Given a model M and an ultrafilter $\mathcal{D}$, let us abbreviate '$M^I/\mathcal{D}$ is $|I|^+$-saturated' by writing '$\mathcal{D}$ saturates M'. In the proof just sketched, the saturation properties of good ultrafilters are tempered by regularity as follows. A good ultrafilter on I will saturate any M which is itself $|I|^+$-saturated (this may be taken as a definition of good ultrafilter, but see also 1.1 below). If $\mathcal{D}$ is regular, then $\mathcal{D}$ saturates a model M if and only if it saturates all $N \equiv M$ (see Keisler [8] Theorem 2.1a). Since a good ultrafilter saturates some model in every elementary class (e.g. any one which is sufficiently saturated), a good regular ultrafilter saturates all models.

The usual definition of good filters is combinatorial. Call a function *monotonic* if $u \subseteq v$ implies $f(v) \subseteq f(u)$, and *multiplicative* if $f(u) \cap f(v) = f(u \cup v)$. In the following definition, it would suffice to consider all monotonic functions.

Definition 1.1 (Good filters, Keisler)**.** Let $\mathcal{D}$ be a filter on I. We say $\mathcal{D}$ is κ-good if for every $\rho < \kappa$, every function $f : [\rho]^{<\aleph_0} \rightarrow \mathcal{D}$ has a multiplicative refinement, i.e. there is $g : [\rho]^{<\aleph_0} \rightarrow \mathcal{D}$ which is multiplicative and such that $g(u) \subseteq f(u)$ for all $u \in [\rho]^{<\aleph_0}$. We say $\mathcal{D}$ is *good* if it is $|I|^+$-good.

This has proved to be a very fruitful definition. The existence of good regular ultrafilters, proved by Keisler under GCH and by Kunen unconditionally, may be understood as asserting existence of ultrafilters which are 'maximal' or 'complex' in at least two senses: in the sense that all functions have multiplicative refinements, or in the sense of being strong enough to saturate any model. As a result, proposed weakenings of this definition have traditionally taken either a more set-theoretic form or a more model-theoretic form. An interesting example of the first is the notion of an 'OK' ultrafilter; see Dow 1985 [4] p. 146 for the history. Note that the cardinal parameter differs from Definition 1.1, i.e. a κ^+-good filter is κ-O.K.

Definition 1.2 (OK filters)**.** Let $\mathcal{D}$ be a filter on I. We say $\mathcal{D}$ is κ-OK if every monotonic function $f : [\kappa]^{<\aleph_0} \rightarrow \mathcal{D}$ which satisfies $|u| = |v| \implies f(u) = f(v)$ has a multiplicative refinement. We say $\mathcal{D}$ is OK if it is $|I|$-OK.

It has been surprisingly difficult to distinguish OK from good. It follows from the existence of an $\aleph_1$-complete (non-principal) ultrafilter that there exist regular ultrafilters on any sufficiently large λ which are OK but not good (see for example Theorem 4.2 (4) $\not\rightarrow$ (5) and Theorem 7.4 of [15]). We do not know of any ZFC proofs.

However, in his paper Dow raises a stronger question: "the question of whether there can be α^+-OK ultrafilters which are not α^+-good."

Question 1.3 (Dow, cf. [4] 4.7)**.** Do there exist α^+-OK ultrafilters which are not α^+-good?

This question frames much of our present work. The theorem already quoted answers it assuming a measurable cardinal, and in fact allows for an arbitrary separation:

Theorem B ([15] 7.4, in the present language)**.** Assume $\aleph_0 < \kappa < \lambda$, $2^\kappa \leq \lambda$, κ measurable. Then there exists a regular ultrafilter $\mathcal{D}$ which is λ-O.K. but not $(2^\kappa)^+$-good.

However, Question 1.3 remains open in ZFC. One of the themes of this paper will be the apparent richness of the region between OK and good. To explain this, we return to the second direction mentioned after Definition 1.1, weakenings of goodness arising from model theory.

Recall that a regular ultrafilter $\mathcal{D}$ saturates a model M iff $\mathcal{D}$ saturates all $N \equiv M$. This means we can simply speak of $\mathcal{D}$ saturating a complete theory T (if $\mathcal{D}$ saturates one, equivalently all, of its models), and we may naturally compare theories T, T', by asking whether any regular $\mathcal{D}$ which saturates T must saturate T'. If so, following [8], we say that $T' \trianglelefteq T$ in *Keisler's order*. Among the regular ultrafilters, the good ultrafilters are those which can saturate any theory; moreover, there exist theories which are only saturated by good ultrafilters. This tells us Keisler's order has a maximum class. An early surprise was that this maximum class includes all theories of infinite linear order.[1]

Theorem C (Shelah 1978 [24] VI.2.6, in our language)**.** If $\mathcal{D}$ is a regular ultrafillter on I and $\mathcal{D}$ saturates $(\omega, <)$, then $\mathcal{D}$ is good.

In particular, following [17], one can define the cut spectrum of a regular ultrafilter $\mathcal{D}$ on I. Say that $N = (\omega, <)^I/\mathcal{D}$ has a (κ_1, κ_2)-cut if κ_1, κ_2 are regular and there exist sequences $(\langle a_\alpha : \alpha < \kappa_1\rangle, \langle b_\beta : \beta < \kappa_2\rangle)$ of elements of N such that for all $\alpha < \alpha' < \kappa_1$ and $\beta < \beta' < \kappa_2$, $a_\alpha < a_{\alpha'} < b_{\beta'} < b_\beta$, but there does not exist c such that $a_\alpha < c < b_\beta$ for all $\alpha < \kappa_1$ and $\beta < \kappa_2$.

Definition 1.4 (The cut spectrum of $\mathcal{D}$ [17] 2.1)**.** For $\mathcal{D}$ a regular ultrafilter on I,

$$\mathcal{C}(\mathcal{D}) = \{(\kappa_1, \kappa_2) : \kappa_1 + \kappa_2 \leq |I| \text{ and } (\omega, <)^I/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2)\text{-cut.}\}$$

Then when $\mathcal{D}$ is regular, $\mathcal{C}(\mathcal{D}) = \emptyset$ if and only if $\mathcal{D}$ is good, and we may try to understand the possible distance of a given ultrafilter from goodness by asking whether its cut spectrum is nonempty (and if so how).

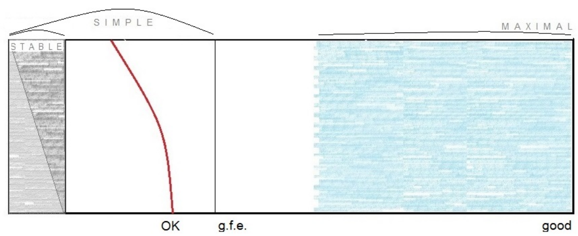
Here too model theory can help in proposing weakenings of goodness, by again leveraging Keisler's order. Choose a theory or a family of theories which appear to be, at least a priori, less difficult to saturate than linear order (or any other theory in the maximum class). Identify a property of ultrafilters which corresponds to saturation of that theory (or family of theories) and try to compare this new property to goodness. This was the approach taken in our recent paper [17], which also led to a proof that the cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ are equal. That work began from the question of whether theories with a certain model-theoretic tree property, called SOP_2, were maximal in Keisler's order. We first proved that a necessary condition for a regular ultrafilter $\mathcal{D}$ on I to saturate some theory with SOP_2 is the following: whenever $\mathcal{T}$ is a tree (i.e. a partially ordered set such that the set of predecessors of any given element is well ordered), any strictly increasing $|I|$-indexed path in the ultrapower $N = \mathcal{T}^I/\mathcal{D}$ has an upper bound in N. When this holds, say that $\mathcal{D}$ has *treetops* (really, $|I|^+$-treetops). We then investigated the distance of this property from goodness by asking: if $\mathcal{D}$ has treetops, is $\mathcal{C}(\mathcal{D}) = \emptyset$?

The surprising answer was yes [17, Theorem 10.1]. Its proof involved a systematic analysis of properties of cuts in $\mathcal{C}(\mathcal{D})$ with the local aim of eventually ruling all cuts out. We believe, however, that in contexts much more general than that paper, carefully revisiting this analysis under weaker hypotheses than treetops may give

[1] A consequence of regularity, requiring a brief argument, is that a regular ultrafilter $\mathcal{D}$ saturates $(\omega, <)$ if and only if it saturates $(X, <)$ where X is any infinite linear order.

much more information. A property of particular interest appears to be "uniqueness," essentially the property that if (κ, θ) and (κ, θ') belong to $\mathcal{C}(\mathcal{D})$ then $\theta = \theta'$. This property motivated many of the problems in our recent open questions paper [23], and it is the focus of Sections 2 and 4 of the present paper. Its conjectural relation to the existence of internal maps (another key driver of problems in [23], which will be explained in due course below) will connect it to our remaining topic and to Section 3.

Towards explaining this remark, we now discuss how a further open question in this framework fits an interesting model theoretic picture into Dow's question 1.3 above. To begin, here is a partial map of classes of theories in Keisler's order. The filled-in regions are classified, and the lines represent some known divisions. Along the top are some properties of theories, and along the bottom are some properties of ultrafilters.



After [17] established the maximality of SOP_2, a key question is whether or not all non-simple theories are maximal. This amounts to asking whether there are theories with a model theoretic property called TP_2 which are non-maximal. By T_{feq} we will mean the model completion of the theory of a parametrized family of independent equivalence relations, as in [30] Definition 1.7. (Any non-simple theory has at least one of SOP_2 or TP_2.) We know that among the TP_2 theories T_{feq} is minimal, and that the following property of a regular ultrafilter is necessary and sufficient for saturating T_{feq} [13, Lemma 6.7–Theorem 6.10]. (Given an ultrapower $N = M^I/\mathcal{D}$, fix in advance a lifting $M^I/\mathcal{D} \to M^I$ so that for $a \in N$, $t \in I$ the coordinate projection $a[t]$ is well defined.)

Definition 1.5 (Good for equality, defined in [13] and named in [14])**.** Call the regular ultrafilter $\mathcal{D}$ on I *good for equality* if whenever M is an infinite model and $A \subseteq M^I/\mathcal{D}$, $|A| \leq I$, there is a map $f : A \to \mathcal{D}$ such that for all $t \in I$, the sequence $\langle a[t] : a \in A$ satisfies $t \in f(a) \rangle$ is without repetition.

The picture above also reflects that in [13] a property of ultrafilters called 'flexible' which is equivalent to OK, and which has model theoretic content, was discovered. Moreover, good for equality implies flexible (= OK) [13, Lemma 8.7 and 8.8]. For our present purposes, this tells us that the conjectural distance between good for equality and good is contained in that between OK (=flexible) and good.[2] So Dow's question 1.3 may be illuminated by advances in understanding the structure of Keisler's order. For example, recently [20, Theorem 8.2], which separated the simple from the non-simple theories in Keisler's order under a large cardinal hypothesis, also gave a new OK but not good for equality – thus, not good – ultrafilter; and moreover the so-called optimal ultrafilters built there yield a new

[2]Note that the line corresponding to "OK" is not if-and-only-if. That is, it is necessary that any regular ultrafilter which saturates a theory to the right of the drawn line (in model-theoretic language, a non-low simple theory or a non-simple theory) be OK. However, this is not sufficient.

family of examples answering 1.3, assuming existence of a supercompact cardinal, as explained in [20, Conclusion 5.18].[3]

Having explained these three interrelated areas arising from the problem of weakening goodness – the question of good versus OK, the problem of the cut spectrum of $\mathcal{D}$ and in particular the question of uniqueness, and the question of good versus good for equality – we now outline our main results. In Section 2, we formalize and investigate uniqueness spectra, primarily of regular ultrapowers. We prove that uniqueness can hold precisely on certain initial segments of cardinals and that it may alternate. In Section 3, we show that in the distance between good for equality and good there arises an automorphism problem which aligns the increasing complexity of first order theories with the increasing complexity of internal maps in ultrapowers. Investigating this problem we are able to give a new characterization of goodness: $\mathcal{D}$ is good if and only if $\mathcal{D}$ is so-called automorphic for all unstable theories (defined below). As a result we can re-frame several open questions. In Sections 4–5, we present an axiomatic approach, showing that even under very weak notions of order and filters one can recover certain uniqueness phenomena. We discuss some model-theoretic examples and limitations of such results.

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2. The uniqueness spectrum

In this section we define and investigate lower cofinality spectra in ultrapowers. The specific phenomenon we study is the following. Specializing to the case of ultrapowers, [17, Theorem 3.2] says that under the hypothesis of treetops the following 'uniqueness' phenomenon arises.[4]

Fact 2.1 (cf. [17] Theorem 3.2)**.** Let $\mathcal{D}$ be a regular ultrafilter on I, with $|I|^+$-treetops. Then for each regular $\kappa \leq |I|$, there is precisely one θ such that $(\omega, <)^I/\mathcal{D}$ has a (κ, θ)-cut.

Note that regularity of $\mathcal{D}$ implies that $(\omega, <)^I/\mathcal{D}$ has a (κ, θ)-cut iff it has a (θ, κ)-cut so without loss of generality we have focused on the first coordinate.

Definition 2.2 (Internal, in ultrapowers)**.** Let $N = M^I/\mathcal{D}$ be an ultrapower. Let M^+ denote the expansion of the theory of M by adding all possible relations, functions, and constant symbols. Then the internal functions and relations are precisely those definable in $N^+ := (M^+)^I/\mathcal{D}$, recalling that ultrapowers commute with expansion and reduction.[5]

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[3]Somewhat more is currently known about the structure of Keisler's order that what is shown on the map above: for example, on the simple theories, in the region to the left of the "OK" line: see [21]. A further discussion of connections between model-theoretic properties of theories and set-theoretic properties of ultrafilters may be found in [15] §4.

[4]Fact 2.1 and Corollary 2.3 both translate a result originally stated for so-called cofinality spectrum problems to ultrafilters. The specifics of such a translation are justified in [17] 10.17–10.21. Regarding 'treetops,' recall from the introduction that this means: whenever $\mathcal{T}$ is a tree, any strictly increasing $|I|$-indexed path in the ultrapower $N = \mathcal{T}^I/\mathcal{D}$ has an upper bound in N.

[5]Equivalently, we say that a relation $R \subseteq N^k$ is *internal* if for each $t \in I$ we may expand $M = M_t$ by interpreting a new k-ary relation symbol P in such a way that $\prod_t (M_t, P)/\mathcal{D} = (N, R)$. We say that a partial function $f : N^k \to N$ is internal if its graph is an internal relation which is the graph of a function.

One strong reason that uniqueness can arise is if there is an internal order preserving map between any two monotonic κ-indexed sequences in the ultrapower. This may seem like a lot to ask, but in fact when the ultrafilter is good this is what happens:[6]

Fact 2.3 (special case of [17] Corollary 3.8)**.** Let $\mathcal{D}$ be a good regular ultrafilter on I. Suppose κ is regular and $\kappa \leq |I|$, and let $N = (\omega, <)^I/\mathcal{D}$. Let $\overline{a} = \langle a_\alpha : \alpha < \kappa\rangle$ and $\overline{b} = \langle b_\alpha : \alpha < \kappa\rangle$ be two strictly monotonic sequences of elements of N. Then there is a monotonic, internal partial one-to-one map f in N whose domain includes $\{a_\alpha : \alpha < \kappa\}$, and such that $f(a_\alpha) = b_\alpha$ for all $\alpha < \kappa$.

Remark 2.4. The proof of Fact 2.3, i.e. of [17] Corollary 3.8, also shows that existence of the map f corresponds to realization of a type in a larger language. Let $M = (\omega, <)$. Let M^+ be M expanded to a model of sufficient set theory: in this model, we have not only ω but also the set of all partial functions with domain an initial segment of ω, and range a subset of $\omega \times \omega$ strictly increasing in both coordinates. This set naturally has the structure of a tree, partially ordered by inclusion; call it $(\mathcal{T}, \triangleleft)$. Note that given an element of $\mathcal{T}$, its range may be thought of as the graph of a partial function between two strictly increasing subsequences of ω. Ultrapowers commute with reducts, so we have that $N = M^I/\mathcal{D}$ expands naturally to $N^+ \equiv M^+$. In this ultrapower, ω is nonstandard and so is $\mathcal{T}$. Consider the type of an element of $\mathcal{T}$ whose range is the graph of a partial function extending $a_\alpha \mapsto b_\alpha$. For a more general argument, see 3.11 below.

This discussion directs our attention to three related spectra.

Convention 2.5. For the remainder of the section, fix some sufficiently large (so uncountable) regular χ. By $\mathcal{H}(\chi)$ we mean[7] the sets hereditarily of cardinality $< \chi$. Let $\mathfrak{A} = (\mathcal{H}(\chi), \in)$. We use $\mathfrak{M}$ to denote a sufficiently saturated model of $Th(\mathfrak{A})$. In this case "internal" has the usual meaning.

Definition 2.6 (Lower cofinality spectra of ultrafilters or models of set theory)**.**

(1) For $\mathfrak{M}$ as in 2.5 and $\ell = 0, 1, 2$, let $\mathrm{LcfSp}_\ell(\mathfrak{M})$ be the set of $\kappa = \mathrm{cf}(\kappa)$ such that κ is an ordinal of $\mathfrak{M}$ and whenever $\bar{a}$, $\bar{b}$ are two strictly increasing sequences of length κ in $\mathfrak{M}$:
   (a) (if $\ell = 0$) $\bar{a}$, $\bar{b}$ have the same coinitiality in $\mathfrak{M}$.
   (b) (if $\ell = 2$) there is an internal order preserving map π in $\mathfrak{M}$ such that $\pi(a_\alpha) = b_\alpha$ for $\alpha < \kappa$.
   (c) (if $\ell = 1$) there is an internal order preserving map in $\mathfrak{M}$ such that for some unbounded $\mathcal{U} \subseteq \kappa$ we have: if $\alpha < \delta$ are from $\mathcal{U}$, then $\pi(a_\alpha) < b_\delta$ and $\pi^{-1}(b_\alpha) < a_\delta$.

(2) Let $\mathcal{D}$ be a regular ultrafilter on I. For $\ell = 0, 1, 2$, let $\mathrm{LcfSp}_\ell(\mathcal{D})$ be the set of $\kappa = \mathrm{cf}(\kappa) \leq |I|$ such that $\kappa \in \mathrm{LcfSp}_\ell((\omega, <)^I/\mathcal{D})$.

We write LcfSp for LcfSp_0.

Discussion 2.7. In Definition 2.6(2), the assumption that $\mathcal{D}$ is a regular ultrafilter entails that we could replace $(\omega, <)$ there by $\mathfrak{M}$, or by another infinite model of

[6] By [17, Theorem 10.1], the hypothesis "$\mathcal{D}$ has with $|I|^+$-treetops" is here replaced by "$\mathcal{D}$ is good" in the quotation of 2.3.

[7] More precisely, define $\mathcal{H}(\chi)$ to be the set of all x such that the cardinality of the transitive closure of x is $< \chi$. Then $\mathfrak{A} = (\mathcal{H}(\chi), \in)$ will be a model of ZFC minus the power set axiom.

linear order. The reason to cut off $\mathrm{LcfSp}_\ell(\mathcal{D})$ at $|I|$ is because we are often interested in measuring how far the ultrafilter is from good, as explained in the introduction. In general, however, we might simply have said:
"Given $\mathcal{D}$ a not necessarily regular ultrafilter on I, $\ell = 0, 1, 2$ and any regular cardinal κ, we say that $\kappa \in \mathrm{LcfSp}_\ell(\mathcal{D})$ when (for $\ell = 0, 2, 1$ respectively) 2.6(1)(a), (b) or (c) hold in the case where $\mathfrak{M}$ there is assumed to be of the form $\mathfrak{B}^I/\mathcal{D}$ for some κ^+-saturated $\mathfrak{B} \equiv \mathfrak{A}$."

Observation 2.8.

(1) Let $\mathcal{D}$ be an ultrafilter[8] on I. Then

$$\mathrm{LcfSp}_2(\mathcal{D}) \subseteq \mathrm{LcfSp}_1(\mathcal{D}) \subseteq \mathrm{LcfSp}_0(\mathcal{D}).$$

(2) If $\mathcal{D}$ is κ^+-good, $\kappa = \mathrm{cf}(\kappa)$, then $\kappa \in \mathrm{LcfSp}_2(\mathcal{D})$.
(3) If $\mathcal{D}$ is not $\aleph_1$-complete, then $\aleph_0 \in \mathrm{LcfSp}_2(\mathcal{D})$.
(4) If $\mathfrak{M}$ is κ^+-saturated then $\kappa \in \mathrm{LcfSp}_2(\mathfrak{M})$.

Proof. (1) follows from the definitions. To see that the second inclusion holds, let π be as in Definition 2.6, let $A = \{a_\alpha : \alpha < \kappa\}$ and let $C = \{c : c > a_\alpha \text{ for all } \alpha < \kappa\}$ be the set of A-nonstandard elements. Likewise let $B = \{b_\alpha : \alpha < \kappa\}$ and let D be the set of B-nonstandard elements. Let $C_{\leq c}$ denote $\{x \in C : x \leq c\}$. Observe that for no $c \in C$ can we have that $\mathrm{dom}(\pi) \cap C_{\leq c} \subseteq A$, as then $\{x : (\exists y \in \mathrm{dom}(\pi))(y > x)\}$ would define the cut given by $(A, C_{\leq c})$. Likewise, for no $c \in C$ can we have that for some $d \in D$, $\mathrm{range}(\pi \restriction C_{\leq c}) \cap D_{\leq d} \subseteq B$, as then we could define the cut $(B, D_{\leq d})$. This shows that for any (c, d) with $c \in C$ and $d \in D$ there is (c', d') with $c > c' \in C$, $d > d' \in D$ and $\pi(c') = d'$, which is enough to show that the co-initiality of C and of D are the same.

(2) is by Fact 2.3 above.

In (3), we can always build a monotonic partial internal map between any two monotonic ω-indexed sequences.

(4) follows from the observation that the existence of such a map may be expressed as a partial type, see Remark 2.4. □

Next we prove that LcfSp can be any cofinite initial segment of the regular cardinals $\leq |I|$. This requires two ingredients. The first is a theorem of the second author that explains how to set the coinitiality of the diagonal embedding[9] of κ in $(|I|, <)^I/\mathcal{D}$, denoted $\mathrm{lcf}(\kappa, \mathcal{D})$, for finitely many values of κ (all regular and $\leq |I|$).

Definition 2.9 (see [24] Definition 3.5 p. 357)**.** For an ultrafilter $\mathcal{D}$ on I and a regular cardinal κ we define $\mathrm{lcf}(\kappa, \mathcal{D})$ to be the smallest cardinality λ such that there is a subset of

$$\{a \in \kappa^I/\mathcal{D} \colon \kappa^I/\mathcal{D} \models \alpha < a \text{ for each } \alpha < \kappa\}$$

which is unbounded from below, and has cardinality λ.

Theorem D ([24] Theorem VI.3.12)**.** Suppose $\aleph_0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda^+$, each λ_i is regular, and $\lambda_{\ell+1} \leq \mu_\ell \leq 2^\lambda$, μ_ℓ regular, for $\ell < n$. Then for some regular λ_1-good but not $(\lambda_1)^+$-good ultrafilter $\mathcal{D}$ over λ, $\mathrm{lcf}(\kappa, \mathcal{D}) = \mu_\ell$ whenever $\lambda_\ell \leq \kappa < \lambda_{\ell+1}$.

[8]See previous discussion for the non regular case.

[9]By regularity of $\mathcal{D}$, the cut spectrum computed with $(\omega, <)$ or $(|I|, <)$ is the same, but the meaning of 'the diagonal embedding of κ' is clearer in the second case.

The second ingredient is our recent proof that the failure of goodness is always witnessed by a symmetric cut.

Theorem E (from [17] Theorem 10.26)**.** Let $\mathcal{D}$ be a regular ultrafilter on λ which is λ-good and not λ^+-good. Then $\mathcal{C}(\mathcal{D})$ has a (λ, λ)-cut.

In other words, recalling from the introduction that $\lambda \leq |I|$ implies $\mathcal{C}(\mathcal{D})$ is λ-good if and only if $\mathcal{C}(\mathcal{D}) \cap (\lambda \times \lambda) = \emptyset$, "the first cut is symmetric":

Corollary 2.10 ([17] Theorem 10.25)**.** For $\mathcal{D}$ a regular ultrafilter, if $\kappa = \min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathcal{C}(\mathcal{D})\}$ then $(\kappa, \kappa) \in \mathcal{C}(\mathcal{D})$.

The interaction of Theorems D and E gives the result about initial segments.

Claim 2.11. Suppose $\aleph_0 < \kappa < \lambda$ where κ is regular and $\lambda = \kappa^{+n}$ for some finite $n \geq 1$. Then there exists a regular ultrafilter $\mathcal{D}$ on λ such that

$$\mathrm{LcfSp}(\mathcal{D}) = \{\mu \colon \mu = \mathrm{cf}(\mu) \text{ and } \mu < \kappa\}.$$

Proof. Apply Theorem D in the case where for each $0 \leq \ell < n$, $\lambda_{1+\ell} := \kappa^{+\ell}$ and for each $0 \leq \ell < n$, $\mu_{1+\ell} = \kappa^{+\ell+1}$. In this ultrafilter $\mathcal{D}$, we know that $(\kappa^{\ell}, \kappa^{+\ell+1}) \in \mathcal{C}(\mathcal{D})$ for $0 \leq \ell < n$. Since $\mathcal{D}$ is κ-good, every regular $\mu < \kappa$ belongs to $\mathrm{LcfSp}(\mathcal{D})$. By Theorem D $\mathcal{D}$ is not κ^+-good, so by Theorem E we know that $\mathcal{C}(\mathcal{D})$ has a (κ, κ)-cut. Since $(\kappa, \kappa^+) \in \mathcal{C}(\mathcal{D})$, this proves $\kappa \notin \mathrm{LcfSp}(\mathcal{D})$. For $1 < \ell < n$, we have that both $(\kappa^{+\ell}, \kappa^{+\ell+1})$ and $(\kappa^{+\ell+1}, \kappa^{+\ell+2})$ belong to $\mathcal{C}(\mathcal{D})$, so clearly $\kappa^{+\ell} \notin \mathrm{LcfSp}(\mathcal{D})$, which completes the proof. □

Corollary 2.12. For any uncountable κ there is a regular ultrafilter $\mathcal{D}$ on κ such that $\kappa \notin \mathrm{LcfSp}(\mathcal{D})$ witnessed by the existence of a (κ, ρ_1)-cut and a (κ, ρ_2)-cut where $\rho_1 \neq \rho_2$ and $\min\{\rho_1, \rho_2\} \geq \kappa$.

Proof. Just as in the proof of Claim 2.11, using $\lambda_1 = \kappa$ and $\mu_1 = \kappa^+$. The resulting $\mathcal{D}$ will be κ-good and not κ^+-good so there will be a (κ, κ)-cut by Theorem E and a (κ, κ^+)-cut by construction. □

Now we ask: it possible to alternate? That is, can we find a regular ultrafilter $\mathcal{D}$ on λ and $\lambda_0 < \cdots < \lambda_n \leq \lambda$ such that $\lambda_n \in \mathrm{LcfSp}(\mathcal{D})$ iff n is even? To obtain alternations, we will appeal to weakly compact cardinals (which will give a clean and direct proof that alternation is possible, though it is unlikely they are essential to this result). Here the reason for $\ell = 1$ appears.

Definition 2.13 (see e.g. Kanamori [7] Theorem 7.8 p. 76)**.** The cardinal κ is said to be *weakly compact* if for every $f : \kappa \times \kappa \to \{0, 1\}$ there is $\mathcal{U} \subseteq \kappa$, $|\mathcal{U}| = \kappa$ and $\mathbf{t} \in \{0, 1\}$ such that for all $\epsilon < \zeta$ from $\mathcal{U}$, $f(\epsilon, \zeta) = \mathbf{t}$. If κ is weakly compact and uncountable, it follows that for any $n < \aleph_0$, $\rho < \kappa$ and $f : [\kappa]^n \to \rho$ there is $\mathcal{U} \subseteq \kappa$, $|\mathcal{U}| = \kappa$ such that $\langle f(\alpha_1, \ldots, \alpha_n) \colon \alpha_1, \ldots, \alpha_n \text{ from } \mathcal{U} \rangle$ is constant.

First we show that for I small relative to some weakly compact κ, subsequent ultrapowers over I cannot destroy uniqueness for κ.

Claim 2.14. Let $\mathcal{D}$ be an ultrafilter on I. Suppose $\kappa > |I|$ is a weakly compact cardinal and $\kappa \in \mathrm{LcfSp}_1(\mathfrak{M})$. Then $\kappa \in \mathrm{LcfSp}_1(\mathfrak{M}^I/\mathcal{D})$.

Proof. Let $N = \mathfrak{M}^I/\mathcal{D}$. Suppose we are given $f^{\ell}_{\alpha} \in {}^I\mathfrak{M}$ for $\alpha < \kappa, \ell \in \{1, 2\}$, so that for $\ell = 1, 2$ the sequence $\langle f^{\ell}_{\alpha}/\mathcal{D} \colon \alpha < \kappa \rangle$ is strictly increasing in N. By Łos' theorem, for each pair $\alpha < \beta < \kappa$ let

$$A^{\ell}_{\alpha,\beta} := \{s \in I \colon f^{\ell}_{\alpha}(s) < f^{\ell}_{\beta}(s)\} \in \mathcal{D}.$$

Recall that any weakly compact cardinal is strongly inaccessible, so there are $|\mathcal{P}(I)| < \kappa$ possible choices for $A^\ell_{\alpha,\beta}$. As κ is weakly compact, we may assume that for each ℓ there is $A_\ell \in \mathcal{D}$ and $\mathcal{U}_\ell \subseteq \kappa$, $|\mathcal{U}_\ell| = \kappa$ such that $\alpha < \beta \ \wedge \ \alpha \in \mathcal{U}_\ell \ \wedge \ \beta \in \mathcal{U}_\ell \implies A^\ell_{\alpha,\beta} = A_\ell$. Let $A_* = A_1 \cap A_2 \in \mathcal{D}$. By construction and Łos' theorem, for each $s \in A_*$, $\ell \in \{1, 2\}$ the sequence $\bar{f}^\ell = \langle f^\ell_\alpha(s) \colon \alpha \in \mathcal{U}_\ell\rangle$ is strictly increasing. After renaming if necessary, we may assume each $\mathcal{U}_\ell = \kappa$. By hypothesis, for some unbounded subset $\mathcal{U}$ of κ, in $\mathfrak{M}$ there is an order preserving map π_s such that $\alpha < \delta \in \mathcal{U}$ implies that $\pi_s(f^1_\alpha(s)) < f^2_\delta(s)$ and also that $\pi_s^{-1}(f^2_\alpha(s)) < f^1_\delta(s)$. When $s \notin A_*$, let π_s be the identity. Let π be the internal order-preserving map of N given by $\langle \pi_s \colon s \in I\rangle \in {}^I\mathfrak{M}$. So whenever $\alpha < \delta \in \mathcal{U}$,

$$N \models \text{“} \ \pi(f^1_\alpha/\mathcal{D}) \ < \ f^2_\delta/\mathcal{D} \ \wedge \ \pi^{-1}(f^2_\alpha/\mathcal{D}) < f^1_\delta/\mathcal{D})\text{”}$$

which satisfies Definition 2.6(1)(c) so completes the proof. □

Second we show that if I is small relative to some κ for which uniqueness fails (witnessed by large λ_1, λ_2), subsequent ultrapowers over I will not resolve this.[10]

Claim 2.15. Let $\mathcal{D}$ be an ultrafilter on I. Suppose that $\mathfrak{M}$ has (κ, λ_1) and (κ, λ_2)-cuts where $\kappa, \lambda_1, \lambda_2$ are regular, $|\mathcal{P}(I)| < \min\{\kappa, \lambda_1, \lambda_2\}$ and and $\lambda_1 \neq \lambda_2$. Then $\kappa \notin \mathrm{LcfSp}_0(\mathfrak{M}^I/\mathcal{D})$.

Proof. While the claim is stated for easy quotation (“witnesses to $\kappa \notin \mathrm{LcfSp}(\mathfrak{M})$ persist in an ultrapower provided the index set is small”), in fact all that we need to prove is that (κ, λ)-cuts are not filled in an ultrapower $N = \mathfrak{M}^I/\mathcal{D}$ provided that κ, λ are regular and $\kappa > 2^{|I|}$ and $\lambda > |I|$. Let $(\bar{a}, \bar{b})$ be a (κ, λ)-cut in $\mathfrak{M}$. For each $\alpha < \kappa$, let $a^*_\alpha \in N$ be the image of a_α under the diagonal embedding, and likewise let b^*_β be the image of b_β for $\beta < \lambda$. Suppose for a contradiction that $c \in \mathfrak{M}^I/\mathcal{D}$ is such that $\alpha < \kappa \ \wedge \ \beta < \lambda$ implies $N \models a^*_\alpha < c < b^*_\beta$. For each $\alpha < \kappa$, let $A_\alpha := \{t \in I \colon \mathfrak{M} \models a_\alpha < c[t]\} \in \mathcal{D}$. This amounts to coloring the elements of κ with at most $|\mathcal{P}(I)| < \kappa$ colors, so by the regularity of κ, there must be some $\mathcal{U} \in [\kappa]^\kappa$ and $A_* \in \mathcal{D}$ such that $\alpha \in \mathcal{U} \implies A_\alpha = A_*$. Now by Łos' theorem, for each $\beta < \lambda$, there is some $t_\beta \in A_*$ such that $\mathfrak{M} \models c[t] < b_\beta$. By regularity of λ, there are $\mathcal{V} \in [\lambda]^\lambda$ and $\mathbf{t} \in A_* \subseteq I$ such that $\beta \in \mathcal{V}$ implies $t_\beta = \mathbf{t}$. But then $c[\mathbf{t}]$ realizes the cut $\langle \bar{a}, \bar{b}\rangle$ in $\mathfrak{M}$, a contradiction. □

Conclusion 2.16. Suppose $\mu_0 < \cdots < \mu_n$ are regular cardinals $< \lambda$ and μ_k is weakly compact when k is even. Then for some regular ultrafilter $\mathcal{D}$ on λ we have that for each $\ell \leq \lfloor \frac{n}{2} \rfloor$, $\mu_{2\ell} \in \mathrm{LcfSp}_1(\mathcal{D}) \subseteq \mathrm{LcfSp}_0(\mathcal{D})$ and $\mu_{2\ell+1} \notin \mathrm{LcfSp}_0(\mathcal{D})$.

Proof. Let $\mu_{n+1} = \lambda$. For each $\ell \leq \lfloor \frac{n+1}{2} \rfloor$, let $\mathcal{D}_\ell$ be a regular ultrafilter on μ_ℓ such that:

- if ℓ is even, then D_ℓ is good, i.e. μ^+_ℓ-good.
- if ℓ is odd, then for any model $\mathfrak{M}$ of $Th(\mathfrak{A})$, the ultrapower $\mathfrak{M}^{\mu_\ell}/\mathcal{D}_\ell$ has both a $(\mu_\ell, \lambda_{\ell+1})$ and a $(\mu_\ell, \lambda_{\ell+2})$-cut for some $\lambda_{\ell+1} \neq \lambda_{\ell+2} > \mu_{\ell-1}$.

Having chosen such a sequence of ultrafilters, let

$$\mathcal{D} = \mathcal{D}_n \times \mathcal{D}_{n-1} \times \mathcal{D}_{n-2} \times \cdots\cdots \times \mathcal{D}_0$$

where the products are taken from left to right. By Claim 2.15, the failures of uniqueness built at odd stages persist in the product, noting that $\ell \geq 2$ and μ_ℓ

[10]Some different results on the case of (κ_1, κ_2)-cuts in ultrapowers where $\kappa_1 + \kappa_2 > 2^\lambda$, and the index model M is quite saturated, have been recently obtained by Golshani and Shelah [5].

weakly compact implies that $2^{\mu_{\ell-1}} < \mu_\ell$. By Claim 2.14 the cardinals added to LcfSp_1 at even stages persist in the product as well. This completes the proof. $\square$

Conclusion 2.17. This analysis shows that uniqueness at a *given* cardinal κ is consistently strictly weaker than κ^+-goodness, since goodness cannot alternate.

To complement this, we conclude by recording the addition of the relevant new conditions onto [17] Theorem 10.26 using the language of LcfSp just introduced. This requires a short fact from our recent paper about open problems on ultrafilters. Its proof checks that from an internal map between cofinal sequences of the two sides of a symmetric cut, one can conclude the cut is definable and therefore realized.

Fact 2.18 ([23] 3.2)**.** Let $\mathcal{D}$ be a regular ultrafilter on λ and let $\kappa \leq \lambda$ be regular. Suppose that in $(\omega, <)^I/\mathcal{D}$ there is a monotonic partial internal map between cofinal subsequences of any two strictly monotonic κ-indexed sequences [*i.e. suppose* $\kappa \in \mathrm{LcfSp}_1(\mathcal{D})$]. Then $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$.

Theorem 2.19. For $\mathcal{D}$ a regular ultrafilter on λ the following are equivalent.

(a) $\mathcal{D}$ has λ^+-treetops.
(b) $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ implies $\kappa \in \mathrm{LcfSp}_1(\mathcal{D})$.
(c) $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ implies $\kappa \in \mathrm{LcfSp}_2(\mathcal{D})$.
(d) $\kappa \leq \lambda \implies (\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$, i.e. $\mathcal{C}(\mathcal{D})$ has no symmetric cuts.
(e) $\mathcal{D}$ is λ^+-good.

Proof. (a) iff (d) iff (e) is the full statement of [17] Theorem 10.26, quoted in part as Theorem E above.

$(e) \implies (c)$ is Fact 2.3.
$(c) \implies (b)$: Observation 2.8(1).
$(b) \implies (d)$: is Fact 2.18. $\square$

We emphasize the interesting question of whether there can exist a regular ultrafilter $\mathcal{D}$ on λ which is *not* good, yet has uniqueness for all regular $\kappa \leq \lambda$ (evidently not all witnessed by the existence of internal order-preserving maps).

Problem 2.20. Prove that for some infinite cardinal λ there is a regular ultrafilter $\mathcal{D}$ on λ such that $\{\kappa : \kappa = \mathrm{cf}(\kappa) \ \land \ \kappa \leq \lambda\} \subseteq \mathrm{LcfSp}_0(\mathcal{D})$ but $\mathcal{D}$ is not λ^+-good.

3. Automorphic ultrafilters

In this section we first review how the distance between good for equality and good represents the increasing strength of internal partial automorphisms in theories. We then introduce the idea of "automorphic ultrafilters" as a way of stratifying this conjecturally nonempty region by mapping the class of all complete countable first order theories into it.

Using this language we prove a new characterization of good ultrafilters (via unstable theories), and reframe several open questions.

Convention 3.1. Throughout this section,

(a) $\mathcal{D}$ is a regular ultrafilter on I, which we sometimes identify with λ;
(b) λ will denote $|I|$;
(c) if we are given a model M and $A \subseteq M^I/\mathcal{D}$, we will say "$A$ is small" to mean $|A| \leq |I|$.

Recall the definition "$\mathcal{D}$ is good for equality," Definition 1.5 above. It had been observed that this definition can be restated in terms of existence of certain internal maps in ultrafilters.

Fact 3.2 ([14] Theorem 5.21)**.** For a regular ultrafilter $\mathcal{D}$ on λ the following are equivalent:

(1) $\mathcal{D}$ is good for equality.
(2) For any infinite M, $N = M^\lambda/\mathcal{D}$ and $A, B \subseteq N$ with $|A| = |B| = \lambda$, there is an internal partial map $f : N \to N$ which is injective and which takes A to B (in this case we say "$\mathcal{D}$ admits internal maps between sets of size λ").

Proof. For completeness, and to motivate Definition 3.6, we sketch a proof.

(2) implies (1): Fix an infinite model M. First observe that since $\mathcal{D}$ is regular, it is always possible to find *some* small set $A \subseteq M^I/\mathcal{D}$ admitting a map $h : A \to \mathcal{D}$ such that for all $t \in I$, the sequence $\langle a[t] : t \in h(a)\rangle$ is without repetition. Call such a map h a "good distribution for A." [For example, we may begin with a regularizing family $\{X_i : i < \lambda\}$, so by definition for all $t \in I$, $Z_t := \{i < \lambda : t \in X_i\}$ is finite. Then choose $\{a_i : i < \lambda\} \subseteq M^I$ so that for each $t \in I$, $\langle a_i[t] : i \in Z_t\rangle$ is without repetition, which is possible as M is infinite. Letting $a_i = \prod_{t\in I} a_i[t]/\mathcal{D}$ for each $i < \lambda$, the set $A = \{a_i : i < \lambda\}$ and the map h taking $a_i \mapsto X_i$ are as desired.] Suppose then that (2) holds. Let A be the small set just built and let B be any other small set. Let $f : N \to N$ be the internal map given by (2). Enumerate $B = \langle b_i : i < \lambda\rangle$ so that $f(a_i) = b_i$. Since f is internal, we may assume that we can find F_t $(t \in I)$ such that each $F_t : M \to M$ and $\prod_{t\in I}(M, F_t)/\mathcal{D} = (N, f)$. (We can also ask only that the F_t are relations and let $X = \{t : F_t \text{ is a function }\} \in \mathcal{D}$. So in the first case, $X = I$.) Then the map $g : B \to \mathcal{D}$ taking $b_i \mapsto X \cap h(a_i)$ is a good distribution for B. Since B was arbitrary, this proves (1).

(1) implies (2): Let A and B be as in the claim, $|A| = |B| = \kappa \leq \lambda$. Let $\{a_i : i < \kappa\}$ list A with no repetition, and let $\{b_i : i < \kappa\}$ list B with no repetition. Let $h : A \to \mathcal{D}$ be a good distribution for A and let $g : B \to \mathcal{D}$ be a good distribution for B. For each $t \in I$, the map $a_i[t] \mapsto b_i[t]$ is a bijection from $\{a_i[t] : t \in h(a_i) \cap g(b_i)\}$ to $\{b_i[t] : t \in h(a_i) \cap g(b_i)\}$, so let $f_t : M \to M$ be any bijection extending this one. Then the map $f := \prod_{t\in I} f_t/\mathcal{D}$ is as desired. □

Remark 3.3. Notice that the proof of 3.2 shows something stronger, namely that we may choose the map f to take $a_i \mapsto b_i$ after fixing any enumeration of A, B without repetition.

Convention 3.4. We will say $\mathcal{D}$ is κ^+-good for equality, κ not necessarily equal to λ, when we may take $|A| = |B| = \kappa$ in Fact 3.2.

Compare Fact 2.3 above.

Now we introduce a way of studying this region via model theory. A natural question, suggested by Scanlon, following a talk of the first author, is the following:

Question 3.5. What can be said about regular ultrafilters which admit internal maps between small elementary submodels in any ultrapower of $M \models T$, for a given complete countable T?

We will first make this problem concrete and then give an answer in the case where T is unstable, Theorem 3.23.

Definition 3.6 (Automorphic ultrafilters)**.** Let $\mathcal{D}$ be a regular ultrafilter on λ and T a complete countable theory.

(1) We say that $\mathcal{D}$ is *automorphic* for T if whenever $M \models T$, $||M|| \leq \lambda$, $N = M^\lambda/\mathcal{D}$, M_0, M_1 are elementary submodels of N with $||M_0|| = ||M_1|| \leq \lambda$ and $f' : M_0 \to M_1$ is an (external) isomorphism, then there exists an internal function f such that:
   (a) f extends f', thus f maps M_0 to M_1.
   (b) f is an internal partial one-to-one map which respects all formulas of the language.
   (c) dom(f) and range(f) are internal sets.

(2) Let Δ be a finite set of formulas of T. We say that $\mathcal{D}$ is Δ-*automorphic* for T if (1) above holds with condition (1)(b) replaced by "f is an internal one-to-one partial map which preserves the truth of all formulas in Δ." So automorphic is Δ-automorphic in the special case where Δ is all formulas of the language.

In the language of Definition 3.6, Fact 3.2 is naturally restated as follows.

Conclusion 3.7. Let $\mathcal{D}$ be a regular ultrafilter on λ. Then $\mathcal{D}$ is good for equality if and only if it is automorphic for the theory of an infinite set.

Corollary 3.8. If $\mathcal{D}$ is automorphic for *any* theory with infinite models it is necessarily automorphic for the theory of an infinite set, and thus, good for equality.

The analogous restatement of Fact 2.3 will require some intermediate claims.

First, we will use the following saturation properties of ultrafilters which are good for equality. (In fact, more is true, namely, $\mathcal{D}$ being good for equality is necessary and sufficient for saturating T_{feq}. However, we will use this fact only indirectly, in the language of 3.7 and 3.8 and of the next Fact.)

Fact 3.9. If $\mathcal{D}$ is a regular ultrafilter which is good for equality then $\mathcal{D}$ saturates the theory of the random graph, and in addition $\mathcal{D}$ saturates any countable stable theory.

Proof. See the summary theorem [15] Theorem 4.2 p. 8154, specifically (5) $\rightarrow$ (3) $\rightarrow$ (2) $\rightarrow$ (1) of that theorem. This shows that any ultrafilter which is good for equality has three other properties. In Theorem G p. 8153 of the same paper, it is recorded that those properties called (3) and (1) are sufficient for saturating the theory of the random graph and for saturating all stable theories, respectively. □

Recall that in model-theoretic terminology, a φ-type is a partial type consisting of positive and negative instances of a single formula.

Fact 3.10 (Local saturation suffices, [11] Theorem 12)**.** Suppose $\mathcal{D}$ is a regular ultrafilter on I and T a countable complete first order theory. Then for any $M^I/\mathcal{D}$, the following are equivalent:

(1) $M^I/\mathcal{D}$ is λ^+-saturated.
(2) $M^I/\mathcal{D}$ realizes all φ-types over sets of size $\leq \lambda$, for each formula φ in the language of T.

Observation 3.11. Let $\mathcal{D}$ be a regular ultrafilter on λ, T a complete first-order theory of cardinality $\leq \lambda$, $M \models T$, $N = M^\lambda/\mathcal{D}$, and Δ a finite set of formulas of $\mathcal{L}_T$. Let M_0, M_1 be two elementary submodels of N of size $\leq \lambda$ which are externally isomorphic via some function f. Then the existence of an internal partial map

> g such that $M_0 \subseteq \mathrm{dom}(g)$, $M_1 \subseteq \mathrm{range}(g)$, $g : M_0 \to M_1$ and g is a partial one-to-one map which extends f and respects all formulas in Δ

can be expressed in terms of a partial φ-type [i.e. a type in positive and negative instances of a single formula] over a set of size $\leq \lambda$ in a related first-order theory. Thus, internal maps of this kind will exist in any good regular ultrafilter.

Proof. We sketch two different ways to represent the existence of g in terms of realization of a type (over a set of size $\leq \lambda$) in an expansion of the model M to a model M^+ in a larger, countable language. Since ultrapowers commute with reducts, we may then expand N naturally to $N^+ \equiv M$, and any good ultrafilter will ensure that N^+ is saturated, therefore that g exists. This justifies the last clause of the observation.

First, let M^+ be a model of sufficient set theory, say, $(\mathcal{H}(\chi), \in)$ for some sufficiently large χ, so $\omega \in M^+$ and $M \in M^+$. Consider the set $\mathcal{T}$ of partial one to one maps which respect all formulas in Δ (since Δ is finite this is a first order statement). Similarly to 2.4 above, $\mathcal{T}$ is partially ordered by inclusion, and the existence of a map g as in the statement of the claim corresponds to realizing a type describing a certain element of $\mathcal{T}$.

Second, we can consider $T = Th(M_*)$ where M_* is the following two-sorted structure: the first sort contains a resplendent model[11] $M \models T$, the second sort A contains an infinite set. We add a new ternary relation symbol $f^\Delta(x, y, z)$ and add infinitely many axioms to ensure the following: (1) for each $x \in A$, $f^\Delta(x, y, z) = f^\Delta_x(y, z)$ is a partial function which is a partial automorphism of M and respects all of the formulas in Δ, and (2) for any $n < \omega$ and sets $\langle b_1, \ldots, b_n \rangle, \langle c_1, \ldots, c_n \rangle$ realizing the same Δ-type over the empty set in M, there is $a \in A$ such that $\bigwedge_{i \leq n} f^\Delta_a(b_i, c_i)$. □

Conclusion 3.12. If $\mathcal{D}$ is a regular ultrafilter on λ, then $\mathcal{D}$ is automorphic for every complete countable theory iff $\mathcal{D}$ is good.

Proof. Suppose $\mathcal{D}$ is good. Let T be complete and countable, and fix suitable M_0, M_1 and f. By the proof of Observation 3.11, for each finite set Δ of formulas of the language of T we may write down a partial φ-type p_Δ in an expanded language expressing the existence of an internal partial one-to-one map extending f. The union $q = \bigcup_\Delta p_\Delta$ of these partial types is a consistent partial type whose realization would tell us that $\mathcal{D}$ is automorphic (in this instance) for T. By the statement of 3.11, as $\mathcal{D}$ is good, each of the types p_Δ are realized. By Fact 3.10, their union q is also realized. Since M_0, M_1, f were arbitrary, this shows $\mathcal{D}$ is automorphic for T.

In the other direction, suppose $\mathcal{D}$ is automorphic for the theory $T_{dlo} := Th(\mathbb{Q}, <)$. Then in the notation of Definition 2.6, $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ implies $\kappa \in \mathrm{LcfSp}_2(\mathcal{D})$. By Theorem 2.19 (c) implies (e), $\mathcal{D}$ is good. □

Combining Conclusion 3.7 and Conclusion 3.12, we have a possible spectrum of complexity focused on the non-simple theories: it arises with the minimum non-simple theory in Keisler's order and is completely resolved by the time we get to the Keisler-maximal theory.

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[11]Call M *resplendent* if whenever a Σ^1_1 formula is satisfiable in some elementary extension of M, it is already satisfiable in M. Each model has a resplendent elementary extension of the same cardinality. We assume this as otherwise M may be rigid.

We now work towards a characterization of those ultrafilters which are automorphic for unstable theories. We will use the characteristic sequences of [12].

Definition 3.13. For a given formula φ and T, recall:

(1) the *characteristic sequence of hypergraphs* for φ is given by $\langle P_n : n < \omega \rangle$ where
$$P_n(y_1, \dots, y_n) = (\exists x) \bigwedge_{1 \leq i \leq n} \varphi(x, y_i)$$
(2) we call a set A a *positive base set* if, identifying the predicates P_n with their interpretations in the monster model, we have that $A^n \subseteq P_n$ for all $n < \omega$. So A is a positive base set iff $\{\varphi(x, a) : a \in A\}$ is a consistent partial type.
(3) We call the formula φ *2-compact* if the characteristic sequence depends on 2, or equivalently, if any set of positive instances of φ is consistent iff every subset of size 2 is consistent.

Fact 3.14 (essentially 5.2 of [10])**.** Let T be a complete countable theory. To show that $\mathcal{D}$ saturates T, it would suffice to show that for every formula $\varphi(\bar{x}, \bar{y})$ of T and for every $A \subseteq (M^\lambda/\mathcal{D})^{\ell(\bar{y})}$ such that $|A| \leq \lambda$ and A is a positive base set for the characteristic sequence of φ, there exists a map $g : A \to \mathcal{D}$ such that writing
$$m_t = |\{a[t] \ : \ a \in A, \ t \in g(a)\}| < \aleph_0 \ \text{ for } t \in \lambda$$
we have that for all $m \leq m_t$, P^M_m is a complete hypergraph on the vertex set
$$\{a[t] \ : \ a \in A, \ t \in g(a)\}.$$

Proof. We start with two reductions. First, by Fact 3.10, it suffices to show we can realize all φ-types over small sets. Second, we may assume that these φ-types consist only of positive instances of the given formula. (Why? We can always code φ as a formula $\theta(x; y, z, w) = (\varphi(x, y) \wedge z = w) \vee (\neg\varphi(x, y) \wedge z \neq w)$ with the property that a φ-type may always be expressed as a set of positive instances of θ. Since we are quantifying over all formulas, this will be enough.) So in what follows let us fix a formula $\varphi(x, y)$ of T (note that $\ell(\bar{x})$, $\ell(\bar{y})$ need not be 1), fix $M \models T$, $N = M^\lambda/\mathcal{D}$ and let $p(x) = \{\varphi(x, a) : a \in A\}$ be a φ-type we wish to realize, which by the above we may assume consists only of positive instances of our given formula. Let $\langle P_n : n < \omega \rangle$ be the characteristic sequence of φ. By definition of the characteristic sequence, A is a positive base set. Suppose A has a distribution of the kind stated in the Fact. For each $t \in I$, the set $\{\varphi(x, a[t]) : a \in A, t \in g(a)\}$ is a consistent partial type. Let b_t realize it. Then by Łos' theorem, $b := \langle b_t : t \in I \rangle/\mathcal{D}$ realizes the type $p(x)$, as desired. □

Corollary 3.15. If φ is a formula of T and φ is 2-compact, Fact 3.14 reduces to saying: it would suffice to show that any positive base set A in $M^\lambda/\mathcal{D}$ with $|A| \leq \lambda$, there is a sequence $\langle C_t : t \in I \rangle$ such that each C_t is finite and is a complete graph for the edge relation P_2 in M and $\prod_t C_t/\mathcal{D} \supseteq A$. In this case we say "$A$ is covered by an ultraproduct of complete P_2-graphs."

Before continuing we recall some facts about Keisler's order. As explained in the introduction, for $\mathcal{D}$ regular, "$\mathcal{D}$ saturates T" means that $M^I/\mathcal{D}$ is $|I|^+$-saturated for some (equivalently, by regularity, every) model M of T. The reader may wish to refer to the picture on page 4.

Definition 3.16. Keisler's order is the pre-order on complete countable theories given by: $T_1 \trianglelefteq T_2$ if and only if for all regular ultrafilters $\mathcal{D}$, if $\mathcal{D}$ saturates T_2 then $\mathcal{D}$ saturates T_1.

Fact 3.17 (For history and discussion of these results, see [15] §4.)**.**

(1) The theory of $(\mathbb{Q}, <)$ is maximal in Keisler's order.
In fact, any theory with a definable linear order is maximal. See [15] *Theorem F, pps. 8152-8153 or* [24] *Theorem 4.3 p. 371.*
(2) The maximal class in Keisler's order consists precisely of those theories T such that for a regular ultrafilter $\mathcal{D}$, $\mathcal{D}$ saturates T if and only if $\mathcal{D}$ is good.
See [2] *Theorem 6.1.8 and* [8] *Theorem 3.4.*
(3) If T_1 is stable and T_2 is unstable, then $T_1 \triangleleft T_2$ in Keisler's order.
See [24] *Theorem 0.3 p. 323.*
(4) The theory of the random graph is minimum among the unstable theories in Keisler's order.
See [13] *Conclusion 5.3.*
(5) The (unstable) theory T_{feq} is minimum among the non-simple theories in Keisler's order.
See [17] *Theorem 13.1.*

Claim 3.18. Let $\mathcal{D}$ be a regular ultrafilter on I. Suppose $\mathcal{D}$ is automorphic for the random graph. Then $\mathcal{D}$ is good.

Proof. Let $M = (\mathbb{Q}, <)$. Let "small" mean $\leq \lambda = |I|$. By quantifier elimination, an ultrapower $N = M^I/\mathcal{D}$ is $|I|^+$-saturated if and only if every positive φ-type over a set of size $\leq |I|$ is realized where $\varphi = \varphi(x; y, z) = y > x > z$. This formula is 2-compact, so we apply Corollary 3.15. We would like to show that for every small positive base set $A \subseteq N^2$ w.r.t. the characteristic sequence of φ, A is covered by an ultraproduct of complete P_2-graphs.

Let $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{D}$ be a regularizing family (so the intersection of any infinitely many elements of this family is empty). Enumerate A as $\langle a_\alpha : \alpha < \lambda\rangle$. Recalling that each $a_\alpha \in A \subseteq N^2$ is a pair (a^1_α, a^2_α), define $d : A \to \mathcal{D}$ by:

$$a_\alpha \mapsto \{t \in I : M \models (\exists x)(a^1_\alpha < x < a^2_\alpha)\} \cap X_\alpha.$$

Henceforth we forget the a's are pairs and write simply $a_\alpha[t]$ for the element of M^2 corresponding to $a^1_\alpha[t], a^2_\alpha[t]$. For each $t \in I$, let $B_t = \{a_\alpha[t] : \alpha < \lambda, t \in d(a_\alpha)\}$. By the definition of the X_α's, each B_t is a finite set of 'vertices' of P^M_1, thus a finite graph in P^M_2.

Since the random graph G is universal, for each $t \in I$ there is a partial isomorphism h_t whose domain is B_t (considered as a P^M_2-graph) and whose range is some finite graph G_t in G. For each $t \in I$ and $\alpha < \lambda$ define $r_{\alpha,t} \in G$ to be $h_t(a_\alpha[t])$. For each $\alpha < \lambda$ let $r_\alpha := \langle r_{\alpha,t} : t \in I\rangle/\mathcal{D} \in G^I/\mathcal{D}$. By Los' theorem, $\{r_\alpha : \alpha < \lambda\}$ is a complete graph in $G^I/\mathcal{D}$. Henceforth we refer to its coordinate projections as $r_\alpha[t]$.

Next, choose in the ultrapower of the random graph a set of distinct vertices $\{c_\alpha : \alpha < \lambda\}$ which is a complete graph in $G^I/\mathcal{D}$ and which is covered by an ultraproduct of complete graphs, as follows. By regularity, some ultraproduct of finite sets, say $\langle n_t : t \in I\rangle$ will have size $\geq \lambda$ mod $\mathcal{D}$. Since the random graph is universal for finite graphs, for each t we may find a complete graph $H_t \subseteq G$ on n_t vertices. Then $H = \prod_t H_t/\mathcal{D}$ is a complete graph of size $\geq \lambda$. Let $C = \{c_\alpha : \alpha < \lambda\} \subseteq H$ be a subset of size λ.

Now we have assumed $\mathcal{D}$ is automorphic for the random graph, so there is an internal partial one-to-one map g sending $r_\alpha \mapsto c_\alpha$ for all $\alpha < \lambda$. Consider the map d_* defined by, for each $\alpha < \lambda$,

$$r_\alpha \mapsto \ (d(a_\alpha) \cap \{t \in I : c_\alpha[t] \in H_t\}) \ \in \mathcal{D}.$$

This has accomplished our goal because for each $t \in I$, $\{c_\alpha[t] : t \in d_*(r_\alpha)\}$ is a complete graph and $h_t \restriction \{r_\alpha[t] : t \in d_*(r_\alpha)\}$ is a partial graph isomorphism onto it. So necessarily $\{r_\alpha[t] : t \in d_*(r_\alpha)\}$ is a complete graph for almost all t, which means $\{a_\alpha[t] : t \in d_*(r_\alpha)\}$ is a complete P_2^M-graph for almost all t. We have shown explicitly that A is covered by an ultraproduct of complete P_2^M-graphs, which completes the proof. $\square$

Definition 3.19. For $\ell = 1, 2$ suppose T_1, T_2 are complete countable theories. We say that T_2 *captures the atomic relational patterns of* T_1 when for every relation $R(x_1, \dots, x_n)$ of τ_{T_1} there are a formula $\varphi(\bar{x}_1, \dots, \bar{x}_n)$ of $\mathcal{L}_{T_2}$ and a constant m, such that $i \leq n \implies \ell(\bar{x}_i) = m$, such that whenever $M_\ell \models T_\ell$ for $\ell = 1, 2$, and whenever $k < \omega$ and $\langle a_i : i < k \rangle \in {}^k M_1$ is a finite sequence of elements, there is a $\langle \bar{b}_j : j < k \rangle \in {}^k({}^m M_2)$ such that for all $\eta \in {}^n k$,

$$M_1 \models R(a_{\eta(0)}, \dots, a_{\eta(n-1)}) \iff M_2 \models \varphi(\bar{b}_{\eta(0)}, \dots, \bar{b}_{\eta(n-1)}).$$

Observation 3.20. Let $\mathcal{D}$ be a regular ultrafilter and let T_1 be a complete countable theory which eliminates quantifiers, such as $T_{dlo} := Th(\mathbb{Q}, <)$ or T_{rg}, the theory of the random graph. Suppose T_2 is a complete countable theory which captures the atomic relational patterns of T_1. If $\mathcal{D}$ is automorphic for T_2, then $\mathcal{D}$ is automorphic for T_1.

Claim 3.21. Let T_2 be a theory with the independence property, i.e. with a formula $\varphi(x, \bar{y})$ such that in some model $M_2 \models T_2$, there is a sequence $\langle \bar{b}_i : i < \omega \rangle$ of elements of ${}^{\ell(\bar{y})}M_2$ such that for any $\sigma \subseteq \omega$,

$$\{\varphi(x, \bar{b}_i) \colon i \in \sigma\} \cup \{\neg\varphi(x, \bar{b}_j) \colon j \in \omega \setminus \sigma\}$$

is a consistent partial type. Then T_2 captures the atomic relational patterns of T_{rg}, the theory of the random graph.

Proof. Let $\langle a_i : i < k \rangle$ be a finite set of elements in a model of T_{rg}. We may choose by induction a sequence $\langle b^0_i b^1_i \colon i < \omega \rangle$ of pairs of elements of M_2, all distinct, with the property that $\varphi(b^0_i, b^1_j)$ iff $a_i R a_j$ holds in the random graph. [Let b^1_i be the element b_i from the definition of independence property, and choose each b^1_i to realize the appropriate φ-type.] Note that by compactness, we may choose any finite initial segment of such a sequence in any model of T_2, simply because once we've found such a finite sequence in the given M_2 its existence is first order expressible (fixing φ) and so transfers to any other elementarily equivalent model. $\square$

The analogous result is also straightforward. (Note that the last line of the previous proof applies here too: Definition 3.19 asks for finite patterns, so it will indeed be satisfied in any model of a T_2 with s.o.p., not only the M_2 from 3.22.)

Claim 3.22. Let T_2 be a theory with the strict order property, so by compactness, for some formula $\varphi(\bar{x}, \bar{y})$ and $M_2 \models T_2$ there is an indiscernible sequence $\langle \bar{b}_i : i < \mathbb{Q} \rangle$ of elements of ${}^{\ell(\bar{y})}M_2$ such that for any two disjoint sets $\sigma, \tau \subset \mathbb{Q}$,

$$\{\varphi(\bar{x}, \bar{b}_i) \colon i \in \sigma\} \cup \{\neg\varphi(\bar{x}, \bar{b}_j) \colon j \in \tau\}$$

is consistent if and only if for all $\alpha \in \sigma$ and for all $\beta \in \tau$, $\alpha < \beta$. Then T_2 captures the atomic relational patterns of $Th(\mathbb{Q}, <)$.

Theorem 3.23. Let $\mathcal{D}$ be a regular ultrafilter on λ. If $\mathcal{D}$ is automorphic for *some* unstable theory, then $\mathcal{D}$ is good.

Proof. Let T be such an unstable theory, so either T has the strict order property or T has the independence property. If T has the independence property, then it captures the atomic relational patterns of the theory of the random graph: apply Claim 3.21 followed by Observation 3.20 followed by Claim 3.18. Likewise, if T has the strict order property, then it captures the atomic relational patterns of $T' = Th(\mathbb{Q}, <)$, so $\mathcal{D}$ is automorphic for T'. Now apply Theorem 2.19(c) $\rightarrow$ (e). □

Conclusion 3.24. Let $\mathcal{D}$ be a regular ultrafilter on λ. To the equivalent conditions of Theorem 2.19 above, we may add:

(f) $\mathcal{D}$ is automorphic for some countable unstable theory.
(g) $\mathcal{D}$ is automorphic for every countable unstable theory.
(h) $\mathcal{D}$ is automorphic for the theory of linear order.
(i) $\mathcal{D}$ is automorphic for the theory of the random graph.

Proof. Recall that condition (e) of that Theorem is that $\mathcal{D}$ is good. Since the theory of linear order and the theory of the random graph are both unstable, we have that (g) $\rightarrow$ (h) $\rightarrow$ (f) and (g) $\rightarrow$ (i) $\rightarrow$ (f). Theorem 3.23 gives (f) $\rightarrow$ (e) and Conclusion 3.12 gives (e) $\leftrightarrow$ (g). □

Corollary 3.25. Let T be a countable complete first order theory and $\mathcal{D}$ a regular ultrafilter. If $\mathcal{D}$ is automorphic for T then $\mathcal{D}$ saturates T.

Proof. Suppose T is stable. As all ultrapowers of finite models are saturated, we may assume T has infinite models. By Corollary 3.8, $\mathcal{D}$ is good for equality. So by Fact 3.9, $\mathcal{D}$ saturates all countable stable theories, including T.

Suppose that T is unstable. If $\mathcal{D}$ is automorphic for T, then by Theorem 3.23, $\mathcal{D}$ is good, i.e. λ^+-good. We know from [8] Theorem 1.4 that if M is a model of *any* countable theory and $\mathcal{D}$ is regular and λ^+-good then $M^\lambda/\mathcal{D}$ is λ^+-saturated. Thus, $\mathcal{D}$ saturates T. □

Corollary 3.26. For a complete countable theory T the following are equivalent:

(a) Any regular ultrafilter $\mathcal{D}$ which saturates T is automorphic for T.
(b) Any regular ultrafilter $\mathcal{D}$ which saturates T is good.
(c) T is in the maximal Keisler class.

Proof. (b) iff (c) is Keisler's characterization of the maximal class in Keisler's order.

(b) implies (a): If $\mathcal{D}$ is good then $\mathcal{D}$ is automorphic for T by Conclusion 3.12.

(a) implies (b): There are two cases. In the first case, T is unstable. Then any regular $\mathcal{D}$ which saturates T is also automorphic for T, so apply Conclusion 3.24 to conclude (b). In the second case, we assume for a contradiction that T is stable. In this case, the assumption (a) entails that every regular ultrafilter $\mathcal{D}$ which saturates T is automorphic for T, and thus by 3.8 must be good for equality. However, it is known that there exist regular ultrafilters (in ZFC) which saturate all stable theories but are not good for equality (see e.g. [19] Theorem 12.1). This contradiction shows the second case cannot occur, so we finish the proof. □

Corollary 3.27. There is more than one non-simple class in Keisler's order iff there is a regular ultrafilter $\mathcal{D}$ on some λ which is automorphic for infinite sets but not automorphic for the random graph.

Corollary 3.28. There is a non-maximal non-simple theory in Keisler's order iff there is a regular ultrafilter $\mathcal{D}$ which saturates T_{feq} but not automorphic for T_{feq}.

We conclude with a curious restatement. By [13] saturation of T_{feq}, the minimum non-simple theory, depends on formulas which are 2-compact and whose associated P_2 may, after adding finitely many parameters, be assumed to be stable. Thus, invoking 5.2 of [10] or just 3.14, we record here that the problem of showing Keisler's order has more than one class on the non-simple theories – in other words, the problem of proving that "good for equality" does not imply "good" – has the following surprising form.

Definition 3.29. Let $\mathcal{D}$ be a regular ultrafilter on λ. Say that a graph G is $\mathcal{D}$-coverable if every complete induced subgraph of $G^\lambda/\mathcal{D}$ of size $\leq \lambda$ is covered by an ultraproduct of complete graphs.

Conclusion 3.30. To prove that there are at least two non-simple classes in Keisler's order, i.e., to prove that good for equality does not imply good:

(1) it would suffice to prove that there exists a regular ultrafilter $\mathcal{D}$ such that every stable graph is $\mathcal{D}$-coverable and the random graph is not $\mathcal{D}$-coverable.
(2) it would suffice to prove that there exists a regular ultrafilter $\mathcal{D}$ which is automorphic for the theory of equality but not for the theory of linear order.

4. Lower cofinality spectrum problems

Here we take an axiomatic approach.

More precisely, in this section we build a bare-bones framework consisting of two orders and a single tree and prove we may still recover a version of uniqueness in Lemma 4.21 below. Once again, two motivations converge here: one model-theoretic and one set-theoretic. The first notes that the analysis of cofinality spectra has strong consequences for cuts in models of Peano arithmetic; for instance, in [22] we use cofinality spectrum problems to prove that a model of PA is κ-saturated iff it has cofinality at least κ and the reduct to the language of order has no (κ', κ')-cuts for $\kappa' < \kappa$. Contrast this with the case in real closed fields, where there is great freedom in determining which pairs of cardinals (κ_1, κ_2) appear as the cofinalities of cuts [26]. It is therefore natural to ask: at what point along the route from o-minimality to Peano arithmetic does substantial control of cuts appear?

The second arises from the fact that our proof that $\mathfrak{p} = \mathfrak{t}$, in [17] Theorem 14.1, proceeded by an analysis of cut spectra in pairs of models with sufficient set theory. Yet that ZFC proof required a forcing argument, revolving around the fact that if we assume for a contradiction that $\mathfrak{p} < \mathfrak{t}$, a so-called peculiar cut would appear, contradicting our (ZFC) analysis of the cut spectrum. It would be very nice to remove the forcing argument from this proof, and we conjecture that an axiomatic analysis of the argument is the way to proceed.

To begin, we take the definition of 'peculiar cut' apart.

Definition 4.1 (Cuts in partial orders)**.** For a partial order $(L, <_L)$,

(1) We say that the sets (A, B) represent a cut in $(L, <_L)$ when:
   (a) $(\forall a \in A)(\forall b \in B)(a <_L b)$, i.e. $A <_L B$.

   (b) there does not exist $c \in L$ such that $A <_L c <_L B$.
(2) We say that the pair of sequences $(\overline{a}, \overline{b})$ represent a cut in $(L, <_L)$ when:
   (a) $\overline{a}$ is $<_L$-increasing
   (b) $\overline{b}$ is $<_L$-decreasing
   (c) $(\mathrm{range}(\overline{a}), \mathrm{range}(\overline{b}))$ represent a cut in $(L, <_L)$.
(3) For a partial order $(L, <_L)$ and disjoint nonempty sets A, B,
   (a) we say a pair (A, B) represents a *rising cut* when: for every $c \in L$

$$\bigwedge_{a \in A} a <_L c \implies \bigvee_{b \in B} b \leq_L c.$$

   (b) we say a pair (A, B) represents a *falling cut* when: for every $c \in L$

$$\bigvee_{a \in A} c \leq_L a \Longleftarrow \bigwedge_{b \in B} c <_L b.$$

   (c) we say a pair (A, B) represents a *peculiar cut* if it represents both a rising and a falling cut.
(4) Just as in (3), but replacing the sets by sequences.

Convention 4.2. Throughout this section, if we say "(A, B) represents a cut" we will mean that $A \neq \emptyset$ or $B \neq \emptyset$ unless otherwise stated. If the given cut is additionally a rising or falling cut, it follows that $B \neq \emptyset$ and/or $A \neq \emptyset$, respectively.

Example 4.3. Assume L is a linear order. Then every cut is a peculiar cut.

Example 4.4. Let $(\mathcal{T}, \triangleleft)$ be a partially ordered set such that the set of predecessors of any given node is linearly ordered. Then every cut is a falling cut.

Definition 4.5. Let $(L, <_L)$ be a partial order and $\overline{a}$ a strictly monotonic sequence. Then $\mathrm{cf}(\overline{a})$ denotes the cofinality of $\overline{a}$, i.e. the cofinality is the ordinal $\lg(\overline{a})$ i.e. the minimum size of a cofinal subsequence in the relevant order. This is always either $0, 1$ or a regular infinite cardinal, and note that by the previous convention we generally avoid the case of 0.

We now investigate the coinitiality of a given increasing sequence. Recall that what makes the next Claim 4.6 nontrivial is that $(L, <_L)$ is allowed here to be a partial order, so a priori we may have multiple descending sequences of different cofinalities approaching the given $\bar{a}$.

Claim 4.6. Let $(L, <_L)$ be a partial order and $\kappa = \mathrm{cf}(\kappa) \geq \aleph_0$. Suppose that $(\overline{a}, \overline{b}^1)$ represents a cut, $(\overline{a}, \overline{b}^2)$ represents a rising cut and $\mathrm{cf}(\overline{a}) = \kappa$. Then

$$\mathrm{cf}(\overline{b}^1) = \mathrm{cf}(\overline{b}^2).$$

Proof. Let $\theta_\ell = \mathrm{cf}(\overline{b}^\ell)$ for $\ell = 1, 2$, so each θ_ℓ is a regular cardinal or 1. Suppose $\theta_1 \neq \theta_2$. Let $\overline{a} = \langle a_\epsilon \colon \epsilon < \kappa \rangle$. For every $\alpha < \theta_1$,

$$\bigwedge_{\epsilon < \kappa} a_\epsilon < b^1_\alpha$$

so as $(\overline{a}, \overline{b}^2)$ represents a rising cut, for each $\alpha < \theta_1$ there is $\beta(\alpha) < \theta_2$ such that $b^2_{\beta(\alpha)} <_L b^1_\alpha$. As $\theta_1 \neq \theta_2$, it follows they are not both 1, so there is $\beta(*) < \theta_2$ such that the set

$$\mathcal{U} = \{\alpha < \theta_1 \colon \beta(\alpha) \leq \beta(*)\}$$

is unbounded in θ_1. [Recall $\theta_1 \neq \theta_2$ are both regular. If $\theta_1 = 1$ we are done, and if they are both infinite, then either $\theta_1 < \theta_2$ or $\theta_2 < \theta_1$, and in each case we are done. If θ_1 is infinite and $\theta_2 = 1$, then let b_* witness that $\theta_2 = 1$. Now either we can choose the assignment $\alpha \mapsto \beta(\alpha)$ so that a cofinal sequence of elements of $\bar{b}^1$ are $<_L$-strictly above b_*, in which case the result is true, *or not*, in which case cofinally many elements of $\bar{b}^2$ are equal to b_*. Since Definition 4.1 requires that $\bar{b}^2$ be a monotonic sequence, this means $\bar{b}^2$ is eventually constant and equal to b_*, and so $\theta_1 = 1 = \theta_2$, contradicting our assumption.]

Thus $b^2_{\beta(*)}$ satisfies

$$\alpha < \theta_1 \wedge \epsilon < \kappa \implies a_\epsilon <_L b^2_{\beta(*)} <_L b^1_{\min(\mathcal{U} \setminus \alpha)} \leq_L b^1_\alpha$$

This contradicts the assumption that $(\overline{a}, \overline{b}^1)$ represents a cut. □

The parallel version of 4.6 holds in the other direction:

Claim 4.7. Let $(L, <_L)$ be a partial order and $\kappa = \mathrm{cf}(\kappa) \geq \aleph_0$. Suppose that $(\overline{a}^1, \overline{b})$ represents a cut, $(\overline{a}^2, \overline{b})$ represents a falling cut and $\mathrm{cf}(\overline{b}) = \kappa$. Then

$$\mathrm{cf}(\overline{a}^1) = \mathrm{cf}(\overline{a}^2)$$

Proof. Analogously to the proof of 4.6. □

Example 4.8. Let N be a nonstandard model of Peano arithmetic, and let $x|y$ be the partial order given by "x divides y." Let p be a prime and let $\psi(x)$ say that p is the only prime divisor of x. Let $\bar{a}$ be cofinal in the standard elements $\psi(\mathbb{N})$ and let $\bar{b}^2$ be a sequence of elements of $\psi(N)$ so that $(\bar{a}, \bar{b}^2)$ represents a $|$-cut, thus a rising $|$-cut. So if $\bar{b}^1$ is any sequence of elements of N such that $(\bar{a}, \bar{b}^1)$ represents a $|$-cut, necessarily $\mathrm{cf}(\bar{b}^1) = \mathrm{cf}(\bar{b}^2)$.

Discussion 4.9. In each case the directionality of Claims 4.6 and 4.7 was crucial; if we suppose e.g. $\kappa = \mathrm{cf}(\kappa) \geq \aleph_0$, and let $(\overline{a}, \overline{b}^1)$, $(\overline{a}, \overline{b}^2)$ represent a cut and a falling cut, respectively where $\mathrm{cf}(\overline{a}) = \kappa$ then it need not be the case the case that $\mathrm{cf}(\overline{b}^1) = \mathrm{cf}(\overline{b}^2)$. There is also as yet no leverage for comparing κ-indexed sequences.

In order to compare different cuts (perhaps in two different partial orders), we now introduce a tree. For the intent of the following Definition 4.10, see 4.12. We have used $\mathbf{r}$ for a *lower cofinality spectrum problem* to distinguish from $\mathbf{s}$, the default name for a *cofinality spectrum problem* in [17]. Some examples will be built in the next section.

Definition 4.10. We say $\mathbf{r}$ is an LCSP (lower cofinality spectrum problem) when $\mathbf{r}$ consists of:

$$(M, L_1, <_1, L_2, <_2, \mathcal{T}, \trianglelefteq_{\mathcal{T}}, F_1, F_2, r_1, r_2)$$

which satisfy:

(1) M is a model.
(2) for $\ell = 1, 2$, $(L_\ell, <_\ell)$ is a partial order definable in M, with a root, i.e. minimum element, $\mathrm{rt}(L_\ell)$ named by the constant r_ℓ. We may write L_ℓ for $(L_\ell, <_\ell)$.
(3) $(\mathcal{T}, \triangleleft_{\mathcal{T}})$ is a tree definable in M, i.e. a partially ordered set where the set of predecessors of any given element is linearly ordered and $x \trianglelefteq_{\mathcal{T}} y$ means $(x \triangleleft_{\mathcal{T}} y) \vee (x = y)$.

(4) for $\ell = 1, 2$, F_ℓ is a homomorphism from $(\mathcal{T}, \triangleleft_{\mathcal{T}})$ to $(L_\ell, <_\ell)$, i.e. $s \triangleleft_{\mathcal{T}} t \implies F_\ell(s) <_\ell F_\ell(t)$.
(5) (*Guided extension*) if $s \in \mathcal{T}$ and $F_\ell(s) <_\ell a^1_\ell <_\ell a^2_\ell <_\ell a^3_\ell$ for $\ell = 1, 2$ then for some $t \in \mathcal{T}$ we have: $s \triangleleft_{\mathcal{T}} t$ and $a^1_\ell <_\ell F_\ell(t) <_\ell a^3_\ell$ for $\ell = 1, 2$.
(6) (*Weak surjection*) if $a^1_\ell <_\ell a^2_\ell <_\ell a^3_\ell$ for $\ell = 1, 2$ then for some $t \in \mathcal{T}$ we have $a^1_\ell <_\ell F_\ell(t) <_\ell a^3_\ell$ for $\ell = 1, 2$.
(7) $\mathcal{T}$ has a root $\mathrm{rt}(\mathcal{T})$, and for $\ell = 1, 2$ we have that $F_\ell(\mathrm{rt}(\mathcal{T})) = \mathrm{rt}(L_\ell)$.
(8) We will assume unless otherwise stated that $\mathbf{r}$ is *nontrivial*, meaning that L_1, L_2 and $\mathcal{T}$ are infinite.

Convention 4.11. In what follows, $\mathbf{r}$ will denote an L.C.S.P.

Discussion 4.12 (Intent of 4.10)**.** Continuing with the nominal analogy of "LCSP" to the "CSP" of [17], these remarks compare 4.10 to [17] 2.3-2.4. The reader of just this paper may feel free to skip these remarks.

4.10(3) Note that we do not ask for the tree to be pseudofinite or well ordered, only that the set below any node is linearly ordered.
4.10(4) The projection functions $F_\ell(x)$ are the parallel to $x(\max \mathrm{dom}(x), \ell)$ for CSP. Other than the requirements on the given projection functions F_ℓ, the linear orders in the tree and the orders $(L, <_L)$ may be quite different; e.g. one may be discrete, the other dense.
4.10(5) This is a substitute for successor: whereas for c.s.p.s it was important that the orders be pseudofinite, here we do not even assume the partial orders are discrete. We simply ask that for any element $s \in \mathcal{T}$ and any nontrivial interval above its projection to L_ℓ, we may find $t \in \mathcal{T}$ above s whose projection is in the given interval.
4.10(6) If in 4.10(5) we let $s = \mathrm{rt}(\mathcal{T})$ and allow $F_\ell(s) \leq a_\ell$ for $\ell = 1, 2$, then 4.10(6) follows. So condition (6) adds something only when $\bigwedge_{\ell=1}^{2} F_\ell(a^1_\ell) = \mathrm{rt}(\mathcal{T})$.

We now define the cut spectrum and several related invariants of $\mathbf{r}$, analogues of [17] Definition 2.8.

Definition 4.13 (The cut spectrum of an LCSP)**.** For an LCSP $\mathbf{r}$, $w \subseteq \{\nearrow, \swarrow\}$, $\ell \in \{1, 2\}$, let $\mathcal{C}_{\mathbf{r}}(w, \ell)$ be the set of pairs (κ, λ) such that for some $(\overline{a}, \overline{b})$:

(1) $(\overline{a}, \overline{b})$ is a pair of sequences of elements of L_ℓ
(2) $\kappa, \lambda \in \mathrm{Reg} \cup \{1\}$ but $\{\kappa, \lambda\} \neq \{1\}$
(3) $\overline{a}$ is strictly $<_\ell$-increasing of length κ
(4) $\overline{b}$ is strictly $<_\ell$-decreasing of length λ
(5) $a_\alpha <_\ell b_\beta$ when $\alpha < \kappa, \beta < \lambda$
(6) for no $c \in L_\ell$ do we have that $\alpha < \kappa \wedge \beta < \lambda \implies a_\alpha <_\ell c <_\ell b_\beta$
(7) if $(\nearrow \in w)$ then the cut is rising, i.e. for any $c \in L_\ell$:

$$\text{if } \bigwedge_{\alpha<\kappa} a_\alpha <_\ell c \;\text{ then }\; \bigvee_{\beta<\lambda} b_\beta \leq_\ell c$$

(8) if $(\swarrow \in w)$ then the cut is falling, i.e. for any $c \in L_\ell$:

$$\text{if } \bigwedge_{\beta<\lambda} c <_\ell b_\alpha \;\text{ then }\; \bigvee_{\alpha<\kappa} c \leq_\ell a_\alpha$$

Definition 4.14 (Related invariants)**.**

(B) Let $\mathfrak{p}_{\mathbf{r}}(w,\ell) = \min\{\kappa+\lambda \colon (\kappa,\lambda) \in \mathcal{C}_{\mathbf{r}}(w,\ell)\}$, i.e. the cardinality of first occurrence of a (rising, falling, or peculiar, depending on w) cut in L_ℓ.
Let $\mathfrak{p}_{\mathbf{r}}(w) = \min\{\mathfrak{p}_{\mathbf{r}}(w,1), \mathfrak{p}_{\mathbf{r}}(w,2)\}$.

(C) *Size of paths whose projections have no natural upper bound.*
Let $x \subseteq \{1,2\}$. Let $\mathfrak{T}_{\mathbf{r},x}$ be the set of regular cardinals κ such that for some sequence $\bar{t} = \langle t_\alpha \colon \alpha < \kappa \rangle$ of elements of $\mathcal{T}$,
  (a) $t_\beta \trianglelefteq_{\mathcal{T}} t_\alpha$ when $\beta < \alpha < \kappa$
  (b) for $\ell \in x \subseteq \{1,2\}$, <u>if</u> there are $a'_\ell, a_\ell \in L_\ell$ such that $F_\ell(t_\alpha) <_\ell a'_\ell < a_\ell$ for each $\alpha < \kappa$, <u>then</u> there is no $t \in \mathcal{T}$ such that: (i) $\alpha < \kappa \implies t_\alpha \trianglelefteq_{\mathcal{T}} t$ and (ii) $\ell \in x \implies F_\ell(t) <_\ell a_\ell$.
  (c) if $\ell \in \{1,2\}$ but $\ell \notin x$ or if no such elements a'_ℓ, a_ℓ exist, then there is no $t \in \mathcal{T}$ such that $\alpha < \kappa \implies t_\alpha \trianglelefteq_{\mathcal{T}} t$.

(D) Let $\mathfrak{t}_{\mathbf{r},x} = \min(\mathfrak{T}_{\mathbf{r},x})$, and $\mathfrak{t}_{\mathbf{r}} = \min\{\mathfrak{t}_{\mathbf{r},\emptyset}, \mathfrak{t}_{\mathbf{r},\{1\}}, \mathfrak{t}_{\mathbf{r},\{2\}}, \mathfrak{t}_{\mathbf{r},\{1,2\}}.\}$

(E) *Spectrum of cardinals which are robust under projection.*
Let $\Theta_{\mathbf{r}}(w,\ell)$ be the set of regular infinite cardinals $\kappa \leq ||M||$ such that <u>if</u> for some $t \in \mathcal{T}$, there is a sequence $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ with $\mathrm{cf}(\bar{a}) = \kappa$ which represents a w-cut of the linear order

$$\mathcal{T}_{\trianglelefteq_{\mathcal{T}}}(\mathbf{r},t) := (\{s \in \mathcal{T} \colon s \trianglelefteq_{\mathcal{T}} t\}, \triangleleft_{\mathcal{T}})$$

i.e. a cut which is rising if $w = \{\nearrow\}$, falling if $w = \{\swarrow\}$, and peculiar if $w = \{\nearrow, \swarrow\}$, <u>then</u> the projection

$$(\langle F_\ell(a_\alpha) \colon \alpha < \mathrm{cf}(\bar{a}) \rangle, \langle F_\ell(b_\beta) \colon \beta < \mathrm{cf}(\bar{b}) \rangle)$$

likewise represents a w-cut of $(L_\ell, <_\ell)$.

(F) $\Theta_{\mathbf{r}}(w) = \Theta_{\mathbf{r}}(w,1) \cap \Theta_{\mathbf{r}}(w,2)$.

(G) *Smoothness.*
  (a) $\mathbf{r}$ *has upper bounds* when for any $t_1, t_2 \in \mathcal{T}$ the set $\mathcal{T}_{\triangleleft_{\mathcal{T}}}(\mathbf{r},t_1) \cap \mathcal{T}_{\triangleleft_{\mathcal{T}}}(\mathbf{r},t_2)$ has a last element.
  (b) For $\ell \in \{1,2\}$ we say $\mathbf{r}$ is ℓ-smooth if for any $a, b \in L_\ell$ we have that $\{x \in L_\ell \colon x \leq_\ell a\} \cap \{x \in L_\ell \colon x \leq_\ell b\}$ has a $\leq_\ell$-last element.
  (c) $\mathbf{r}$ is *smooth* when: if $\ell \in \{1,2\}$, $t \in \mathcal{T}$ and $a_\ell \in L_\ell$ satisfies $F_\ell(t) \not<_\ell a_\ell$, <u>then</u> there is $s \in \mathcal{T}$ such that $s \triangleleft_{\mathcal{T}} t$, $F_\ell(s) <_\ell a_\ell$, and if $s_1 \triangleleft_{\mathcal{T}} t \ \wedge \ F_\ell(s_1) <_\ell a_\ell$ then $s_1 \triangleleft s$.

(H) $\mathbf{r}$ is endless when $(\forall s \in \mathcal{T})(\exists t \in \mathcal{T})(s \triangleleft_{\mathcal{T}} t)$.

Claim 4.15. Assume $\mathbf{r}$ is a LCSP and has upper bounds in the sense of $4.13(G)(a)$. If $t \in \mathcal{T}_{\mathbf{r}}$ then every cut of $(\mathcal{T}_{\mathbf{r},\leq t}, \triangleleft_{\mathcal{T}})$ is a peculiar cut of $(\mathcal{T}_{\mathbf{r}}, \triangleleft_{\mathbf{r}})$.

Proof. Note that every cut $(\bar{a}, \bar{b})$ of $(\mathcal{T}_{\mathbf{r},\leq t}, \triangleleft_{\mathcal{T}})$ is a peculiar cut of $(\mathcal{T}_{\mathbf{r},\leq t}, \triangleleft_{\mathcal{T}})$ since this is a linear order, recalling 4.4. The claim is that it is a peculiar cut in the tree itself. It is a falling cut in the tree by 4.4. To see it is a rising cut, suppose $c \in \mathcal{T}$ is such that $\alpha < \lg(\bar{a}) \implies a_\alpha \triangleleft c$. Let $\mathcal{T}_{\triangleleft c}$ denote the set of elements in $\mathcal{T}$ below c, which is linearly ordered by $\triangleleft$, and likewise for $\mathcal{T}_{\triangleleft b_0}$. Having upper bounds implies that $\mathcal{T}_{\triangleleft c} \cap \mathcal{T}_{\triangleleft b_0}$ has a $\triangleleft$-greatest element c_*. By assumption, $\alpha < \lg(\bar{a}) \implies a_\alpha \triangleleft c_* \triangleleft c$. But by construction $c_* \triangleleft b_0$, and since $(\bar{a}, \bar{b})$ is a cut there must be some $\beta < \lg(\bar{b})$ such that $b_\beta \triangleleft c_*$. So $b_\beta \triangleleft c$, which shows that the cut $(\bar{a}, \bar{b})$ is rising and therefore peculiar, recalling that c was arbitrary. □

Observation 4.16. For $\ell = 1, 2$,

(1) If $\mathbf{r}$ is smooth, then:
  (a) $\mathfrak{T}_{\mathbf{r},\emptyset} \subseteq \mathfrak{T}_{\mathbf{r},\{\ell\}} \subseteq \mathfrak{T}_{\mathbf{r},\{1,2\}}$.
  (b) $\mathfrak{t}_{\mathbf{r},\emptyset} \geq \mathfrak{t}_{\mathbf{r},\{\ell\}} \geq \mathfrak{t}_{\mathbf{r},\{1,2\}}$.
(2) $\mathcal{C}_{\mathbf{r}}(\emptyset,\ell) \subseteq \mathcal{C}_{\mathbf{r}}(\{\nearrow\},\ell) \cap \mathcal{C}_{\mathbf{r}}(\{\swarrow\},\ell) \subseteq \mathcal{C}_{\mathbf{r}}(\{\nearrow\},\ell) \cup \mathcal{C}_{\mathbf{r}}(\{\swarrow\},\ell)$.

Proof. (1) (b) follows from (a), and for (a), by 4.13(C)(c) if $\ell \in x$ then the sequence $\bar{t}$ is unbounded in $\mathcal{T}$, so 4.13(C)(b) will be immediately satisfied. Note that smoothness says more: given any strictly increasing $\bar{t}$ in $\mathcal{T}$, if $\langle F_\ell(t) : t \in \bar{t}\rangle$ is bounded by a_ℓ for $\ell = 1,2$, then if there is any upper bound t_* of $\bar{t}$ in $\mathcal{T}$, even if $F_\ell(t_*) \geq a_\ell$ for $\ell = 1,2$ then smoothness will give an element t contradicting 4.13(C)(b).

(2) Immediate from the definition. □

Definition 4.17. Let $\mathbf{r}$ be an LCSP.

(1) We call $\mathbf{r}$ *reflective* when: if $\ell \in \{1,2\}$ and $a <_\ell b$ then for some $c <_\ell d$ the partial orders $(a,b)_{L_\ell} := (\{x \in L_\ell \colon a <_\ell x <_\ell b\}, <_\ell)$ and $(c,d)_{L_\ell}$ are anti-isomorphic.
(2) We call $\mathbf{r}$ *symmetric* when $(L_1,<_1)$ and $(L_2,<_2)$ are isomorphic, and *strongly symmetric* if they are equal.

Claim 4.18. Let $\mathbf{r}$ be an LCSP and $\ell \in \{1,2\}$. If $\mathbf{r}$ is reflective and ℓ-smooth then:

(1) $(\kappa_1,\kappa_2) \in \mathcal{C}_{\mathbf{r}}(\emptyset,\ell) \iff (\kappa_2,\kappa_1) \in \mathcal{C}_{\mathbf{r}}(\emptyset,\ell)$.
(2) $(\kappa_1,\kappa_2) \in \mathcal{C}_{\mathbf{r}}(\{\nearrow\},\ell) \iff (\kappa_2,\kappa_1) \in \mathcal{C}_{\mathbf{r}}(\{\swarrow\},\ell)$.
(3) $(\kappa_1,\kappa_2) \in \mathcal{C}_{\mathbf{r}}(\{\nearrow,\swarrow\},\ell) \iff (\kappa_2,\kappa_1) \in \mathcal{C}_{\mathbf{r}}(\{\nearrow,\swarrow\},\ell)$.

Claim 4.19. Let $\mathbf{r}',\mathbf{r}$ be LCSPs, $w \subseteq \{\nearrow,\swarrow\}$.

(1) Suppose that $(L_\ell,<_\ell)^{\mathbf{r}} = (L_\ell,<_\ell)^{\mathbf{r}'}$ for $\ell = 1,2$. Then for $\ell = 1,2$:
  (a) $\mathcal{C}_{\mathbf{r}}(w,\ell) = \mathcal{C}_{\mathbf{r}'}(w,\ell)$.
  (b) $\mathfrak{p}_{\mathbf{r}}(w,\ell) = \mathfrak{p}_{\mathbf{r}'}(w,\ell)$.
(2) For $\ell, m \in \{1,2\}$, if $(L_\ell,<_\ell)^{\mathbf{r}}$ and $(L_m,<_m)^{\mathbf{r}'}$ are isomorphic then $\mathcal{C}_{\mathbf{r}}(w,\ell) = \mathcal{C}_{\mathbf{r}'}(w,m)$.
(3) For $\ell, m \in \{1,2\}$, if $(L_\ell,<_\ell)^{\mathbf{r}}$, $(L_m,<_m)^{\mathbf{r}'}$ are anti-isomorphic then $(\kappa_1,\kappa_2) \in \mathcal{C}_{\mathbf{r}}(w,\ell) \iff (\kappa_2,\kappa_1) \in \mathcal{C}_{\mathbf{r}'}(w,m)$.
(4) Assume $\mathbf{r}$ is a symmetric LCSP. Then $\mathcal{C}_{\mathbf{r}}(w,1) = \mathcal{C}_{\mathbf{r}}(w,2)$, so we may write simply $\mathcal{C}_{\mathbf{r}}(w)$.

Claim 4.20. Let $\mathbf{r}$ be an LCSP.

(1) If there is in $(L_\ell,<_\ell)^{\mathbf{r}}$ a strictly increasing sequence $\langle a_\alpha \colon \alpha < \delta\rangle$, δ a limit ordinal *then* there is such a sequence in $(L_{3-\ell},<_{3-\ell})^{\mathbf{r}}$ when $\delta \leq \mathfrak{t}_{\mathbf{s},\{\ell\}}$.
(2) If δ is a limit, $\mathbf{r}$ is endless and $\langle a_\alpha \colon \alpha < \delta\rangle$ is $<_\ell$-increasing and $\delta < \mathfrak{t}_{\mathbf{r},\{\ell\}}$ *then* there is a $<_{3-\ell}$-increasing sequence of length $\delta + \omega$.

Proof. (1) We choose $t_\alpha \in \mathcal{T}$ by induction on α such that:

- $\beta < \alpha \implies t_\beta \triangleleft_{\mathcal{T}} t_\alpha$
- if α is a successor then $a_{2\alpha} <_\ell F_\ell(g_\alpha) < a_{2\alpha+2}$

In the case $\alpha = 0$ apply 4.10(6), and in the case α successor apply 4.10(5).

For α limit we apply the hypothesis that $\delta \leq \mathfrak{t}_{\mathbf{r},\{\ell\}}$, 4.13(D). Note that we use $2\delta = \delta$ for every limit δ, so $\alpha < \delta \implies 2\alpha < \delta$.

This completes the induction, and $\langle F_{3-\ell}(t_\alpha) \colon \alpha < \delta\rangle$ is $<_{3-\ell}$-increasing, as desired.

(2) Having chosen $\langle t_\alpha \colon \alpha < \delta\rangle$ which is $<_{\mathcal{T}}$-increasing as above, we can choose t_0, $\bigwedge_{\alpha<\delta} t_\alpha <_{\mathcal{T}} t_\delta$ as $\delta < \mathcal{T}_{3,\{t\}}$. We can moreover choose $t_{\delta+m}$ by induction on

$n > 0$ such that $n = m+1$ implies $t_{\delta+m} <_{\mathcal{T}} t_{\delta+n}$, using the assumption that $\mathbf{r}$ is endless. Once more, $\langle F_{3-\ell}(t_\alpha) \colon \alpha < \delta + \omega \rangle$ is as required. $\square$

The main result of this section is the following general version of Uniqueness. Informally, Lemma 4.21 says: Suppose there is a ordinary (κ, θ_1)-cut, a priori not necessarily either rising or falling, in the first order L_1 and likewise an ordinary (κ, θ_2)-cut in the second order L_2, and suppose that κ is one of the cardinals whose "projections are $\nearrow$-true," meaning that any rising cut in the tree whose left side has cofinality κ projects to a rising cut in both orders. *Then* $\theta_1 = \theta_2$.

Lemma 4.21. If $(\kappa_\ell, \theta_\ell) \in \mathcal{C}_{\mathbf{r}}(\emptyset, \ell)$ for $\ell = 1, 2$ and $\kappa_1 = \kappa = \kappa_2 < \mathfrak{t}_{\mathbf{r},\{1,2\}}$ and $\kappa \in \Theta_{\mathbf{r}}(\{\nearrow\}, 1) \cap \Theta_{\mathbf{r}}(\{\nearrow\}, 2)$, *then* $\theta_1 = \theta_2$.

Proof. Let $(\overline{a}^\ell, \overline{b}^\ell)$ witness $(\kappa_\ell, \theta_\ell) \in \mathcal{C}_{\mathbf{r}}(\{\emptyset\}, \ell)$ for $\ell = 1, 2$, so $\kappa_\ell, \theta_\ell \geq 1$.

First, we try to choose t_α by induction on $\alpha < \kappa$ such that:

- $t_\alpha \in \mathcal{T}$
- $\beta < \alpha \implies t_\beta \lhd_{\mathcal{T}} t_\alpha$
- $a^\ell_{2\alpha+2} \lhd_{\mathcal{T}} F(t_\alpha) \unlhd_{\mathcal{T}} a^\ell_{2\alpha+4}$ for $\ell = 1, 2$

For $\alpha = 0$, apply 4.10 clause (6), since the induction asks that our witness is above a^ℓ_2. For $\alpha = \beta + 1$, apply 4.10 clause (5) with t_β here in place of s there.

For α a limit ordinal $< \kappa$, we have that $\alpha < \mathfrak{t}_{\mathbf{r},\{1,2\}}$ so $\mathrm{cf}(\alpha) \notin \mathfrak{T}_{\mathbf{r},\{1,2\}}$. Moreover, $\langle t_\beta \colon \beta < \alpha \rangle$ is $\lhd_{\mathcal{T}}$-increasing and $\ell \in \{1,2\} \wedge \beta < \alpha \implies F_\ell(t_\beta) <_\ell a^\ell_{2_\alpha} <_\ell a^\ell_{2\alpha+1} <_\ell a^\ell_{2\alpha+2}$. Thus, by the definition 4.13(C)-(D) of "$\mathrm{cf}(\alpha) < \mathfrak{t}_{\mathbf{r},\{1,2\}}$," there is t'_α such that $\beta < \alpha \implies t_\beta \unlhd_{\mathcal{T}} t'_\alpha$ and $\ell \in \{1,2\} \implies F_\ell(t'_\alpha) <_\ell a^\ell_{4\alpha+2}$. Now choose t_α as in the case $\alpha = \beta + 1$.

Second, having chosen the sequence $\overline{t} = \langle t_\alpha \colon \alpha < \kappa \rangle$, since this sequence is $\unlhd$-increasing and F is a homomorphism in the sense of 4.10(4), we have that $\bigwedge_{\ell=1}^{2} F(t_\alpha) < b^\ell_1 < b^\ell_0$. Since $\kappa < \mathfrak{t}_{\mathbf{r}}$ there is $t \in \mathcal{T}$ such that $\alpha < \kappa \implies t_\alpha \lhd_{\mathcal{T}} t$ and $\bigwedge_{\ell=1}^{2} F_\ell(t) <_\ell b^\ell_0$. By definition of tree, the set $\mathcal{T}_{\lhd t}$ of elements $\lhd$-below $\mathcal{T}$ is a linear order in which $\langle t_\alpha \colon \alpha < \kappa \rangle$ is increasing and bounded. Hence for some $\theta \in \mathrm{Reg} \cup \{1\}$, for some $\lhd_{\mathcal{T}}$-decreasing sequence $\overline{s} = \langle s_\alpha \colon \alpha < \theta \rangle$ in $\mathcal{T}_{\lhd t}$ we have that the pair $(\overline{t}, \overline{s})$ represents a cut. So it will suffice by transitivity of equality to prove that $\theta_\ell = \theta$ for $\ell = 1, 2$.

For each $\alpha < \theta$ and $\ell \in \{1, 2\}$, let $c^\ell_\alpha = F_\ell(s_\alpha) \in L_\ell$. Since F_ℓ is a homomorphism, necessarily $\overline{c}^\ell = \langle c^\ell_\alpha \colon \alpha < \theta \rangle$ is $<_\ell$-decreasing and each of its elements is $<_\ell$-above $\overline{a}^\ell$. Since $\mathcal{T}_{\lhd t}$ is a linear order, $(\overline{t}, \overline{s})$ represents a rising, and even peculiar, cut of $\mathcal{T}_{\lhd t}$, recalling 4.3. As $\kappa \in \Theta_{\mathbf{r}}(\{\nearrow\}, \ell)$ for $\ell = 1, 2$, we have by definition 4.13(E) that the pair $(\overline{a}^\ell, \overline{c}^\ell)$ represents a rising cut of $(L_\ell, <_\ell)$ for $\ell = 1, 2$.

Thus recalling $(\overline{a}^\ell, \overline{b}^\ell)$ represents a cut, by 4.6 we have that $\mathrm{cf}(\overline{c}^\ell) = \mathrm{cf}(\overline{b}^\ell)$. But $\theta = \mathrm{cf}(\overline{s}) = \mathrm{cf}(\overline{c}^\ell)$, $\theta_\ell = \mathrm{cf}(\overline{b}^\ell)$ so $\theta = \theta_\ell$, as desired. $\square$

Corollary 4.22. Assuming symmetry, the same holds swapping the occurrences of " $\nearrow$ " for " $\swarrow$ " in the statement of Lemma 4.21.

A dual claim to 4.21 is:

Corollary 4.23. If $(\kappa_\ell, \theta_\ell) \in \mathcal{C}_{\mathbf{r}}(\{\swarrow\}, \ell)$ for $\ell = 1, 2$ and $\kappa_1 = \kappa = \kappa_2 < \mathfrak{t}_{\mathbf{r},\{1,2\}}$ and $\kappa \in \Theta_{\mathbf{r}}(\{\emptyset\}, 1) \cap \Theta_{\mathbf{r}}(\{\emptyset\}, 2)$, *then* $\theta_1 = \theta_2$.

Proof. In the proof of 4.21, make the obvious changes: (a) "Since $\mathcal{T}_{\triangleleft}$ is a linear order, $(\bar{t}, \bar{s})$ represents a *falling*, and even peculiar, cut of $\mathcal{T}_{\triangleleft}$, recalling 4.3." (b) "As $\kappa \in \Theta_{\mathbf{r}}(\{\swarrow\}, \ell)$ for $\ell = 1, 2$, we have by definition 4.13(E) that the pair $(\bar{a}^{\ell}, \bar{c}^{\ell})$ represents a *falling* cut of $(L_{\ell}, <_{\ell})$ for $\ell = 1, 2$." ☐

Conclusion 4.24. Let $\mathbf{r}$ be an LCSP and $w_0 \subseteq w_1 \subseteq \{\nearrow, \swarrow\}$, where either:

$$w_0 = \emptyset \ \wedge \ \{\nearrow\} \subseteq w_1 \text{ or } w_0 = \{\nearrow\} \ \wedge \ w_0 \subseteq w_1 \subseteq \{\nearrow, \swarrow\}$$

Then:

(1) There is a function $\mathrm{lcf}(-, \mathbf{r})$ with domain $\{\kappa \colon \kappa \in \Theta_{\mathbf{r}}(w_1) \ \wedge \ \kappa < \mathfrak{t}_{\mathbf{r},\{1,2\}}\}$ such that:
  (a) $\mathrm{lcf}(-, \mathbf{r})$ is a regular cardinal $\geq \mathfrak{p}_{\mathbf{r}}$
  (b) if $\ell \in \{1, 2\}$, $\kappa < \mathfrak{t}_{\mathbf{r},\{1,2\}}$, $\kappa \in \Theta_{\mathbf{r}}(w_1)$ and $\bar{a}$ is a strictly $<_{\ell}$-increasing sequence of length κ then:
    (i) for some $\bar{b}$ of length $\mathrm{lcf}(\kappa, \mathbf{r})$, the pair $(\bar{a}, \bar{b})$ is a w_1-cut of $(L_{\ell}, <_{\ell})$.
    (ii) if $\bar{b}'$ is such that $(\bar{a}, \bar{b}')$ is a w_0-cut of $(L_{\ell}, <_{\ell})$ then $\mathrm{cf}(\bar{b}') = \kappa$.

(2) If $\mathbf{r}$ is symmetric, the parallel statement holds for decreasing sequences with $\swarrow$ in place of $\nearrow$.

Proof. (1) by Lemma 4.21, (2) by Corollary 4.23. ☐

Some examples of LCSPs are given in the next section.

5. Examples

This section gives several examples of LCSPs, in increasing order of model-theoretic interest. Recall the discussion about real closed fields versus Peano arithmetic from §4. Since we have already established a mild analogue of uniqueness in this case (4.21), it is interesting to see that the main example of this section, 5.11, shows that the basic nontrivial example of an LCSP is already much less complex model-theoretically than models of set theory or PA: it is "half-dependent," i.e. not 2-independent, Definition 5.8 below. It would be interesting to investigate further how model-theoretically uncomplicated such an example may be.

Example 5.1 (The standard finite example)**.** For a finite number $\mathbf{n} > 0$, we define $\mathbf{r} = \mathbf{r}^1_{\mathbf{n}}$ by:

(1) $L^{\mathbf{r}}_{\ell} = \mathbf{n} = \{0, \ldots, \mathbf{n} - 1\}$
(2) $<^{\mathbf{r}}_{\ell}$ is the natural order on $\mathbf{n}$
(3) $\mathrm{rt}(L^{\mathbf{r}}_{\ell}) = 0$
(4) Let $\mathcal{T}^{\mathbf{r}}$ be the set of all η such that for some $m \in [1, \mathbf{n})$,
  - $\eta = \langle \eta(\ell, i) \colon \ell \in \{1, 2\}, i \leq m \rangle$, so let $\mathrm{lev}(\eta) = m$.
  - $\eta(\ell, 0) = 0$
  - $\eta(\ell, i) < \mathbf{n}$
  - $i < j < m \implies \eta(\ell, i) < \eta(\ell, j)$
(5) $\eta <^{\mathbf{r}}_{\mathcal{T}} \sigma$ iff $\eta = \sigma \restriction \{(\ell, i) \colon \ell \in \{1, 2\}, i < \lg(\eta)\}$
(6) $F_{\ell}(\eta) = \eta(\ell, \mathrm{lev}(\eta))$
(7) $\mathrm{rt}(\mathcal{T}^{\mathbf{r}}) = \langle 0, 0 \rangle$.

Claim 5.2. For every $\mathbf{n} > 0$, $\mathbf{r}_{\mathbf{n}}$ is an LCSP, which moreover:

(1) is smooth, and also fact ℓ-smooth for $\ell = 1, 2$.

(2) is reflective; in fact every interval in $(L^{\mathbf{r}}_\ell, <^{\mathbf{r}}_\ell)$ is anti-isomorphic to itself.
(3) is strongly symmetric.

Example 5.3 (The pseudofinite case)**.** Ultraproducts of the $M_{\mathbf{r}^1_{\mathbf{n}}}$'s, or just of models of $\bigcap_n \bigcup_{m>n} Th(M_{\mathbf{r}_m})$, are smooth, strongly symmetric LCSPs.

Example 5.4 (The standard infinite example)**.** For non-zero ordinals α, β we define $\mathbf{r} = \mathbf{r}^2_{\alpha,\beta}$ as follows, with $\ell \in \{1, 2\}$:

(1) $L^{\mathbf{r}}_\ell = L_\ell(\mathbf{r})$ is α
(2) $<^{\mathbf{r}}_\ell$ is the usual order
(3) $\mathrm{rt}(L^{\mathbf{r}}_\ell) = 0$
(4) $\eta \in \mathcal{T}_{\mathbf{r}}$ iff:
  (a) $\eta = \langle \eta(\ell, i) \colon \ell \in \{1, 2\}, i \leq \mathrm{lev}(\eta)\rangle$
  (b) $\mathrm{lev}(\eta)$ is an ordinal $\leq \beta$
  (c) $i < j \leq \mathrm{lev}(\eta) \implies \eta(\ell, i) < \eta(\ell, j)\}$
(5) $<_{\mathcal{T}_{\mathbf{r}}}$ is defined as above: $\eta <_{\mathcal{T}_{\mathbf{r}}} \nu$ iff:
  (a) $\eta, \nu \in \mathcal{T}_{\mathbf{r}}$
  (b) $\mathrm{lev}(\eta) \leq \mathrm{lev}(\nu)$
  (c) $\ell \in \{1, 2\} \wedge i < 1 + \mathrm{lev}(\eta)$ implies $\eta(\ell, i) = \nu(\ell, i)$
  (d) $F_\ell(\eta) = \eta(\ell, \mathrm{lev}(\eta))$.

Claim 5.5. Let $\alpha > 0$ and let β be a limit ordinal. Let $\mathbf{r} = \mathbf{r}^2_{\alpha,\beta}$ as in 5.4. Then:

(1) $\mathbf{r}$ is an LCSP.
(2) $\mathbf{r}$ is smooth and also ℓ-smooth for $\ell = 1, 2$.
(3) if α, β are limit ordinals then $\mathbf{r}$ is endless
(4) $\mathbf{r}$ is strongly symmetric
(5) however, $\mathbf{r}$ is not reflective if α is infinite.

Example 5.6 (A general case)**.** For $\ell = 1, 2$, let $\mathfrak{B}_\ell$ be a Boolean algebra with with $0_{\mathfrak{B}}$ but no $1_{\mathfrak{B}}$, or just a partial order with a minimal element $0_{\mathfrak{B}}$. Let $\beta > 0$ be a limit ordinal. Let $\mathbf{r} = \mathbf{r}^3_{\mathfrak{B},\beta}$ be as follows:

(1) $L_\ell[\mathbf{r}]$ is $\mathfrak{B}_\ell$
(2) $<^{\mathbf{r}}_\ell$ is $\leq_{\mathfrak{B}_\ell}$
(3) $\mathrm{rt}(L^{\mathbf{r}}_\ell) = 0_{\mathfrak{B}_\ell}$
(4) $\mathcal{T}_{\mathbf{r}}$ is the set of η such that for some $\mathrm{lev}(\eta) < \beta$,
  (a) $\eta = \langle \eta(\ell, i) \colon \ell \in \{1, 2\}, i \leq \mathrm{lev}(\eta)\rangle$
  (b) $\eta(\ell, i) \in \mathfrak{B}_\ell$
  (c) $i < j \leq \lg(\eta) \implies \eta(\ell, i) <_{\mathfrak{B}_\ell} \eta(\ell, j)$
(5) $F_\ell(\eta) = \eta(\ell, \mathrm{lev}(\eta))$
(6) $\mathrm{rt}(T_{\mathcal{R}}) = \langle 0_{\mathfrak{B}_1}, 0_{\mathfrak{B}_2}\rangle$.

We now work towards Conclusion 5.19, existence of an LCSP $\mathbf{r}$ such that the first order theory $Th(M_{\mathbf{r}})$ is $\frac{1}{2}$-dependent.

Definition 5.7. We say the formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ is 2-independent in the first order complete theory T when in $\mathfrak{C}_T$ there exist $\langle \bar{a}_i : i < \omega\rangle$, $\langle \bar{b}_j : j < \omega\rangle$ with $\ell(\bar{a}_i) = \ell(y)$ for $i < \omega$ and $\ell(\bar{b}_i) = \ell(z)$ for $j < \omega$, such that for any function $F : \omega \times \omega \to \{0, 1\}$ the set of formulas

$$\{\varphi(x, \bar{a}_i, \bar{b}_j) \colon F(i, j) = 1\} \cup \{\neg\varphi(x, \bar{a}_k, \bar{b}_l) \colon F(k, l) = 0\}$$

is consistent.

Definition 5.8. We say that the theory T is half-dependent if no formula φ is 2-independent in any model of T. [12]

Fact 5.9 ([29] 5.66)**.** T is $\frac{1}{n}$-dependent when for every m, ℓ and finite $\Delta \subseteq \mathcal{L}(\tau_T)$, for infinitely many $k < \omega$, we have $|A| \leq k \implies |\mathbf{S}^m_\Delta(A)| < 2^{(k/\ell)^n}$.

Definition 5.10. Let τ_* be the vocabulary consisting of:

- P_1, P_2, P_3 unary predicates
- $<_1, <_2, <_3$ binary predicates
- c_1, c_2, c_3 individual constants
- F_1, F_2 are unary function symbols
- F is a binary function
- R_1, R_2 three-place predicates
- G a three-place function.

In the next Definition 5.11, we allow the interpretation of function symbols to be partial functions. [This is written to be a finite universal theory.]

Definition 5.11. Let K be the set of τ_*-models M such that

(1) P_1^M, P_2^M, P_3^M is a partition of $|M|$
(2) $<_\ell^M$ is a linear order of P_ℓ^M with first element c_ℓ^M for $\ell = 1, 2$
(3) $<_3^M$ is a partial order of P_3^M which is a tree with root c_3^M
(4) F^M is a two place function from P_3^M into itself; $F^M(s_1, s_2)$ is the maximal common $\leq_3$-lower bound
(5) F_ℓ^M is a function from P_3 into P_ℓ, monotonic increasing i.e. such that $s <_3^M t \implies F_\ell(s) <_\ell^M F_\ell(t)$, and $F_\ell^M(c_3) = c_\ell$.
(6) for $\ell = 1, 2$, the relation R_ℓ^M satisfies:
  (a) $R_\ell^M \subseteq \{(a_1, a_2, t) \colon t \in P_3^M, a_\ell \in P_\ell^M, a_\ell \leq F_\ell^M(t))\}$.
  (b) if $s <_3^M t$ and $a_\ell \in P_\ell^M$ then $(a_1, a_2, s) \in R_\ell^M$ iff
$$(a_1, a_2, t) \in R_\ell^M \wedge \bigwedge_{\ell=1}^{2} a_\ell \leq F_\ell^M(s)$$
  (c) if $(a_1, a_2, t) \in R_1^M$, $a_1' \leq_1^M a_1$ and $a_2 \leq_2^M a_2' \leq F_2^M(t)$, then $(a_1', a_2', t) \in R_1^N$.
  (d) if $(a_1, a_2, t) \in R_2^M$, $a_1 \leq_1^M a_1' \leq F_1^M(t)$, and $a_2' \leq_2^M a_2$, then $(a_1', a_2', t) \in R_2^N$.
(7) $\mathrm{dom}(G_\ell)$ is named by a predicate X_ℓ
(8) $\mathrm{dom}(G_\ell) \subseteq \{(a_\ell, t) \colon t \in P_3^M, a_\ell \in P_\ell^M, a_\ell \leq_\ell^M F_\ell^M(t)\}$ for $\ell = 1, 2$
(9) for $t \in P_3^M$ and $a_1 \in P_1^M$, $a_2 \in P_2^M$ the following are equivalent.
  (a) $G_1^M(a_1, t) = a_2$
  (b) $G_2^M(a_2, t) = a_1$
  (c) $(\exists s)(s \leq_3 t \wedge F_1(s) = a_1 \wedge F_2(s) = a_2)$
  and both conditions imply that $(a_1, a_2, t) \in R_1^M \wedge (a_1, a_2, t) \in R_2^M$.
(10) for every $t \in P_3^M$ and $\ell \in \{1, 2\}$, $a \mapsto G_\ell^M(a, t_*)$ is a partial increasing function from P_ℓ^M into $P_{3-\ell}^M$.

$K_{<\aleph_0}$ is the class of finite $M \in K$.

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[12] The notation is from [29], the idea being that reciprocals were a useful way to keep track of the negation.

Example 5.12. The following structure belongs to K from 5.11. Let P_1, P_2 name copies of $(\mathbb{N}, <)$ with the usual order called $<_1$, $<_2$ respectively. Let P_3 be the tree whose elements are finite sequences of natural numbers, partially ordered by inclusion, called $\triangleleft$. Let F take any two elements of P_3 to their longest common initial segment. Let $\mathrm{Pr} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Gödel pairing function (this is external and used only in defining the model). For $\ell = 1, 2$ define F_ℓ by induction on $\lg(t)$. Let $F_\ell(\emptyset) = 0$. For $t \in \mathcal{T}$ with $\lg(t) = 1$, if $t(0) = \mathrm{Pr}(a_1, a_2)$ let $F_\ell(t) = a_\ell$. For $t \in \mathcal{T}$ with $s \triangleleft t$ and $\lg(t) = n + 1 = \lg(s) + 1$, suppose that $s(n-1) = \mathrm{Pr}(a_1, a_2)$ and $t(n) = \mathrm{Pr}(b_1, b_2)$ let $F_\ell(t) = a_\ell + b_\ell$. [Clearly this satisfies the definition of an LCSP.] For any $(a_1, a_2, t) \in P_1 \times P_2 \times P_3$ and $\ell = 1, 2$, let $R_\ell(a_1, a_2, t)$ hold when $a_1 \leq F_1(t)$ and $a_2 \leq F_2(t)$. For each $t \in P_3$, let $G_1(-, t)$ be the partial order-preserving bijection whose graph is given by $\{(F_1(s), F_2(s)) : s \triangleleft t\}$, and let $G_2(-, t)$ be the corresponding partial order-preserving bijection in the other direction.

In this example, although we have independence arising from e.g. "guided extension" in 4.10(5), freedom in the sense of 5.8 is curtailed because the F_ℓ must be homomorphisms.

Note again that the functions G_ℓ in 5.11 are allowed to be partial and the relations are allowed to be nonempty; once e.g. R_1, R_2 are nonempty, the closure conditions (c), (d) apply for R_1, R_2 respectively.

Definition 5.13. Suppose $M, N \in K$.

(a) Let $M \subseteq N$ mean: $|M| \subseteq |N|$, $X \in \tau_* \implies X^M \subseteq X^N$, and M is closed under F^N_ℓ, F^N but not necessarily under G^M_ℓ.
(b) For $X \subseteq M$, $\mathrm{cl}(X, M)$ is the closure of $X \cup \{c^M_\ell : \ell = 1, 2, 3\}$ and the partial functions of M.

Claim 5.14. If $M \in K$ and $t_* \in P^M_3$, then there is N such that, for $\ell = 1, 2$,

(1) $N \in K$, $M \subseteq N$
(2) $||M|| < ||N|| \leq ||M|| + |P^M_1| + |P^M_2|$
(3) $G^N_\ell(a_\ell, t_*)$ is well defined iff $a_\ell \in P^N_\ell$ and $a_\ell \leq^N_\ell F^N_\ell(t_*)$.

Claim 5.15.

(1) K is the class of models of a finite universal theory, call it T_0.
(2) If $M \in K$, $X \subseteq M$ is finite then $\mathrm{cl}(X, M)$ [meaning closure under F, F_1, F_2] has $\leq 6|X| + 3$ elements. If we close also under the Gs, then the closure has $\leq (6|X| + 3)^2$ elements.
(3) If $M \subsetneq N \in K$ are finite, then for some M_1 we have that $M \subsetneq M_1 \subseteq N$ and one of the following occurs:
   (a) there are $\ell \in \{1, 2\}$ and $a_\ell \in P^N_\ell \setminus M$, $|M_1| = |M| \cup \{a_\ell\}$
   (b) there is $t \in P^N_3 \setminus M$ such that $|M_1| = |M| \cup \{t\}$

Proof. (1) By 5.11.

(2) Recall that F gives the greatest lower bound of two elements of P_3. Then $X_3 := \mathrm{cl}(X \cap P_3, M) \cap P_3 = \{F(s, t) \colon s, t \in X \cap P_3\}$ has $\leq 2|X \cap P_3|$ members. Subsequent closure under F_1 adds at most X_3 elements and likewise for F_2. The three extra elements are because we are obligated to include the constants if they are not already in X.

(3) If $P^M_1 \neq P^N_1$ choose $\ell = 1$, $a_\ell \in P^N_1 \setminus P^M_1$ and $M_1 := N \upharpoonright (|M| \cup \{a_\ell\})$ are as required in (a).

If $P_1^M = P_1^N$ but $P_2^M \neq P_2^N$ choose $\ell = 2$, $a_\ell \in P_2^N \setminus P_2^M$ and $M_1 = N \restriction (|M| \cup \{a_\ell\})$ are as required in (a).

If $\bigwedge_{\ell=1}^{2} P_\ell^M = P_\ell^N$ then necessarily $P_3^M \neq P_3^N$. Choose $t \in P_3^N \setminus P_3^M$ such that $|\{s \in P_3^N : s <_3^N t\}|$ is minimal, and finite. Let $M_3 = N \restriction (|M| \cup \{t\})$, so this is as required in (b). □

Claim 5.16. We have that $K_{<\aleph_0}$:

(1) is nonempty, in fact there is $M_* \in K_{\aleph_0}$ embeddable uniquely into any $M \in K$.
(2) has the JEP over the individual constants.
(3) has the disjoint amalgamation property, with universe given by the union.

Proof. (1) Define M to be the model with set of elements $\{1, 2, 3\}$. For $\ell = 1, 2$, let $P_\ell^M = \{\ell\}$, let $F_\ell^M(3) = \ell$, let $R_\ell^M = \{(1, 2, 3)\}$, and let $G_\ell^M(\ell, 3) = \ell$. For $\ell = 1, 2, 3$, let $c_\ell^M = \ell$, and finally let $F^M(3, 3) = 3$.

(2) Follows from (1) and (3).

(3) Assume $M_0 \subseteq M_1 \in K$, $M_0 \subseteq M_1$, $M_1 \cap M_2 = M_0$ and the M_ℓ are finite. Recall we aren't necessarily assuming closure under the Gs. By the previous claim, without loss of generality $M_\ell \setminus M_0$ has a single element, so the conditions are easily verified. □

Conclusion 5.17. T_0 has a model completion T_1 which has elimination of quantifiers and is categorical in $\aleph_0$.

Proof. The class $K_{<\aleph_0}$ of finite members of K is countable, closed under isomorphism and under substructure ("HP"), uniformly locally finite (we may bound the size of a model generated by n elements by 5.15(2)), and has JEP and AP by 5.16. Quoting Hodges [6] Thoerem 7.4.1 p. 349, its Fraissé limit M has a theory T_1 which is ω-categorical and has quantifier elimination. This theory is complete and model complete (for model completeness, see e.g. [6] Theorem 8.3.1 p. 374 (e) $\to$ (a)). Since M embeds all finite submodels of elements of K and since any finite subset of any $N \equiv M$ occurs as a subset of an element of $K_{<\aleph_0}$, it is straightforward to show by compactness that any model of T_0 may be embedded in some model of T_1 and vice versa. So T_1 is the model companion of T_0 and has the required properties. □

Claim 5.18. (1) If $M \subseteq N$ are from $K_{<\aleph_0}$, and $a \in N \setminus M$, then there is a unique M_1 such that $M_1 \in K$, $M \subseteq M_1 \subseteq N$, and for $\ell \in \{1, 2\}$, $a \in P_\ell^N \implies |M_1| = |M| \cup \{a\}$ while $a \in P_3^N$ implies $|M_1| = |M| \cup \{a, F_1^M(a), F_2^M(a)\}$.

(2) If $M \subseteq \mathfrak{C}_T$, M finite with n elements,
   (a) if $\ell \in \{1, 2\}$, $\{p \in \mathbf{S}(M) : P_\ell(x) \in p\}$ has

$$\leq |P_\ell^M| + |P_\ell^M| \times (2|P_{3-\ell}^n|^{|P_3^M|}) \leq n^2$$

   members, or just $\leq n + (2n)^{n+1} \leq (2n)^{n+2}$.
   (b) if $\ell = 3$ then

$$|\{p \in \mathbf{S}(M) : P_3(x) \in p\}| \leq \max\{|P_1^M|^{|P_2^M|}, |P_2^M|^{|P_1^M|}\} \leq (2n)^{\log n}.$$

Proof. (1) Follows from the axioms.

(2) For clause (a): by part (1) and the choice of T it suffices to bound

$$|\{\mathrm{tp}_{\mathrm{qf}}(b, M, N) : M \subseteq N \in K, N \setminus N = \{n \in P_\ell^M \setminus P_\ell^M\}$$

What freedom do we have? First, $N \models$ "$c_\ell <_\ell b$" so there are $|P^M_\ell|$ possible cuts which b realizes in $(P^M_\ell, <^N_\ell)$ over P^M_ℓ. Second, for each $t \in P^M_3$ we will need to decide R_1, R_2 (and thus G_1, G_2). There are $\leq 2|P^M_{3-\ell}|$ possibilities, so all together we have $|P^M_\ell| \times (2|P^M_{3-\ell}|)^{|P^M_3|}$.

For clause (b): We have $\leq 2|P^M_\ell|$ choices for $F^N_\ell(b)$. For the R_ℓ, assuming $i \in \{1,2\}$, $|P^M_i| \leq |P^M_{3-i}|$, so the number of possibilites for the R_ℓ is $\leq 2|P_{3-i}|^{|P^M_i|}$, so the count follows. □

Conclusion 5.19. T_1 is $\frac{1}{2}$-dependent.

Proof. For some real constant c, for every sufficiently large n, if $A \subseteq \mathfrak{C}_T$, $|A| = n$ then $|\mathbf{S}(A)| \leq c^{n \log n}$ by 5.18(2). Hence by 5.9 the result follows. □

Claim 5.20. If M is a model of T, then there is a unique endless LCSP $\mathbf{r}$ such that $M_\mathbf{r} = M$, $(L^\mathbf{r}_\ell, <^\mathbf{r}_\ell) = (P^M_\ell, <^M_\ell)$ for $\ell = 1, 2$, $\mathcal{T}_\mathbf{r} = (P^M_3, <^M_3)$. Moreover, $\mathbf{r}$ is smooth.

Proof. We check Definition 4.10. The only non-obvious condition is (5), which follows by model completeness of T, as does "endless."

To see that $\mathbf{r}$ is smooth: for $\mathbf{i} = 3$, this holds by F^N. For $\mathbf{i} = 1, 2$, this is trivial as $(P^M_\ell, <^M_\ell)$ is a linear order. For $\mathbf{i} = 4$, the point is that $a \mapsto G^M_\ell(a, t)$ is an isomorphism from $(\mathcal{T}_\mathbf{r})_{\leq t}$ onto $(L_\ell, <_\ell)_{\leq F^M_\ell(t)}$. □

In the example above (5.12) we have the independence property from guided extension but not, as just shown, 2-independence, motivating the following problem.

Problem 5.21. Determine whether meaningful analogues of Uniqueness hold in any NIP theory extending the theory of linear order.

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