OPEN PROBLEMS ON ULTRAFILTERS AND SOME CONNECTIONS TO THE CONTINUUM

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Abstract. We discuss a range of open problems at the intersection of set theory, model theory, and general topology, mainly around the construction of ultrafilters. Along the way we prove a uniqueness theorem for a weak notion of cut.

For Hugh Woodin on the occasion of his birthday.

Recently we discovered a surprising connection [15] between a model-theoretic problem from the sixties known as Keisler's order and a question about cardinal invariants of the continuum arising from work of Rothberger in the forties [20], whether $\mathfrak{p} = \mathfrak{t}$. Keisler's order compares theories via the saturation properties of their regular ultrapowers. Its study has brought certain connections of model theory and set theory sharply into focus, which has been very helpful in illuminating model-theoretic structure. Indications from continuing work on Keisler's order are that the connection of the two problems was more than a lucky coincidence, and that the potential interest of the order and of regular ultrafilters for set theory and general topology also goes potentially much beyond this one example.

We therefore present here a series of representative open problems in set theory, motivated by these interactions with model theory, and mainly around the construction of regular ultrafilters. Along the way we prove a uniqueness theorem for a weak notion of cut, Theorem 2.4, and give a new sufficient condition for goodness of an ultrafilter, Lemma 3.2. We note that the emphasis here on regular ultrafilters reverses a long trend of emphasis in set theory: already by the 1960s, the model-theoretic and set-theoretic use of ultrafilters had diverged, with model theorists focusing on regular ultrafilters for their uniform saturation properties and set theorists often focusing (over uncountable sets) on ultrafilters which were non regular (e.g. were normal). We hope such a paper will help to centralize and communicate intuition and to stimulate work from both sides in this fascinating and largely unexplored area.

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1. The cut spectrum

We begin with the problem of determining the possible *cut spectrum* of a regular ultrafilter $\mathcal{D}$. (After stating the problem, we will explain all of the terms used.)

General Problem 1.1 (cf. [15] 2.9)**.** *For $\lambda \geq \aleph_0$ and $\mathcal{D}$ a regular ultrafilter on I, $|I| = \lambda$, determine the possible values of*

$$\begin{aligned}\mathcal{C}(\mathcal{D}) =\{&(\kappa_1, \kappa_2) \; : \; \kappa_1, \kappa_2 \textit{ are infinite and regular,} \\ &\kappa_1 + \kappa_2 \leq |I| \textit{ and } (\omega, <)^I/\mathcal{D} \textit{ has a } (\kappa_1, \kappa_2)\textit{-cut}\}.\end{aligned}$$

First, say that a linearly ordered set has a (κ_1, κ_2)-pre-cut if there is a strictly increasing sequence $\bar{a} = \langle a_\alpha : \alpha < \kappa_1 \rangle$ of cofinality κ_1 and a strictly decreasing sequence $\bar{b} = \langle b_\beta : \beta < \kappa_2 \rangle$ of cofinality κ_2 such that for $\alpha < \alpha' < \kappa_1$ and $\beta < \beta' < \kappa_2$ we have $a_\alpha < a_{\alpha'} < b_{\beta'} < b_\beta$. We say the set has a (κ_1, κ_2)-cut if there are $\bar{a}, \bar{b}$ as above and in addition, there is no c such that for all $\alpha < \kappa_1$ and $\beta < \kappa_2$, $a_\alpha < c < b_\beta$. (Some authors use "gap" in place of "unfilled cut." For us a cut is always unfilled.) We say a filter $\mathcal{D}$ is "on I" to mean that $\mathcal{D} \subseteq \mathcal{P}(I)$ and $I \in \mathcal{D}$.

Recall that a filter $\mathcal{D}$ on I is called *regular* if there exist a family of $|I|$ elements of $\mathcal{D}$, called a regularizing family, such that the intersection of any infinite subfamily is empty. Equivalently, each $t \in I$ belongs to at most finitely many elements of the regularizing family. Such ultrafilters always exist: given an infinite cardinal λ, let $I = [\lambda]^{<\aleph_0}$ and consider the family $\mathcal{F} = \{X_\beta : \beta < \lambda\}$ where $X_\beta = \{v \in I : \beta \in v\}$. Then $\mathcal{F}$ has the desired regularity property and also the finite intersection property, so it may be extended to an ultrafilter.[1]

Regularity of $\mathcal{D}$ entails that 1.1 would not change if we used an infinite linear order other than ω. More precisely, we might have defined:

Definition 1.2. *Given an ultrafilter $\mathcal{D}$ on I and a linear order X, let*

$$\begin{aligned}\mathcal{C}(\mathcal{D}, X) =\{&(\kappa_1, \kappa_2) : \kappa_1, \kappa_2 \textit{ infinite and regular,} \\ &\kappa_1 + \kappa_2 \leq |I| \textit{ and } X^I/\mathcal{D} \textit{ has a } (\kappa_1, \kappa_2)\textit{-cut}\}.\end{aligned}$$

Then by regularity we have the following. (Morally speaking, this fact is a consequence of Keisler's stronger result on saturation, [8] Theorem 2.1a, discussed below. A proof is included in the Appendix.)

Fact 1.3. *If $\mathcal{D}$ is a regular ultrafilter and $(N, <^N)$, $(M, <^M)$ are any two infinite linearly ordered sets, then $\mathcal{C}(\mathcal{D}, N) = \mathcal{C}(\mathcal{D}, M)$.*

[1] Donder [2] proved that in the core model all uniform ultrafilters are regular, and these will be our main focus here. Still, we note that one outcome of our recent investigations has been a certain reuniting of work on regular and irregular ultrafilters (e.g. κ-complete ultrafilters where κ is measurable or compact). Essentially, one can build a good regular filter $\mathcal{D}_0$ on I such that the quotient $\mathcal{P}(I)/\mathcal{D}_0$ is isomorphic to a certain complete Boolean algebra $\mathfrak{B}$. One can then build an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$, which need not be regular, to have certain prescribed properties. Combining $\mathcal{D}_*$ and $\mathcal{D}_0$ one obtains a regular ultrafilter on I which inherits some of the given features. This approach is outlined in [17] §1.3 p. 619 and Definition 2.11 p. 624 through the end of §2.2, p. 628.

Discussion 1.4. *Thus, we are justified in speaking of $\mathcal{C}(\mathcal{D})$ and using $(\omega, <)$ for our computations.*

In particular, if $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$, no new gaps may appear when replacing ω by other infinite ordinals, and we may use orders such as $(\mathbb{Q}, <)$ to prove facts about $\mathcal{C}(\mathcal{D}, \omega)$. Moreover, we will have $(\kappa, \mu) \in \mathcal{C}(\mathcal{D})$ if and only if $(\mu, \kappa) \in \mathcal{C}(\mathcal{D})$. So we will usually assume $\kappa \leq \mu$.

Some useful facts about regular ultrapowers of linear order: the regular ultrapower $(\omega, <)^I/\mathcal{D}$ will have cofinality $> \lambda$, and will contain both strictly increasing and strictly decreasing sequences of length λ by [8, Theorem 1.5a]. So for every regular $\kappa \leq \lambda$ there is *some* $\theta \leq 2^\lambda$ such that $X^I/\mathcal{D}$ contains a (κ, θ)-cut.[2]

In what follows we will assume all first-order languages are countable and all theories complete unless otherwise stated. Given a model M, if $M^I/\mathcal{D}$ is $|I|^+$-saturated let us say that "$\mathcal{D}$ saturates M." The strong incompleteness visible in the regularizing family gives regular ultrafilters an important model-theoretic property, namely that if $M \equiv N$ then $\mathcal{D}$ saturates M if and only if $\mathcal{D}$ saturates N (see Keisler [8, Corollary 2.1a]). So when $\mathcal{D}$ is regular, it is well defined to say that "$\mathcal{D}$ saturates T." Keisler's order is precisely the pre-order $\trianglelefteq$ on complete, countable theories given by:

Definition 1.5 (Keisler's order)**.** *$T_1 \trianglelefteq T_2$ if for every regular ultrafilter $\mathcal{D}$, if $\mathcal{D}$ saturates T_2 then $\mathcal{D}$ saturates T_1.*

Keisler's order induces a dual pre-order on regular ultrafilters:

Definition 1.6. *$\mathcal{D}_1 \trianglelefteq^{uf} \mathcal{D}_2$ if for every complete countable theory T, if $\mathcal{D}_1$ saturates T then $\mathcal{D}_2$ saturates T.*

We will keep Keisler's order and its dual in mind throughout the paper.[3]

The real interest of $\mathcal{C}(\mathcal{D})$ now begins with the fact that there exist ultrafilters for which $\mathcal{C}(\mathcal{D}) = \emptyset$. This is explained in some detail in [15] pps. 240-242. Briefly, however, Keisler had defined and proved the existence of so-called good ultrafilters under GCH [6], an assumption later eliminated by Kunen [10].

Definition 1.7. *Say that the ultrafilter $\mathcal{D}$ on I is κ^+-good if every monotonic $f : [\kappa]^{<\aleph_0} \rightarrow \mathcal{D}$ has a multiplicative refinement, where monotonic means that $u \subseteq v \in [\lambda]^{<\aleph_0}$ implies $f(u) \supseteq f(v)$ and a multiplicative refinement of f is a function f' such that $f(u) \supseteq f'(u) \in \mathcal{D}$ for all finite u and $f'(u) \cap f'(v) = f'(u \cup v)$ for all finite u, v. Say that $\mathcal{D}$ is* good *if it is $|I|^+$-good.*

Keisler then proved that the maximum class in Keisler's order exists and consists precisely of those theories such that if $\mathcal{D}$ is regular and saturates T then $\mathcal{D}$ must be good [8]. This implies that $\trianglelefteq^{uf}$ has a maximum class consisting precisely of the good regular ultrafilters. Shelah proved in [21] VI.2.6 that the theory of infinite dense linear order belongs to the maximum Keisler class.[4] Together with Fact

[2]It follows, in model theoretic terminology, that the only failures of $|I|^+$-saturation in such an ultrapower come from the cuts visible to $\mathcal{C}(\mathcal{D})$.

[3]The relation $\trianglelefteq$ is usually considered as a partial order on the equivalence classes. For further information on the classification around Keisler's order, the reader may consult the original paper of Keisler 1967 [8] or the recent papers [13], [16].

[4]This was surprising: after all, Keisler's proof of the existence of a maximal class essentially shows that sufficiently rich theories, such as Peano arithmetic, may explicitly code failures of goodness as failures of saturation. With the theory of linear ordering, there is no such coding.

1.3, these theorems imply that if $\mathcal{D}$ is a regular ultrafilter on I, then $(\omega,<)^I/\mathcal{D}$ is $|I|^+$-saturated if and only if $\mathcal{D}$ is good. In other words,

Fact 1.8. *If $\mathcal{D}$ is a regular ultrafilter, then $\mathcal{C}(\mathcal{D})=\emptyset$ if and only if $\mathcal{D}$ is good.*

Now, Fact 1.8 suggests that a first way to observe the strength of an ultrafilter is to consider how far $\mathcal{C}(\mathcal{D})$ is from being empty: "the fewer the cuts in $\mathcal{C}(\mathcal{D})$, the more powerful the ultrafilter." This idea motivated many of our results in [15]. (A different[5] way to measure the strength of an ultrafilter is by the set of theories which it saturates, as in $\trianglelefteq^{uf}$.)

At this point, we remind the reader of the big picture question, 1.1. As stated, this question is related to one of the most basic questions about regular ultrafilters: the possible values of $\mu(\mathcal{D})$ for regular $\mathcal{D}$, where for $\mathcal{D}$ an ultrafilter on I we define $\mu(\mathcal{D}) = \min\{\kappa : \kappa = |\prod_i n_i/\mathcal{D}|$ for some sequence $\langle n_i : i \in I\rangle$ and $\kappa \geq \aleph_0\}$, i.e. the minimum element in the spectrum of infinite cardinals which are cardinalities of ultraproducts of finite cardinals mod $\mathcal{D}$. By [8], [21], [9], and [23] this and the spectrum is fully understood.[6]

However, the general picture for cuts has proved much more elusive. The main structure theorem of [15], in the context of ultrapowers, tells us that:

Theorem 1.9 ([15] Theorem 10.25)**.** *Let $\mathcal{D}$ be a regular ultrafilter on I, $|I|\geq\lambda$. Then the following are equivalent:*

(a) $\mathcal{C}(\mathcal{D})=\emptyset$.
(b) $\mathcal{C}(\mathcal{D})$ has no symmetric cuts, i.e. $(\kappa,\kappa)\notin\mathcal{C}(\mathcal{D})$ for all $\kappa\leq\lambda$.
(c) $\mathcal{D}$ is good.

Thus, "the first cut is symmetric", meaning that if κ is minimal such that $(\kappa_1,\kappa_2)\in\mathcal{C}(\mathcal{D})$ and $\kappa_1+\kappa_2=\kappa$ then necessarily $(\kappa,\kappa)\in\mathcal{C}(\mathcal{D})$. Several immediate questions arise. First, what other cuts may appear with that first symmetric cut? Is *every* entry in $\mathcal{C}(\mathcal{D})$ witnessed by a corresponding symmetric cut?

Problem 1.10. *Suppose that $\mathcal{D}$ is an ultrafilter on I and $\mathcal{C}(\mathcal{D})\neq\emptyset$, so there is some minimal $\kappa\leq|I|$ such that $(\kappa,\kappa)\in\mathcal{C}(\mathcal{D})$. For which, if any, other cardinals $\mu<\kappa$ may it be the case that also $(\kappa,\mu)\in\mathcal{C}(\mathcal{D})$?*

Problem 1.11. *Suppose $(\kappa_1,\kappa_2)\in\mathcal{C}(\mathcal{D})$. Is $(\kappa_1+\kappa_2,\kappa_1+\kappa_2)\in\mathcal{C}(\mathcal{D})$?*

The paper [15] gave a new sufficient condition for the cut spectrum to be empty: existence of upper bounds of paths through trees.

Theorem 1.12 ([15] Theorem 10.25)**.** *Suppose that $\mathcal{D}$ is a regular ultrafilter on $|I|$ and $\mathcal{D}$ has "$|I|^+$-treetops," meaning that if $(\mathcal{T},\trianglelefteq)$ is any tree, in the ultrapower $N=\mathcal{T}^I/\mathcal{D}$ any $\trianglelefteq^N$-strictly increasing sequence of cofinality $\kappa<|I|^+$ has an upper bound in N. Then $\mathcal{C}(\mathcal{D})=\emptyset$.*

In particular, we may add to Theorem 1.9 the equivalent condition

[5]Nonetheless, we do not know how different: for instance, whether every Keisler class corresponds to a set of cuts, say, whether $\mathcal{C}(\mathcal{D})=\mathcal{C}(\mathcal{D}')$ implies that $\mathcal{D}$ and $\mathcal{D}'$ saturate the same theories. Such questions motivated some recent work separating cut-like properties of ultrafilters from other properties, as in (2) $\not\rightarrow$ (3) of [13] Theorem 4.2. This appears to be a question about how ultrafilters deal with the independence property in model theory, and progress on understanding Keisler's order on the simple unstable theories may help to illuminate this.

[6]Moreover, $\mu(\mathcal{D})$ is an indicator for whether or not $M^I/\mathcal{D}$ is $|I|^+$-saturated whenever Th(M) is countable and stable with the fcp, see for instance [13] §1.1, especially Theorem D(2) there.

(d) $\mathcal{D}$ has λ^+-treetops.

In that paper, we had motivated the connection to $\mathfrak{p}$ and $\mathfrak{t}$ by defining cardinals $\mathfrak{p}_{\mathcal{D}} = \min\{\kappa : (\kappa_1, \kappa_2) \in \mathcal{C}(\mathcal{D})$ and $\kappa_1 + \kappa_2 = \kappa\}$ and $\mathfrak{t}_{\mathcal{D}} = \min\{\lambda :$ in some $\mathcal{D}$-ultrapower of a tree there is a strictly increasing λ-indexed sequence which has no upper bound$\}$. Then Theorem 1.9 essentially follows from showing that for regular $\mathcal{D}$, the cardinals $\mathfrak{p}_{\mathcal{D}}$ and $\mathfrak{t}_{\mathcal{D}}$ are equal.[7] It is natural to ask whether cuts and unbounded paths have a similar relationship above the size of the index set. By a tree we will mean a partially ordered set in which the set of predecessors of any given node is well ordered.

Problem 1.13. *Suppose κ_1, κ_2 are infinite and regular, $\lambda \leq \kappa_1 + \kappa_1 < 2^\lambda$, and $(\omega, <)^I/\mathcal{D}$ has a (κ_1, κ_2)-cut. Let $\kappa = \kappa_1 + \kappa_2$. Is it necessarily the case that for some tree $\mathcal{T}$, in the ultrapower $N = \mathcal{T}^\lambda/\mathcal{D}$ there is a strictly increasing κ-indexed sequence with no upper bound in N?*

Some results on the case of (κ_1, κ_2)-cuts in ultrapowers where $\kappa_1 + \kappa_2 > 2^\lambda$, and the index model M is quite saturated, were obtained by Golshani and Shelah [4].

At this point in our discussion, a very interesting path opens up: the main proof of [15] is amenable to axiomatic analysis. Namely, that proof proceeded via various component lemmas analyzing the structure of $\mathcal{C}(\mathcal{D})$ under treetops (proving symmetry, uniqueness, and so forth). One can consider these component phenomena in isolation and try to determine their strength.[8] In particular, consider:

Definition 1.14 (Uniqueness). *Say the regular ultrafilter $\mathcal{D}$ on I has* uniqueness *for the regular cardinal κ when: if $(\kappa, \theta) \in \mathcal{C}(\mathcal{D})$ and $(\kappa, \theta') \in \mathcal{C}(\mathcal{D})$ then $\theta = \theta'$.*

Uniqueness was established in the case of regular ultrafilters for κ strictly below the goodness of $\mathcal{D}$, or equivalently, for $\mathcal{D}$ with κ^+-treetops, in [15] Theorem 3.1.

Problem 1.15. *Find necessary and sufficient conditions on a regular ultrafilter $\mathcal{D}$ on I so that for all $\kappa = \mathrm{cf}(\kappa) \leq \lambda = |I|$, $\mathcal{D}$ has uniqueness for κ.*

For example, any regular ultrafilter has uniqueness for $\aleph_0$, but it is also possible to construct certain failures of uniqueness, as is done in our paper [19]. Remarkably, it is not known whether uniqueness is strictly weaker than goodness:

Problem 1.16. *For which, if any, I does there exist a regular ultrafilter $\mathcal{D}$ on I such that $\mathcal{D}$ has uniqueness for all regular $\kappa \leq |I|$ but $\mathcal{C}(\mathcal{D}) \neq \emptyset$?*

We shall return to this question. However, it seems likely that Uniqueness is a much more general phenomenon. Before discussing 1.16 further let us give some new evidence for this statement.

[7]Connecting these cardinals to the true $\mathfrak{p}$ and $\mathfrak{t}$ requires several more steps. First, replace the pair of a model and its ultrapower by the more general pair of a model and an elementary extension satisfying certain pseudofiniteness and expansion properties; such a pair is called a *cofinality spectrum problem* $\mathbf{s}$. For any such $\mathbf{s}$, there are corresponding cardinals $\mathfrak{p}_{\mathbf{s}}$ and $\mathfrak{t}_{\mathbf{s}}$, which specialize to the $\mathfrak{p}_{\mathcal{D}}$, $\mathfrak{t}_{\mathcal{D}}$ mentioned above in the special case when the c.s.p. arises as a regular ultrapower. A major theorem of [15] was a proof in ZFC that for any c.s.p. $\mathbf{s}$, $\mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$ is impossible. Assume then for a contradiction that $\mathfrak{p} < \mathfrak{t}$. For a certain well chosen $\mathbf{s}$ arising as a model of set theory and its generic ultrapower in some forcing extension, one can check that $\mathfrak{p}_{\mathbf{s}} \leq \mathfrak{p}$ and $\mathfrak{t} \leq \mathfrak{t}_{\mathbf{s}}$, so $\mathfrak{p} < \mathfrak{t}$ would entail $\mathfrak{p}_{\mathbf{s}} < \mathfrak{t}_{\mathbf{s}}$, a contradiction.

[8]This is part of a broader question of examining that proof, for instance with the goal of removing the forcing argument from the proof that $\mathfrak{p} = \mathfrak{t}$. A beginning in this direction appears in our paper [19].

2. A uniqueness theorem

In this section we prove Theorem 2.4, which leverages Bell's Theorem (stated below) to give uniqueness for a weak notion of cut. Recall that if S is a subset of a partially ordered set, we say S is *centered* if every finitely many elements of S have a common lower bound, and *σ-centered* if it is the union of countably many centered subsets.

Theorem A (Bell [1])**.** *$\mathfrak{p}$ is the first cardinal κ for which $MA_\kappa(\sigma$-centered) fails, i.e., $\mathfrak{p} = \min\{\kappa$: there are a σ-centered poset P and a collection D of κ dense subsets of P such that no filter on P meets all members of $D\}$.*

Let $\mathcal{D}$ be a filter on $\mathbb{N}$, which we will always assume contains all co-finite subsets of $\mathbb{N}$. By "$\mathcal{D}+A$" we mean the filter generated by $\mathcal{D}\cup\{A\}$, which is nontrivial when $A \in \mathcal{D}^+$, i.e. A is not equivalent to $\emptyset \mod \mathcal{D}$. We use $\bigwedge$ and $\bigvee$ to mean infinite conjunction and disjunction, respectively.

Definition 2.1. *Let $\mathcal{D}$ be a filter on $\mathbb{N}$ and let δ, θ be limit ordinals, usually regular cardinals. We omit $\mathcal{D}$ when $\mathcal{D}$ is the filter of cofinite subsets.*

(1) A weakly peculiar *(δ,θ)-$\mathcal{D}$-cut in ${}^\omega\omega$ is a pair $(\langle f_\alpha : \alpha<\delta\rangle, \langle g_\beta : \beta<\theta\rangle)$ of sequences of functions in ${}^\omega\omega$ such that:*
 (a) $(\forall \alpha < \alpha' < \delta)(f_\alpha <_{\mathcal{D}} f_{\alpha'})$
 (b) $(\forall \beta < \beta' < \theta)(g_{\beta'} \leq_{\mathcal{D}} g_\beta)$
 (c) $(\forall \alpha < \delta)(\forall \beta < \theta)(f_\alpha <_{\mathcal{D}} g_\beta)$
 (d) for no $A \in \mathcal{D}^+$ and $f \in {}^\omega\omega$ do we have

$$(\forall \alpha < \delta)(\forall \beta < \theta)(\ f_\alpha <_{\mathcal{D}+A} f <_{\mathcal{D}+A} g_\beta\ ).$$

(2) A left peculiar *(δ,θ)-$\mathcal{D}$-cut in ${}^\omega\omega$ is a weakly peculiar such cut which, in addition, satisfies:*
 (e) if $A \in \mathcal{D}^+$ and $f \in {}^\omega\omega$ and $\bigwedge_{\alpha<\delta} f_\alpha <_{\mathcal{D}+A} f$ then $\bigvee_{\beta<\theta} g_\beta \leq_{\mathcal{D}+A} f$.

Definition 2.2. *We define*

$$\mathcal{C}(\mathcal{D}, {}^\omega\omega) = \{(\kappa_1,\kappa_2) :\ \kappa_1,\kappa_2 \text{ regular and there is a left peculiar } (\kappa_1,\kappa_2)\text{-}\mathcal{D}\text{-cut}\}.$$

As before, when $\mathcal{D}$ is the co-finite filter, we may omit it.

Note that Definition 2.2 differs from the same notation as used in [15, 14.12(2)]. Left peculiar is a nontrivial weakening of peculiar.[9]

The following cardinals will be useful in describing the uniqueness spectrum.

Definition 2.3. *With $\mathcal{C}({}^\omega\omega)$ as in 2.2, we define:*
 (a) $\mathrm{spec}_{\mathfrak{uq}} = \{\kappa :$ if $(\kappa,\theta_\ell) \in \mathcal{C}({}^\omega\omega)$ for $\ell = 0,1$ then $\theta_1 = \theta_2\}$.
 (b) $\mathfrak{uq} = \min\{\mathrm{spec}_{\mathfrak{uq}} \setminus \aleph_1\} = \{\kappa : \kappa \geq \aleph_1, \kappa \in \mathrm{spec}_{\mathfrak{uq}}\}$.
 (c) $n\mathfrak{uq} = \min\{\kappa : \kappa = \mathrm{cf}(\kappa) > \aleph_0, \kappa \notin \mathrm{spec}_{\mathfrak{uq}}\}$.

In the following theorem, it will follow from the remark after equation (3) that $\theta_\ell \leq 2^{\aleph_0}$ for $\ell = 0, 1$. However, the theorem does not follow from the construction in [15] Section 14 because there we collapse the continuum to $\mathfrak{h}$ which is $\geq \mathfrak{p}$ but $\leq \mathfrak{b}$.

Theorem 2.4. $\mathfrak{p} \leq n\mathfrak{uq}$.

[9] For example, it can be shown that if $(\bar{f}, \bar{g})$ is a left-peculiar and right-peculiar (κ_1, κ_2)-cut and $\kappa_1 \neq \kappa_2$ then some c.c.c. forcing notion forces that $(\bar{f}, \bar{g})$ is left peculiar but not right peculiar.

Proof. Suppose that $(\kappa, \theta_\ell) \in \mathcal{C}({}^\omega\omega)$ for $\ell = 0, 1$, exemplified by $(\overline{f}^\ell, \overline{g}^\ell)$ for $\ell = 0, 1$. Assume $\kappa < \mathfrak{p}$ and we will prove that $\theta_0 = \theta_1$. Throughout the proof, let $\mathcal{D}$ denote the co-finite filter.

We may assume there is a sequence $\langle k^*_n : n \in \mathbb{N} \rangle$ of integers such that for $\ell = 0, 1$ and $\alpha < \kappa$, $f^\ell_\alpha(n) \leq k^*_n$ for all but finitely many n and for $\ell = 0, 1$ and $\beta < \theta_\ell$, $g^\ell_\beta \leq k^*_n$ for all but finitely many n. For instance, $k^*_n = g^0_0(n) + g^1_0(n) + 1$ will suffice. We first consider whether, on infinitely many $n \in \mathbb{N}$, we may adjust the spacing of elements so that the increasing sequences of the two cuts (or cofinal subsets of the same) strictly interlace. The claim asserts that this is sufficient to prove the decreasing sequences must have the same cofinality.

Claim 2.5 (A sufficient condition)**.** *To show that $\theta_0 = \theta_1$ it would suffice to show that there exist an infinite $A \subseteq \mathbb{N}$, a club E of κ, a sequence $\bar{k} = \langle k_n : n \in A \rangle \in {}^A\mathbb{N}$, and sequences of functions $\bar{\pi}^0 = \langle \pi^0_n : n \in A \rangle$, $\bar{\pi}^1 = \langle \pi^1_n : n \in A \rangle$ such that:*

(a) *for $\ell = 0, 1$ and $\beta < \theta_\ell$, for all but finitely many $n \in A$, $g^\ell_\beta(n) \in [0, k_n)$.*

(b) *for $\ell = 0, 1$ and every $n \in A$, the function $\pi^\ell_n : [0, k_n) \to [0, k_n)$ is nondecreasing and onto an initial segment.*

(c) *if $\alpha < \beta$ are from E then for all but finitely many $n \in A$,*

$$\max\{\pi^0_n(f^0_\alpha(n)), \pi^1_n(f^1_\alpha(n))\} < \min\{\pi^0_n(f^0_\beta(n)), \pi^1_n(f^1_\beta(n))\}.$$

Proof of Claim 2.5. Suppose such objects exist. Renaming, without loss of generality $E = \kappa$. When $n \notin A$, let π^ℓ_n be the identity and let $k_n = k^*_n$.

First we give a name to each f^ℓ_α after its movement by the maps π^ℓ_n: for each $\ell = 0, 1$ and $\alpha < \kappa$ let $\mathbf{f}^\ell_\alpha \in {}^\omega\omega$ be the function defined (for all but at most finitely many n) by

$$\mathbf{f}^\ell_\alpha(n) = \pi^\ell_n(f^\ell_\alpha(n)). \tag{1}$$

Define $\mathbf{g}^\ell_\beta$ analogously for $\beta < \theta_\ell$. By hypothesis (c) of the Claim,

$$\alpha < \beta < \kappa \implies \mathbf{f}^1_\alpha <_{\mathcal{D}+A} \mathbf{f}^0_\beta \ \wedge\ \mathbf{f}^0_\alpha <_{\mathcal{D}+A} \mathbf{f}^1_\beta. \tag{2}$$

Recall that $f^\ell_\alpha(n) < f^\ell_\beta(n)$ for all but finitely many n by definition of $\mathcal{C}({}^\omega\omega)$, and that π^ℓ_n is nondecreasing and is the identity for $n \in \mathbb{N} \setminus A$. So for $\ell = 0, 1$,

$$\alpha < \kappa \wedge \beta_0 < \beta_1 < \theta_\ell \implies \mathbf{f}^\ell_\alpha <_{\mathcal{D}} \mathbf{f}^\ell_{\alpha+1} \leq_{\mathcal{D}} \mathbf{g}^\ell_{\beta_1} \leq_{\mathcal{D}} \mathbf{g}^\ell_{\beta_0}. \tag{3}$$

Note that $\beta < \beta' < \theta_\ell$ implies only $g^\ell_{\beta'} \leq_{\mathcal{D}} g^\ell_\beta$, not necessarily $<_{\mathcal{D}}$, however the sequence cannot be eventually constant (mod $\mathcal{D}$). If it were, then writing g^ℓ_* for the eventually constant value, the function defined by $g^\ell_*(n) - 1$ for each n such that $g^\ell_*(n) \geq 1$ (and 0 otherwise) would contradict 2.1(1)(d). So for $\ell = 0, 1$,

(4) for every $\beta < \theta_\ell$ there is β' with $\beta < \beta' < \theta_\ell$ such that $\mathbf{g}^\ell_{\beta'} <_{\mathcal{D}} \mathbf{g}^\ell_\beta$.

Let us check that for $\ell = 0, 1$, and every infinite $A' \subseteq A$,

(5) no $\mathbf{h} \in {}^\omega\omega$ satisfies: for all $\alpha < \kappa, \beta < \theta_\ell$ we have $\mathbf{f}^\ell_\alpha \leq_{\mathcal{D}+A'} \mathbf{h} \leq_{\mathcal{D}+A'} \mathbf{g}^\ell_\beta$.

Suppose for a contradiction there were such an $\mathbf{h}$. Let $h_{\max}(n) = \max\{k \leq k_n : \pi^\ell_n(k) \leq \mathbf{h}(n)\}$ and $h_{\min}(n) = \min\left(\{k \leq k_n : \pi^\ell_n(k) \geq \mathbf{h}(n)\} \cup \{k_n\}\right)$. For each $\alpha < \kappa$, we have that $\mathbf{f}^\ell_\alpha <_{\mathcal{D}} \mathbf{f}^\ell_{\alpha+1} \leq_{\mathcal{D}+A'} \mathbf{h}$, so $f^\ell_\alpha <_{\mathcal{D}+A'} h_{\max}$. Likewise, recalling (4), $h_{\min} <_{\mathcal{D}+A'} g^\ell_\beta$. As the π_n are onto an initial segment, $h_{\max} = h_{\min} =: h$. Then h and A' contradict the fact that $(\bar{f}^\ell, \bar{g}^\ell)$ is weakly peculiar, 2.1(1)(d). This proves (5). Let us check that moreover for $\ell = 0, 1$, for any infinite $A' \subseteq A$ and any $\mathbf{h} \in {}^\omega\omega$,

$$\text{if } \bigwedge_{\alpha<\kappa} \mathbf{f}^\ell_\alpha <_{\mathcal{D}+A'} \mathbf{h} \text{ then } \bigvee_{\beta<\theta_\ell} \mathbf{g}^\ell_\beta \leq_{\mathcal{D}+A'} \mathbf{h}. \tag{6}$$

Suppose $\mathbf{h}$ satisfies the "if" clause. As before we may define $h = h_{\max}$, and by a similar argument, for each $\alpha < \kappa$, $f^\ell_\alpha <_{\mathcal{D}+A'} h$. As $(\bar{f}^\ell, \bar{g}^\ell)$ is left peculiar[10], using A' for A in 2.1(2)(e) there is $\gamma < \theta_\ell$ such that $g^\ell_\gamma \leq_{\mathcal{D}+A'} h$. Hence $\mathbf{g}^\ell_\gamma \leq_{\mathcal{D}+A'} \mathbf{h}$, which completes the check of (6).

Now we use the interlacing of the ascending sequences on A. Since θ_0, θ_1 are regular cardinals, to prove the claim it will suffice to show that the descending sequences also interlace on A, i.e. :

(I) if $\gamma < \theta_0$ then for some $\beta < \theta_1$ we have $\mathbf{g}^1_\beta \leq_{\mathcal{D}+A} \mathbf{g}^0_\gamma$, and
(II) if $\gamma < \theta_1$ then for some $\beta < \theta_0$ we have $\mathbf{g}^0_\beta \leq_{\mathcal{D}+A} \mathbf{g}^1_\gamma$.

As these are symmetric, it suffices to prove (I). Let $\gamma < \theta_0$ be given. For all $\alpha < \kappa$, by (2) and (3),

$$\mathbf{f}^1_\alpha <_{\mathcal{D}+A} \mathbf{f}^0_{\alpha+1} \leq_{\mathcal{D}} \mathbf{g}^0_\gamma. \tag{7}$$

Thus, $\mathbf{f}^1_\alpha <_{\mathcal{D}+A} \mathbf{g}^0_\gamma$ for all $\alpha < \kappa$. Applying equation (6) with the present A we find $\beta < \theta_1$ such that $\mathbf{g}^1_\beta \leq_{\mathcal{D}+A} \mathbf{g}^0_\gamma$. This proves (I) and the claim. *Claim 2.5.* $\square$

Continuing the proof of Theorem 2.4, we define a forcing notion $\mathbf{Q}$ as follows:

A. $p \in \mathbf{Q}$ iff $p = (v, m, u, \overline{\pi}^0, \overline{\pi}^1, \bar{k})$ where:
  (i) $m \in \mathbb{N}$
  (ii) $v \subseteq \mathbb{N}$ is finite, with $\max v \leq m$
  (iii) $u \subseteq \kappa$ is finite
  (iv) for $\ell = 0, 1$, $\langle f^\ell_\alpha \restriction [m, \omega) : \alpha \in u\rangle$ is increasing
  (v) $\overline{\pi}^\ell = \langle \pi^\ell_n : n \in v\rangle$
  (vi) $\bar{k} = \langle k_n : n \in v\rangle$, with $k_n \geq g^0_0(n) + g^1_0(n) + 1$ for each $n \in v$, and $\pi^\ell_n : k_n \to k_n$ is nondecreasing onto an initial segment.

B. $p \leq_{\mathbf{Q}} q$ when:
  (i) $v_p \subseteq v_q$, $m_p \leq m_q$, $u_p \subseteq u_q$, $v_q \setminus v_p \subseteq [m_p, m_q)$,
  (ii) $\pi^\ell_{n,p} = \pi^\ell_{n,q}$ for $n \in v_p$, and
  (iii) if $n \in v_q \setminus v_p$ and $\alpha < \beta$ are from u_p then

$$\max\{\pi^0_{n,q}(f^0_\alpha(n)), \pi^1_{n,q}(f^1_\alpha(n))\} < \min\{\pi^0_{n,q}(f^0_\beta(n)), \pi^1_{n,q}(f^1_\beta(n))\}.$$

Now we record some properties of $\mathbf{Q}$. First, letting only u vary, we have that fixing an appropriate five-tuple x, the set $\{p \in \mathbf{Q} : (v_p, m_p, \overline{\pi}^0_p, \overline{\pi}^1_p, \bar{k}_p) = x\}$ is directed. This shows that $\mathbf{Q}$ is σ-centered. Second, if $\alpha < \kappa$ then the set

$$\mathcal{I}^1_\alpha = \{p \in \mathbf{Q} : \alpha \in u_p\}$$

is dense open. Third, if $m \in \mathbb{N}$ the set

$$\mathcal{I}^2_m = \{p \in \mathbf{Q} : m_p \geq m\}$$

[10] The hypothesis 'left peculiar' is essential in proving 2.5. If the cut is only assumed to be weakly peculiar, it is possible to force a counterexample to the claim using $\kappa > \mathfrak{p}$, noting that $\kappa < \mathfrak{p}$ was not used in that proof. In fact, for weakly peculiar cuts, it is possible to force a counterexample to the theorem.

is dense open. Fourth, if $m \in \mathbb{N}$ the set

$$\mathcal{I}^3_m = \{p \in \mathbf{Q} : v_p \not\subseteq \{0, \dots m-1\}\}$$

is dense open, since given p, we may choose $i > m_p$ such that $\langle f^\ell_\alpha(i) : \alpha \in u\rangle$ is increasing for $\ell = 0, 1$. Together, letting $\alpha < \kappa$, $m \in \mathbb{N}$ vary, we have a collection of no more than $\kappa < \mathfrak{p}$ dense open subsets of $\mathbf{Q}$, and $\mathbf{Q}$ is σ-centered.

By Bell's theorem, there exists a filter $\mathbf{G}$ on $\mathbf{Q}$ which is generic for the collection of dense open sets just built. Then $A = \bigcup\{v_p : p \in \mathbf{G}\}$, $E = \kappa$, and $\pi^\ell_m = \pi^\ell_{p,m}$ for some $p \in \mathbf{G}$ satisfy the hypotheses of Claim 2.5. This completes the proof of Theorem 2.4. □

Informally, the connection of $\mathfrak{p}$ to cuts in linear orders, especially in the context of [15], may be described by saying: from our present perspective, $\mathfrak{p}$ expresses some kind of saturation (which is maximal from that perspective). There are natural weaker notions corresponding to saturation of some combinatorially natural theories, such as T_{feq}, the theory of an existentially closed model of a parametrized family of independent equivalence relations, or the theory of the generic tetrahedron-free three-hypergraph, called $T_{3,2}$ in [18] Definition 0.2 . Do these weaker notions also give rise to natural and interesting cardinal invariants? In the remainder of the paper, we will touch only on a part of this question, having to do with T_{feq} and its relation to the existence of certain internal maps.

3. On internal maps

Returning to the context of regular ultrafilters, it turns out that the phenomenon of Uniqueness in the $\mathfrak{p} = \mathfrak{t}$ proof from [15] in fact hides something stronger. If $N = M^I/\mathcal{D}$ is an ultrapower and $g : N \to N$ a partial map, we call g *internal* (or *induced*) if it genuinely comes from the index models, that is, if we could expand the language of N by a new binary relation symbol R and for each $t \in I$, we could choose M_t to be M expanded by the new symbol R interpreted as the graph of a partial map from M to M, in such a way that in the ultraproduct $(M_t)^I/\mathcal{D}$ the relation R is the graph of g.

What was shown in [15] was not only uniqueness, but the a priori stronger fact that uniqueness is witnessed by internal order preserving maps.

Fact 3.1 ([15] Cor. 3.7)**.** *If $\mathcal{D}$ is a regular ultrafilter on I and $\mathcal{D}$ is good (or equivalently: $\mathcal{D}$ has $|I|^+$-treetops) then for any $\kappa \leq |I|$ the following holds:*

> $\bigoplus_\kappa = \bigoplus_\kappa(\mathcal{D})$:
> *for any two strictly increasing κ-indexed sequences $\langle a_\alpha : \alpha < \kappa\rangle$, $\langle b_\alpha : \alpha < \kappa\rangle$ of elements of $N = (\omega, <)^I/\mathcal{D}$, there is an internal partial order-preserving map $f : N \to N$ extending $a_\alpha \mapsto b_\alpha$.*

We might have defined $\bigoplus_{\kappa,M} = \bigoplus_{\kappa,M}(\mathcal{D})$ for M an infinite linear order, with $\bigoplus_\kappa = \bigoplus_{\kappa,\omega}$. However, by an argument similar to that of Fact 1.3, one can use regularity of $\mathcal{D}$ to show that when $\kappa \leq |I|$ the property $\bigoplus_{\kappa,M}$ does not depend on the choice of linear order.

Curiously, this strengthening turns out to already be equivalent to goodness.

Lemma 3.2. *Let $\mathcal{D}$ be a regular ultrafilter on I.*

(a) Let $\kappa \leq |I|$ be regular. If $\bigoplus_\kappa$ from 3.1 holds, then $(\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$.
(b) In particular, if $\bigoplus_\kappa$ holds for all regular $\kappa \leq |I|$ then $\mathcal{D}$ is good.

Proof of Lemma 3.2. Clause (b) follows from clause (a) by Theorem 1.9.

To prove clause (a), recall from the remarks before the proof that having fixed $\mathcal{D}$, we have that $\bigoplus_\kappa \iff \bigoplus_{\kappa,\mathbb{Q}}$ and by Fact 1.3, $\mathcal{C}(\mathcal{D}) = \mathcal{C}(\mathcal{D},\mathbb{Q})$. So it will suffice to show that if $\bigoplus_{\kappa,\mathbb{Q}}$ then $N = M^\lambda/\mathcal{D}$ contains no (κ,κ)-cut. As ultrapowers commute with reducts, we may further assume $M = (\mathbb{Q};+,<)$, i.e. our $\mathbb{Q}$ has the usual structure of an ordered additive group. Suppose then that $(\langle a_i : i < \kappa\rangle, \langle b_i : i < \kappa\rangle)$ is a (κ,κ)-pre-cut of N. Without loss of generality, $\overline{a}$ is strictly increasing and $\overline{b}$ is strictly decreasing. Using the available addition and subtraction, let $d = b_0 + 1$ and define a sequence $\langle b'_i : i < \kappa\rangle$ of elements of N by: $b'_i = d - b_i$. Clearly $\langle b'_i : i < \kappa\rangle$ is a strictly increasing sequence.

By assumption, there is an internal order-preserving partial bijection f on N with $\{a_i : i < \kappa\} \subseteq \mathrm{dom}(f)$ and $\{b'_i : i < \kappa\} \subseteq \mathrm{range}(f)$ such that $i < \kappa \implies f(a_i) = b'_i$. (Actually, an internal map between cofinal subsequences would suffice.) In the language with a symbol for f, clearly $\mathrm{dom}(f)$ is a definable set.

Let $X = \{a \in \mathrm{dom}(f) : a < d - f(a)\}$. Clearly X is an internal set, since it is definable from the internal functions $+$, f. First we check that X is nonempty. For each $i < \kappa$, $a_i \in \mathrm{dom}(f)$ and $d - f(a_i) = d - b'_i = d - (d - b_i) = b_i$. Since $a_i < b_i$ by our original choice of sequences, this shows that $a_i \in X$ for each $i < \kappa$. Second, since f is order preserving, X is an initial segment of $\mathrm{dom}(f)$. Third, for each $i < \kappa$, consider b_i. If $b_i \notin \mathrm{dom}(f)$, clearly $b_i \notin X$. Otherwise, recalling that $a_i < b_i$ and f is order preserving, necessarily $f(a_i) < f(b_i)$. Then $d - f(b_i) < d - f(a_i) = d - b'_i = b_i$. So for each $i < \kappa$, $b_i \notin X$. We have shown that the pre-cut $(\langle a_i : i < \kappa\rangle, \langle b_i : i < \kappa\rangle)$ is internally definable, therefore realized (=filled) in the ultrapower N. This completes the proof. □

Thus, if there exist ultrafilters satisfying Problem 1.16, uniqueness for such ultrafilters will not always be witnessed by internal order preserving maps.

This line of questioning leads us to recall an ultrafilter property, called "good for equality," from work of Malliaris. For simplicity, we focus here on the case $\lambda = |I|$ though the definition makes sense for any infinite $\lambda \leq |I|$.

Fact 3.3 ([11], [12])**.** *The regular ultrafilter $\mathcal{D}$ on I is* good for equality *if and only if for any two sequences $\langle a_i : i < \lambda\rangle$ and $\langle b_i : i < \lambda\rangle$ of distinct elements of $N = \omega^I/\mathcal{D}$, there exists an internal bijection $h : N \to N$ such that $h(a_i) = b_i$ for all $i < \lambda$. Informally, say that $\mathcal{D}$ admits internal maps between small sets.*

For simplicity, we focus here on the case $\lambda = |I|$ though the definition makes sense for any infinite $\lambda \leq |I|$.

The original motivation for this property came from its interesting model-theoretic content: for $\lambda \leq |I|$, "$\mathcal{D}$ is λ-good for equality" is necessary and sufficient for $\mathcal{D}$ to λ^+-saturate the theory T_{feq} mentioned in Section 2. A table connecting the half-dozen known properties of ultrafilters and of the theories which require these properties for saturation is in [13] Section 4.

Here is a very simply stated question: for a regular ultrafilter, can existence of internal maps be separated from existence of internal order-preserving maps? This is probably the most pressing ultrafilter construction problem from the point of view of Keisler's order, and is likely to require very new ideas in ultrafilter construction.

Problem 3.4. *Let $\mathcal{D}$ be a regular ultrafilter on I, $|I| \geq \lambda$. Suppose $\mathcal{D}$ admits internal maps between sets of size λ. Must it be the case that for any two strictly* *increasing sequences* $\langle a_i : i < \lambda \rangle$, $\langle b_i : i < \lambda \rangle$ *of elements of* $N = \omega^I/\mathcal{D}$, *there exists an internal order-preserving partial function* h *with* $\mathrm{dom}(h) \subseteq N$ *and* $\mathrm{range}(H) \subseteq N$ *such that* $h(a_i) = b_i$ *for all* $i < \lambda$*?*

To summarize and restate the construction problem:

General Problem 3.5. *Let* $\mathcal{D}$ *be a regular ultrafilter.*
If $\mathcal{D}$ *is good for equality, is* $\mathcal{C}(\mathcal{D}) = \emptyset$*?*
Equivalently: if $\mathcal{D}$ *is good for equality, must* $\mathcal{D}$ *be good?*
Equivalently: if $\mathcal{D}$ *admits internal maps between small sets in the sense captured by 'good for equality,', must* $\mathcal{D}$ *admit internal order-preserving maps between small sequences in the sense captured by Problem 3.4?*

We note here that a negative answer to Problem 3.5, separating good for equality from good in ZFC, is likely to also solve an old question of Dow from 1985 [3]. We had given a positive answer to this question in [13] Corollary 1.11, assuming a measurable cardinal. Again, we state the question before defining the terms.

Problem 3.6 (Dow 1985 [3])**.** *Do there exist regular ultrafilters which are* λ^+*-OK but not* λ^+*-good?*

The connection of this problem to 3.5 is that any ultrafilter on I which is good for equality is so-called flexible, therefore $|I|$-OK.[11] The condition"$\mathcal{D}$ is λ-OK" weakens "$\mathcal{D}$ is λ^+-good" by asking that only the monotone functions $f : [\lambda]^{<\aleph_0} \to \mathcal{D}$ such that $|u| = |v|$ implies $f(u) = f(v)$ have a multiplicative refinement. (It is unfortunate that the cardinals are off by a plus sign, but for clarity in these brief remarks, we retain the notation of the original sources. That is, Problem 3.6 asks that all 'nice' maps from $[\lambda^+]^{<\aleph_0}$ to $\mathcal{D}$ all have multiplicative refinements, despite the fact that some maps from $[\kappa]^{<\aleph_0}$ to $\mathcal{D}$ do not, for some $\kappa \leq \lambda$. One could presumably solve Problem 3.5 using just $\kappa = \lambda^+$, but we would expect such a solution to also shed light on the case of $\kappa \leq \lambda$.)

We may also consider a cardinal invariant analogue of 3.5. Recalling 2.3, define $\mathrm{spec}_{\mathcal{D},\mathfrak{uqm}} = \{ \kappa : \text{ if } (\kappa, \theta_\ell) \in \mathcal{C}(\mathcal{D}, {}^\omega\omega) \text{ for } \ell = 0, 1, \text{ then } \theta_1 = \theta_2$ and there is an internal order preserving partial map which witnesses it $\}$. Here "$\mathfrak{uqm}$" is for "unique mapping," noting that the definition of internal map makes sense also for reduced products.

Problem 3.7. *Is* $\mathrm{spec}_{\mathfrak{uq}} = \mathrm{spec}_{\mathfrak{uqm}}$*? At the very least, are the derived cardinal invariants equal?*

To conclude the discussion of Problem 3.5, it is worth underlining that there is a rich model theory related to these problems, and a successful ultrafilter construction may provide the most direct way of illuminating this. It concerns the structure of the so-called non-simple theories, the large region on the right in the diagram.

IMAGE GOES HERE

The picture, explained in [18], illustrates the known map of Keisler's order. The lines indicate dividing lines and the white regions are not yet mapped. The red arrow indicates the location of a very surprising recent discovery, the existence of

[11] The original proof goes by showing: there is a minimal TP_2 theory in Keisler's order, [11] Corollary 6.10; $\mathcal{D}$ is good for equality iff it saturates this minimal TP_2 theory, [11] Lemma 6.7 plus [12] Theorem 5.21; if $\mathcal{D}$ saturates any theory with TP_2, in particular the minimal theory, then it is flexible, [11] Lemma 8.8; flexible implies OK, [11] Claim 8.4.

an infinite descending chain, [18] Theorem 6.6. The large shaded region on the right is the maximum class; we don't know whether it may encompass all non-simple theories. This hinges precisely on the question of whether good and good for equality coincide: this is because on one hand, as mentioned in the preceeding footnote, there is a minimum Keisler class among the non-simple theories and its members are precisely those theories T such that if $\mathcal{D}$ is regular and good for T then $\mathcal{D}$ is good for equality, and on the other hand, there is a maximum Keisler class whose members are precisely those theories T such that if $\mathcal{D}$ is regular and good for T then $\mathcal{D}$ is good. Thus, progress on the set-theoretic problem of constructing regular ultrafilters which are good for equality but not good would establish a nontrivial gap in the model-theoretic complexity of theories in these two classes.[12]

4. Bases for goodness

In this section we describe one problem illustrative of the fact that model-theoretic 'patterns' arising from realization or omission of types in ultrapowers may usefully reflect back onto the structure of the underlying ultrafilter. To begin, we explain a result mentioned above.

Fact 4.1 (Keisler [5])**.** *If $\mathcal{D}$ is a regular ultrafilter on λ, then $\mathcal{D}$ is good if and only if for every complete countable theory T and any $M \models T$, $M^\lambda/\mathcal{D}$ is λ^+-saturated (i.e. if and only if for every complete countable theory T, $\mathcal{D}$ is good for T).*

Why? Consider a complete, countable theory T and a regular ultrafilter $\mathcal{D}$ on λ. Given an ultrapower $N = M^\lambda/\mathcal{D}$, we fix in advance a lifting to M^λ so that the coordinate projections $a[t]$ are well defined for any $a \in N$ and $t \in \lambda$. When $\bar{a}$ is a tuple, write $\bar{a}[t]$ for $\langle a_i[t] : i < \lg(\bar{a})\rangle$. The following fact gives a direct connection between realization of types and multiplicative refinements for certain monotonic functions. For a proof, see [13] 1.7-1.8.

Fact 4.2. *Let $M \models T$ and $N = M^\lambda/\mathcal{D}$. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be a regularizing family for $\mathcal{D}$. Let $p(x) = \{\varphi_\alpha(x; \bar{a}_\alpha) : \alpha < \lambda\}$ enumerate some consistent partial type in the ultrapower N. Consider the map d on $[\lambda]^{<\aleph_0}$ given by:*

$$s \mapsto \{t \in \lambda : M \models \exists x \bigwedge \{\varphi_\alpha(x; \bar{a}_\alpha[t]) : \alpha \in s\} \cap \bigcap_{\alpha \in s} X_\alpha.$$

Then d is a monotonic map into $\mathcal{D}$ by Łos' theorem, and p is realized if and only if d has a multiplicative refinement.

Fact 4.1 then follows from showing that in any sufficiently rich theory (set theory, Peano arithmetic) all relevant monotonic functions may be represented by appropriate types: one can directly code a failure of multiplicativity in the ultrafilter as the omission of a type in the ultrapower.

From this discussion, the reader may already guess that Keisler's order reflects a certain structure on the family of all monotonic functions from $[\lambda]^{<\aleph_0} \to \mathcal{D}$ which is largely invisible to current tools. For instance, as was discussed above, the theory $T_* = Th(\mathbb{Q}, <)$ is obviously much less expressive than Peano arithmetic. However, T_* and PA are in the same class, the maximum class, in Keisler's order. This means

[12]In model-theoretic language, this would separate the Keisler-class of T_{feq} (the minimum class among theories with the tree property) and the Keisler-class of theories with the strong tree property SOP_2.

that in order that every monotonic function from $[\lambda]^{<\aleph_0}$ to $\mathcal{D}$ have a multiplicative refinement, it is sufficient to have multiplicative refinements for the functions arising as codes for types describing cuts in linear order.

General Problem 4.3. *Give an internal set-theoretic characterization of a family $\mathcal{F} \subseteq [\lambda]^{<\aleph_0} \to \mathcal{D}$ of monotonic functions which is in some sense minimal for the property that if every $f \in \mathcal{F}$ has a multiplicative refinement then $\mathcal{D}$ is good.*

The following rephrasing of Problem 3.5 may be a useful weakening of 4.3. $\mathcal{H}(\mu)$ are the sets hereditarily of cardinality less than μ for some sufficiently large μ.

Problem 4.4. *Suppose $\mathcal{D}$ is a regular ultrafilter on λ such that $N = (\mathcal{H}(\mu), \epsilon)^\lambda/\mathcal{D}$ is "λ^+-saturated for functions," i.e. if f is a partial function with domain of cardinality $\leq \lambda$ from N to N then there is an internal function which extends it. Is $\mathcal{D}$ good?*

Progress on the set-theoretic Problems 4.3 and 4.4 may help with a longstanding model-theoretic problem: finding a model-theoretic characterization of the maximal Keisler class. At present we have only the set-theoretic characterization given by Keisler, i.e. T is $\trianglelefteq$-maximum if and only if the regular ultrafilters which are good for T are precisely the good regular ultrafilters. The most recent advance on this question, a new sufficient model-theoretic condition for maximality given in [15] Theorem 1.11, led to many of the results and ideas in the present paper, and surely there is much more to be said.

Appendix

Proof of Fact 1.3. Suppose $(\kappa, \theta) \in \mathcal{C}(\mathcal{D}, N)$, witnessed by $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \theta\rangle)$ in the ultrapower $N^I/\mathcal{D}$. Fix a lifting $N^I/\mathcal{D} \to N$, so that for each $t \in I$ the coordinate projections $a_\alpha[t]$, $b_\beta[t]$ are well defined. Let $A = \{a_\alpha : \alpha < \kappa\}$ and $B = \{b_\beta : \beta < \theta\}$. Since $\mathcal{D}$ is regular, we may find a regularizing family and choose $\kappa + \theta$ distinct elements of this family which we index as $\{X_\alpha : \alpha < \kappa\} \cup \{X_\beta : \beta < \theta\}$. Define a map $f : A \cup B \to \mathcal{D}$ by $a_\alpha \mapsto X_\alpha$ for $\alpha < \kappa$ and $b_\beta \mapsto X_\beta$ for $\beta < \theta$. For each $t \in I$, let $A[t] = \{a_\alpha[t] : t \in f(a_\alpha)\}$ and $B[t] = \{b_\beta[t] : t \in f(b_\beta)\}$. For each $t \in I$, both $A[t]$ and $B[t]$ are finite, and so $A[t] \cup B[t]$ is a finite, discretely ordered set in N. Let $\mathbf{i}_t : A[t] \cup B[t] \to M$ be an order preserving injection. (For greater transparency, we may choose this map so that if $x < y$ are from $\mathrm{dom}(\mathbf{i}_t)$ and $N \models (\exists z)(x < z < y)$ then $M \models (\exists z)(\mathbf{i}_t(x) < z < \mathbf{i}_t(y))$; but this isn't necessary.) Fix an element 0 of M. For $\alpha < \kappa$ and $t \in I$, define $c_{\alpha,t}$ to be $\mathbf{i}_t(a_\alpha[t])$ if $t \in X_\alpha$, and 0 otherwise. For $\beta < \theta$ and $t \in I$, define $d_{\beta,t}$ to be $\mathbf{i}_t(b_\beta[t])$ if $t \in X_\beta$, and 0 otherwise. Let $c_\alpha := \langle c_{\alpha,t} : t \in I\rangle/\mathcal{D} \in M^I/\mathcal{D}$, and likewise let $d_\beta = \langle d_{\beta,t} : t \in I\rangle/\mathcal{D} \in M^I/\mathcal{D}$. By Łos' theorem, $(\langle a_\alpha : \alpha < \kappa\rangle, \langle b_\beta : \beta < \theta\rangle)$ is a cut in $N^I/\mathcal{D}$ if and only if $(\langle b_\alpha : \alpha < \kappa\rangle, \langle d_\beta : \beta < \theta\rangle)$ is a cut in $M^I/\mathcal{D}$, and so $(\kappa, \theta) \in \mathcal{C}(\mathcal{D}, M)$. $\square$

By using $M = N$ in Fact 1.3 and choosing each $\mathbf{i}_t$ in the proof of that Fact to be order reversing, $(\kappa, \mu) \in \mathcal{C}(\mathcal{D})$ if and only if $(\mu, \kappa) \in \mathcal{C}(\mathcal{D})$.

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