KEISLER'S ORDER HAS INFINITELY MANY CLASSES

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ABSTRACT. We prove, in ZFC, that there is an infinite strictly descending chain of classes of theories in Keisler's order. Thus Keisler's order is infinite and not a well order. Moreover, this chain occurs within the simple unstable theories, considered model-theoretically tame. Keisler's order is a central notion of the model theory of the 60s and 70s which compares first-order theories (and implicitly ultrafilters) according to saturation of ultrapowers. Prior to this paper, it was long thought to have finitely many classes, linearly ordered. The model-theoretic complexity we find is witnessed by a very natural class of theories, the n-free k-hypergraphs studied by Hrushovski. This complexity reflects the difficulty of amalgamation and appears orthogonal to forking.

A significant challenge to our understanding of unstable theories in general, and simple theories in particular, has been the apparent intractability of the problem of Keisler's order. Determining the structure of this order is a large-scale classification program in model theory. Its structure on the stable theories was known, and recent progress on the unstable case has had surprising applications, described in [24], [25], [27]. The order was long thought to be finite, with perhaps four classes, whose identities were suggested in 1978 (see Problem 0.3 below). In the present paper, we leverage the ZFC theorems of [26] to prove, nearly fifty years after Keisler introduced the order, that Keisler's order has infinitely many classes, and moreover is not a well order. The nature of this infinite hierarchy suggests that the order may encode much more model-theoretic information than was generally thought.

There are a number of recent accounts of Keisler's order, as in the introduction to [26]. We give here a brief sketch to put our main theorem in context.

Keisler's order asks about the saturation properties of certain limits of sequences of models, the regular ultrapowers. If $\mathcal{D}$ is an ultrafilter on the infinite set I, let us call $\mathcal{D}$ *regular* if whenever M is a model in a countable language, whether or not the ultrapower $M^I/\mathcal{D}$ is $|I|^+$-saturated depends only on the theory of M. (The theorem giving this equivalent definition is due to Keisler [12]. In fact, consistently, all ultrafilters are regular [2].) Given a theory T and a regular ultrafilter $\mathcal{D}$, we may therefore say that "$\mathcal{D}$ saturates T" if indeed $M^I/\mathcal{D}$ is $|I|^+$-saturated for some, equivalently every, model of T. Keisler proposed the study of the pre-order on complete, countable theories given by:

Definition 0.1 (Keisler 1967)**.** *$T_1 \trianglelefteq T_2$ iff any regular ultrafilter $\mathcal{D}$ which saturates T_2 must also saturate T_1.*

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The pre-order $\trianglelefteq$ is usually thought of as a partial order on the $\trianglelefteq$-equivalence classes. Keisler's order allows for a comparison of complexity of any two theories, possibly in different languages, as noted in Morley's enthusiastic review [28].

Problem 0.2. *Determine the structure of Keisler's order.*

Given the complexity of ultrafilters, it was widely believed that Keisler's order would be coarse, with a small number of classes, linearly ordered. In order for ultrafilters to be able to distinguish theories, we would expect, informally, that the 'patterns' of types in each must be much more than superficially different, and so that divisions in Keisler's order would correspond to model-theoretic dividing lines. Indeed, the union of the first two classes is precisely the stable theories ([31] VI.5). The following problem suggests, in current language, that the simple unstable theories form a single class, and that the non-simple, non-maximal theories form a single class. Note already the suggestion that one would need a parallel (not simply an extension) of local stability theory in order to approach the problem.

Problem 0.3 ([31] Problem VI.0.1 p. 324)**.** *It would be very desirable to prove that: (1A) $T_1, T_2 \in K_{ind}$ implies T_1, T_2 are $\trianglelefteq$-equivalent, (1B) T_1, $T_2 \in K_{cdt}$ (or we should ask also whether $\kappa_{inp}(T) = \infty$, $\kappa_{sct}(T) = \infty$) implies T_1, T_2 are $\trianglelefteq$-equivalent. This will complete the model-theoretic share of investigating Keisler's order for countable theories. For this it seems reasonable to try to find for $T \in K_{ind}$ a theory parallel to II, Section 2 for stable theories.*

For a long time, there was little progress on the unstable case; [21] gives some history. This was due both to the difficulty of constructing ultrafilters and the state of our understanding of unstable theories. In the last few years, there has been very substantial progress in this area (Malliaris [17]-[20], Malliaris and Shelah [21]-[26]). By our earlier paper [22], there are at least two classes within the simple theories, so at least five classes total.

We now state the main theorem of the present paper:

Main Theorem (6.6 below)**.** *There is an infinite descending sequence of simple rank 1 theories in Keisler's order. More precisely, there are simple theories $\{T^*_n : n < \omega\}$, with trivial forking, such that writing*

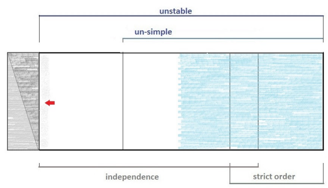
- *$\mathcal{T}_A$ for the class of theories without the finite cover property (f.c.p.)*
- *$\mathcal{T}_B$ for the class of stable theories with the f.c.p.*
- *$\mathcal{T}_C$ for the minimum unstable class, i.e. the Keisler-equivalence class of the random graph*
- *$\mathcal{T}_{max}$ for the Keisler-maximal class, i.e. the Keisler-equivalence class of linear order (or SOP_2)*
- *and $\mathcal{T}_n$ for the Keisler-equivalence class of T^*_n*

for all $m < n < \omega$ we have:

$$\mathcal{T}_A \triangleleft \mathcal{T}_B \triangleleft \mathcal{T}_C \triangleleft \qquad \cdots\cdots\cdots \mathcal{T}_n \triangleleft \mathcal{T}_m \triangleleft \cdots \triangleleft \mathcal{T}_2 \triangleleft \mathcal{T}_1 \triangleleft \mathcal{T}_0 \triangleleft \qquad \mathcal{T}_{max}.$$

This theorem says that Keisler's order, far from being coarse, has a kind of productive fineness: it is sensitive to an entire hierarchy of amalgamation properties within the simple theories, detecting a complexity which is orthogonal to forking. Part of the interest of Keisler's order is that it appears particularly sensitive to gradations in the complexity of the independence property, which is orthogonal to much of what was known.

The current picture may be illustrated as follows:



The arrow indicates the location of the infinite descending chain. The white regions are not yet mapped. The large shaded region on the right is the maximum class; we don't know whether it may encompass all non-simple theories. We now introduce some key objects of the proof.

Definition 0.4. *$T_{m,k}$ is the model completion of the theory with one symmetric irreflexive $(k+1)$-ary relation R and no complete graphs on $m+1$ vertices. We say (m,k) is* nontrivial *to mean that $m > k \geq 2$, and say that $T_{m,k}$ is nontrivial if (m,k) is. We will assume $T_{m,k}$ is nontrivial unless otherwise stated.*

When $m > k = 1$, so the edge relation is binary, there is a lot of dividing: the theory is non-simple and in fact SOP_3, but in the case of hyperedges the situation is different.[1]

Theorem 0.A. (Hrushovski [9]) *For $m > k \geq 2$, the theory $T_{m,k}$ is simple with trivial forking.*

Simple theories, a generalization of stable theories which include the random graph and pseudofinite fields, are an active area of model-theoretic research as well as a fertile interface for applications of model theory to geometry, combinatorics, and number theory [1], [10], [16]. However, already for internal model-theoretic reasons, it had long seemed plausible that the simple unstable theories might admit a meaningful division into an infinite hierarchy, see e.g. Conjecture 5.6 of [33].

Our argument will be guided by the following informal thesis.

Thesis 0.5. *The theories $T_{m+r,k+r}$ become in some sense less complicated (closer to the random graph) as $r \rightarrow \infty$.*

Note that this thesis does not yet account for each coordinate growing separately.

Some brief remarks on the proof are in order, since much of the interest of the result comes from the structure it reveals. How to see differences among the theories $T_{n,k}$? It may appear that any distances between them are in some sense finitary in nature. The remarkable fact is that there emerges a connection – both by analogy and by proof – to the sense in which set mappings explain how certain distances between infinite cardinals are also finitary in nature. The key theorem for us (all definitions will be given in §1) is the Kuratowski-Sierpinski characterization of the distance between alephs via existence of free sets in set mappings:

$$(\aleph_{\alpha+k}, k, \aleph_\alpha) \rightarrow k+1 \text{ but } (\aleph_{\alpha+k}, k+1, \aleph_\alpha) \not\rightarrow k+2.$$

[1] When a simple theory has 'trivial forking' it is a model-theoretic indication that, at least in one strong sense, complexity is low. A formula $\varphi(x, \bar{a})$ divides if for some indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ with $\bar{a}_0 = \bar{a}$, and for some $n < \omega$, every n formulas from $\{\varphi(x, \bar{a}_i) : i < \omega\}$ are inconsistent. In simple theories forking and dividing coincide. For the $T_{n,k}$, there is quantifier elimination, and so the point is that no formula which is a finite boolean combination of instances of R can divide.

The sense in which free sets escape control of their smaller pieces echoes the characteristic sense in which amalgamation problems in the $T_{n,k}$ escape control of their smaller pieces. The proof will make this correspondence more precise. The effect is that the set theory reflects down to illuminate large differences among purely model theoretic properties which we would otherwise be hard pressed to distinguish: ultrafilters (in which we have inscribed the two cardinals $\mu = \aleph_\alpha, \lambda = \aleph_{\alpha+k}$ in an intelligible way) can show us when there is a significant difference in the complexity of the amalgamation problems presented by these different hypergraphs.

In slightly more detail, the ultrafilter contribution may be sketched as follows. (All definitions will be given below.) Let $\mathcal{P}(I)$ denote the set of all subsets of I. Let μ be an infinite cardinal, $\mu < |I|$. First, we build a certain kind of regular filter $\mathcal{D}_0$ on I, called excellent, in such a way that in the quotient Boolean algebra $\mathfrak{B} \cong \mathcal{P}(I)/\mathcal{D}_0$ the maximal antichains have size μ, a measure of the remaining degrees of freedom. Second, we build another kind of ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$, called perfect, and pull it back to I to obtain the final ultrafilter $\mathcal{D}$, called (λ, μ)-perfected. As part of this picture, there is a way of representing types in the ultrapower as sequences in the intermediate Boolean algebra $\mathfrak{B}$. Now we ask about the cardinal distance between $\lambda = |I|$ and μ. It will turn out that if $\mu = \aleph_\alpha$, $\lambda = \aleph_{\alpha+\ell}$ and $T = T_{k+1,k}$ then whether such a $\mathcal{D}$ saturates T [whether $\mathcal{D}_*$ is able to resolve the corresponding representations of types in $\mathfrak{B}$ despite its relative "narrowness"] will depend on the relative sizes of k and ℓ. So each finite amalgamation constraint, such as the prohibition on tetrahedra, will be properly reflected in a certain finite distance built in to our infinite amplifiers, the ultrafilters.

In light of Theorem 6.6, we may ask: are there incomparable simple theories in Keisler's order? If not, the amalgamation properties highlighted in our theorem carry a great deal more of the structure of simple theories than one might a priori expect. If so, if there are many incomparable classes of simple theories and if they may be internally characterized, this is also likely to be very productive for model theory because it would bring currently invisible undercurrents of complexity in simple unstable theories to the surface.

While the present paper reverses nearly fifty years of thinking about Keisler's order, this reversal comes with the remarkable suggestion, visible in the mechanics of our proof, that Keisler's order may provide a systematic way of detecting the fine 'combinatorial building blocks' of structures.

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Contents

1. Preparation

Our model-theoretic approach is guided by the framework of [26] and in particular its program of stratifying the complexity of simple theories according to their so-called *explicit simplicity.* As explained there, to capture the problem of realizing types in ultrapowers it is helpful to remember that at each index model, Łos' theorem may guarantee that the 'projections' of various finite fragments of the type are correct but it will not, in general, preserve their relative position. An informative translation of the complexity which may arise in such projections is to ask: given a type $p \in \mathbf{S}(N)$, $||N|| = \lambda$ not forking over some small M_*, when can we color the finite pieces of p (or more correctly, sufficiently closed sets containing them) with μ colors so that any time we move finitely many pieces of the same color by piecewise automorphisms which are the identity on M_*, agree on common intersections and introduce no new forking, the union of the images is a consistent partial type?

The main theorems of the present paper will imply that for each finite $k \geq 2$, for $T_{k+1,k}$ it is necessary and sufficient to have μ colors when $\lambda = \mu^{+(k-1)}$. To see that, for instance, the tetrahedron-free three-hypergraph $T_{3,2}$ requires multiple colors, consider a type $\{R(x,a,b), R(x,b,c), R(x,a,c)\}$ in the monster model, where $\models \neg R(a,b,c)$ but piecewise automorphisms of $\{R(x,a,b)\}$, $\{R(x,b,c)\}$, $\{R(x,a,c)\}$ may move the parameters onto a triangle.

Before making further remarks on strategy, we review a family of classical results on set mappings. Proofs of Theorems 1.B, 1.C, 1.D may be found in Erdős, Hajnal, Máté, and Rado [6], as noted. We use λ, κ, μ for infinite cardinals and k, ℓ, m, n for integers.

Definition 1.1. *Let m, n be integers, α an ordinal, and λ, μ infinite cardinals.*

1. *We say $F : [\lambda]^m \rightarrow [\lambda]^{<\mu}$ is a* set mapping *if $F(x) \cap x = \emptyset$ for $x \in [\lambda]^m$.*
2. *We say the set $X \subseteq \lambda$ is* free *with respect to F if $F(x) \cap X = \emptyset$ for every $x \in [X]^m$.*

Notation 1.2. *We write:*

$$(\lambda, m, \mu) \longrightarrow n$$

to mean that for every set mapping $F : [\lambda]^m \rightarrow [\lambda]^{<\mu}$ there is a set X of size n which is free with respect to F, and write

$$(\lambda, m, \mu) \not\longrightarrow n$$

to mean that for some set mapping $F : [\lambda]^m \rightarrow [\lambda]^{<\mu}$ no set of size n is free with respect to F.

A celebrated theorem of Sierpiński [34] states that the continuum hypothesis holds precisely when $\mathbb{R}^3$ admits a decomposition into three sets A_x, A_y, A_z such that for $w = x, y, z$, A_w intersects all lines in the direction of the w-axis in finitely many points. In other words, this property characterizes $\aleph_1$. Kuratowski and Sierpiński then characterized all $\aleph_n$s via set mappings:

Theorem 1.B. (see [6] Theorem 46.1) *For any $m < \omega$ and ordinal α we have that*

$$(\aleph_{\alpha+m}, m, \aleph_\alpha) \longrightarrow m + 1$$

Theorem 1.C. (see [6] Theorem 45.7) *For any $m < \omega$ and ordinal α we have that*

$$(\aleph_{\alpha+m}, m + 1, \aleph_\alpha) \not\longrightarrow m + 2$$

Corollary 1.3. (Monotonicity) *Given $m_0 \leq m \leq m_1$, $n_0 \leq n \leq n_1$,*

(a) if $n > m$ and $(\lambda, m, \mu) \longrightarrow n$ then $(\lambda, m_0, \mu) \longrightarrow n_0$.
(b) if $n_1 > m_1$ and $(\lambda, m, \mu) \not\longrightarrow n$ then $(\lambda, m_1, \mu) \not\longrightarrow n_1$.

Proof. (a) Suppose we are given a set mapping $F : [\lambda]^{m_0} \rightarrow [\lambda]^{<\mu}$. Define $F' : [\lambda]^m \rightarrow [\lambda]^{<\mu}$ by $F'(u) = \bigcup\{F(u') : u' \in [u]^{m_0}\} \setminus u$. Clearly F' is a set mapping. Since $(\lambda, m, \mu) \rightarrow n$, there is a free set v_* for F', $|v_*| = n$. Fix $v_{**} \subseteq v_*$ with $|v_{**}| = n_0 \leq n$. Let us check that v_{**} is as required for F. Let $u \in [v_{**}]^{m_0}$ and $\alpha \in v_{**} \setminus u$. Since F is a set mapping, it suffices to prove that $\alpha \notin F(u)$. Towards this, choose $w \subseteq v_* \setminus (u \cup \{\alpha\})$ of size $m - m_0$, which is possible because $|v_* \setminus (u \cup \{\alpha\})| = n - (m_0 + 1) = (n-1) - m_0 \geq m - m_0$. So $u \cup w \subseteq v_*$ and $|u \cup w| = m_0 + (m - m_0) = m$ elements and $\alpha \in v_* \setminus (u \cup w)$. As v_* was chosen to be a free set for F', necessarily $\alpha \notin F'(u \cup w)$. Now recalling the definition of F', $F'(u \cup w) = \bigcup\{F(x) : x \in [u \cup w]^{m_0}\} \setminus (u \cup w)$. Since $\alpha \notin (u \cup w)$ and we know that $\alpha \notin F'(u \cup w)$, we conclude $\alpha \notin F(u)$ as desired.

(b) This holds by the contrapositive of (a), i.e. (a) applied to m_1, n_1, m, n instead of m, n, m_0, n_0. $\square$

The general situation for free sets of large finite size relative to m is less clear. For instance, it is known that:

Theorem 1.D. (see [6] Theorem 46.2) *For any $n < \omega$ and ordinal α we have that*

- (Hajnal-Máté) $(\aleph_{\alpha+2}, 2, \aleph_\alpha) \longrightarrow n$
- (Hajnal) $(\aleph_{\alpha+3}, 3, \aleph_\alpha) \longrightarrow n$

However, we note there are also consistency results.[2] In the following theorem, $\tau(n+1)$ is the least natural number such that $\tau(n+1) \rightarrow (\tau(n), 7)^5$. The notation $a \rightarrow (b, c)^r$, for $a, b, c, r \in \mathbb{N}$, means that whenever the r-element subsets of an a-element set are colored with two colors, then either there is a b-element subset with all its r-tuples of the first color, or there is a c-element subset with all its r-tuples of the second color. The existence of an a given b, c, r is given by Ramsey's theorem. (Further results are in a forthcoming paper [29].)

Theorem 1.E. (Komjáth and Shelah 2000 [14], Theorem 1) *There is a function $\tau : \omega \rightarrow \omega$ such that whenever μ is regular, $n < \omega$, $\lambda = \mu^{+n}$, $\mu = \mu^{<\mu}$, and $\bigwedge_{\ell < n} 2^{\mu^{+\ell}} = \mu^{+\ell+1}$, for some $(< \mu)$-complete $\mu^{+(n+1)}$-c.c. forcing notion $\mathcal{P}$ of cardinality λ collapsing no cardinals, in $\mathbf{V}^{\mathcal{P}}$ we have $2^\mu = \mu^{+n}$ and there is a set mapping $F : [\lambda]^4 \rightarrow [\lambda]^{<\mu}$ with no free subset of size $\tau(n)$.*

In symbols, under these assumptions,

$$\left(\mu^{+n}, 4, \mu\right) \not\longrightarrow \tau(n)$$

We now briefly motivate our use of these theorems for realizing and omitting types. A first adjustment is that we would like to enclose fragments of types in suitable larger parameter sets, so we will want to replace the condition $x \cap F(x) = \emptyset$ in the definition of set mapping with the condition that $x \subseteq F(x)$ as in 1.4(1) and also to replace "not free" with 1.4(2).

[2] We carry out the present proof entirely in ZFC. It will be very interesting to see whether future work will show such independence results to also be reflected in the model-theoretic structure of simple theories, or whether the connection goes no further than what we develop here.

Definition 1.4. *Let $k < n$ be integers, α an ordinal, and λ, θ infinite cardinals (usually $\theta = \aleph_0$). .*

(1) *We say $F : [\lambda]^k \to [\lambda]^{<\theta}$ is a* strong set mapping *if $x \subseteq F(x)$ for $x \in [\lambda]^k$.*
(2) *We say the set $X \in [\lambda]^n$ is* covered *with respect to F if there exists $x \in [X]^k$ such that $X \subseteq F(x)$.*

Briefly, working in one of the theories $T_{k+1,k}$ ($k \geq 2$), we will want to associate to each finite subtype a larger 'enveloping' set, and to color these envelopes in such a way that within any fixed color class, any time a near-forbidden configuration (e.g. a hyperedge on the parameters in $T_{3,2}$) appears it must already be contained in one of the associated envelopes. In the course of our analysis, we will be able to ensure the individual envelopes correspond to consistent partial types over submodels. As will be explained, this property of absorbing forbidden configurations will then give sufficient leverage for a proof (by contradiction) of amalgamation within each color class. The right formalization for our present arguments is the following. The one-step closure operator cl_1 will be defined in 2.4 below. The existence statement is Lemma 2.5.

Definition 1.5. *Let $\mathrm{Pr}_{n,k}(\lambda, \mu) = \mathrm{Pr}^0_{n,k}(\lambda, \mu)$ be the statement that:*
There is $G : [\lambda]^{<\aleph_0} \times [\lambda]^{<\aleph_0} \to \mu$ such that:
if $w \in [\lambda]^n$,
$\overline{u} = \langle u_v : v \in [w]^k \rangle$, $v \in [w]^k \implies v \subseteq u_v \in [\lambda]^{<\aleph_0}$
and $G \restriction \{(u_v, \mathrm{cl}_1(u_v)) : v \in [w]^k\}$ is constant,
then for some $v \in [w]^k$ we have $w \subseteq \mathrm{cl}_1(u_v)$.

The saturation half of the argument, Theorem 4.1, involves an analysis of model-theoretic amalgamation problems arising in ultrapowers with Definition 1.5 as a key ingredient. A key point is that for the hypergraphs in question we may always take $\mu < \lambda$, and in fact, the subscript k is tied to the cardinal distance of λ and μ. The (λ, μ)-perfect (optimal) ultrafilters of [26] play an important role, as will be explained in due course.

If failures of freeness, which is to say of covering, help with saturation, when will existence of free sets yield omitted types? A priori, given a model $N = (\lambda, R) \models T_{n,k}$, we cannot directly apply Theorem 1.B to omit a type, because that theorem does not guarantee that the free set will occur on an R-complete graph. The right analogue for the non-saturation half of the argument, Conclusion 5.4, will be:

Lemma 1.6 (proved in §2)**.** *Suppose that $n > k \geq 2$ and $(\lambda, k, \mu) \to n$. Then there is a model M of $T_{n,k}$ of size $\geq \lambda$, and λ elements of its domain $\langle b_\alpha : \alpha < \lambda \rangle$, such that writing*

$$\mathcal{P} = \{ w \in [\lambda]^n : (\ \forall u \in [w]^{k+1})(M \models R(\bar{b}_u)) \ \}$$

we have that for any strong set mapping $F : [\lambda]^k \to [\lambda]^{<\mu}$, for some $w \in \mathcal{P}$

$$(\forall v \in [w]^k)(w \not\subseteq F(v)).$$

For orientation, the reader may now wish to read the statement of Theorem 6.1, as well as of the Main Theorem 6.6. Keisler's order is defined in §6. Further background on Keisler's order and saturation of ultrapowers appears in [25] and in the introduction to [26]. Earlier sources are [12], [13].

We now turn to the proofs.

2. Key covering properties

In this section, we give the the existence proof corresponding to Definition 1.5 above, Lemma 2.5. We also prove Lemma 1.6 from p. 7 above. "Pr" abbreviates "property." Locally in this section, we will refer to the property from 1.5 as $\mathrm{Pr}^0_{n,k}(\lambda,\mu)$ to distinguish it from the weaker variant ${\mathrm{Pr}^1}_{n,k}(\lambda,\mu)$ defined below. We will establish results for both Pr^0 and Pr^1 in this section, although only Pr^0 is central for our proofs. The one-step closure operator will be defined in 2.4. (In all later sections in the paper, Pr means Pr^0, as will be stated in Convention 2.8.)

Definition 2.1. *Let* ${\mathrm{Pr}^1}_{n,k}(\lambda,\mu)$ *be the statement that:*
if $N = (\lambda, R) \models T_{n,k}$ *then we can find* $G : [\lambda]^{<\aleph_0} \to \mu$ *such that:*

> *if* $w \in [\lambda]^n$, $(w, R \restriction w)$ *is a complete hypergraph,*
> $\overline{u} = \langle u_v : v \in [w]^k \rangle$, $v \in [w]^k \implies v \subseteq u_v \in [\lambda]^{<\aleph_0}$
> *and* $G \restriction \{(u_v, \mathrm{cl}_1(u_v)) : v \in [w]^k\}$ *is constant,*
> *then for some* $v \in [w]^k$ *we have* $w \subseteq \mathrm{cl}_1(u_v)$.

So the only difference between 2.1 and 1.5 is the clause "$(w, R \restriction w)$ is a complete hypergraph." We focus on the stronger property Pr^0, but Pr^1 is worth stating as it is natural for our setting, and fits well with Lemma 1.6.

We start by upgrading some properties of strong set mappings.[3]

Claim 2.2. *Suppose* $(\lambda, \ell, \mu^+) \not\to \ell + 1$, *witnessed by the set mapping* $F_0 : [\lambda]^\ell \to [\lambda]^{\leq \mu}$. *Let* $F : [\lambda]^\ell \to [\lambda]^{\leq\mu}$ *be any function such that for all* $u \in [\lambda]^\ell$,

(i) $u \subseteq F(u)$
(ii) $F_0(u) \subseteq F(u)$
(iii) closure: for all $v \in [F(u)]^\ell$, $F(v) \subseteq F(u)$.

Then for all $n > \ell$ *and* $w \in [\lambda]^n$, *there exists* $u \in [w]^\ell$ *such that* $w \subseteq F(u)$.

Proof. Let $n > \ell$ be given. Choose $w \in [\lambda]^n$ and choose $v \in [w]^\ell$ such that $|w \cap F(v)|$ is maximal. (As w is finite, the maximum is well defined, and by (i), it is non zero.) If $w \subseteq F(v)$ we are done. Else suppose for a contradiction that we may choose $\alpha \in w \setminus F(v)$. Let $x = v \cup \{\alpha\}$. As $|x| = \ell + 1$, there is $y \in [x]^\ell$ such that $F_0(y) \cap x \neq \emptyset$. As F_0 is a set mapping and $|x| = \ell+1$, this means $F_0(y) \cap x = x \setminus y$. Thus $x \setminus y \subseteq F_0(y) \subseteq F(y)$, the last inclusion by (ii). We are assuming in (i) that $y \subseteq F(y)$. We conclude $x \subseteq F(y)$. A fortiori, $\alpha \in F(y)$.

Now recall that $v \subseteq x$, so $v \in [F(y)]^\ell$. By our assumption (iii), $F(v) \subseteq F(y)$. Thus, remembering w from the beginning of the proof, $F(v) \cap w \subseteq F(y) \cap w$. But we know $\alpha \in w$, $\alpha \notin F(v)$, and $\alpha \in F(y)$. So $F(v) \cap w \subsetneq F(y) \cap w$, contradicting our choice of v. □

Claim 2.3. *If* $(\lambda, \ell, \mu^+) \not\to \ell+1$ *then there is a strong set mapping* $F : [\lambda]^\ell \to [\lambda]^{\leq\mu}$ *such that for any* $n \in [\ell+1, \omega)$ *and any* $w \in [\lambda]^n$, *there is* $u \in [w]^\ell$ *with* $w \subseteq F(u)$.

Proof. Let $F_0 : [\lambda]^\ell \to [\lambda]^{\leq\mu}$ be a set mapping witnessing $(\lambda, \ell, \mu^+) \not\to \ell + 1$. It suffices to build F satisfying the criteria of Claim 2.2. For each $u \in [\lambda]^\ell$, let $A^0_u = u \cup F_0(u)$. Then by induction on $i < \omega$ define $A^{i+1}_u = A^i_u \cup \bigcup\{F_0(v) : v \in [A^i_u]^\ell\}$. Each A^i_u is of size $\leq \mu$, so letting $F(u) = \bigcup\{A^i_u : i < \omega\}$ suffices. □

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[3]Note that in Claim 2.2, the proof would go through with the hypotheses that $(\lambda, \ell, \mu) \to \ell+1$, changing the range of the function to $[\lambda]^{<\mu}$. The statement is for easy quotation in Claim 2.3.

Definition 2.4 (Content and closure)**.** *Suppose* $(\lambda, k-1, \mu^+) \not\to k$. *Let* $F : [\lambda]^{k-1} \to [\lambda]^{\leq \mu}$ *be given by Claim 2.3, in the case* $\ell = k-1$. *Fix functions* $\langle F_i : i < \mu \rangle$ *with* $F_i : [\lambda]^{k-1} \to \mu$, *so that for each* $u \in [\lambda]^{k-1}$, $\langle F_i(u) : i < \mu \rangle$ *lists* $F(u)$ *without repetition. For* $x \subseteq \lambda$ *not necessarily finite, we may now define:*

(1) *The content of* x*:*

$$\mathrm{cont}(x) = \{i < \mu : \textit{ for some } v \in [w]^{k-1} \textit{ and } \alpha \in x \setminus v \textit{ we have } F_i(v) = \alpha\}.$$

So $\mathrm{cont}(x)$ *is finite if* x *is finite, and of cardinality* $|x|$ *otherwise.*

(2) *The* n*-step closure of* x*:* $\mathrm{cl}_0(x) = x$, *and*

$$\mathrm{cl}_{n+1}(x) = \mathrm{cl}_n(x) \cup \{F_i(v) : i \in \mathrm{cont}(\ \mathrm{cl}_n(x)\ ), v \in [\mathrm{cl}_n(w)]^{k-1}\}.$$

Thus, if $x \subseteq \lambda$ is finite, so is its n-step closure $\mathrm{cl}_n(x)$ for each finite n; and if x is infinite, $|\,\mathrm{cl}_n(x)| = |x|$ for each n.

In what follows we will mainly use the one-step closure, $\mathrm{cl}_1(x)$. Notice that $\mathrm{cont}(x)$ asks for which indices i is it the case that F_i takes some $(k-1)$-element subset of x to another element of x. To compute the 1-step closure, we then apply any such F_i to *all* $(k-1)$-element subsets of x.

With these results and definitions in hand we turn towards the existence statements for saturation and non-saturation, Lemma 1.5 and Claim 1.6.

Lemma 2.5. *Suppose* $(\lambda, k-1, \mu^+) \not\to k$. *Then* $\mathrm{Pr}_{n,k}(\lambda, \mu)$ *holds.*

Proof. Let F, $\langle F_i : i < \mu \rangle$ be as in Definition 2.4. We define $G(u_1, u_2)$ to code:

(1) $\mathrm{otp}(u_1)$, $\mathrm{otp}(u_2)$,
(2) $\{\mathrm{otp}(\alpha \cap u_1), \mathrm{otp}(\alpha \cap u_2)) : \alpha \in u_1\}$,
(3) $\{\langle \mathrm{otp}(\alpha_0 \cap u_1), \mathrm{otp}(\alpha_1 \cap u_1), \dots, \mathrm{otp}(\alpha_{k-1} \cap u_1), i\rangle : \alpha_0, \dots, \alpha_{k-1} \in u_1$ with no repetition, $\alpha_{k-1} = F_i(\{\alpha_0, \dots, \alpha_{k-2}\})\ \}$.
(4) $\{\langle \mathrm{otp}(\alpha_0 \cap u_2), \mathrm{otp}(\alpha_1 \cap u_2), \dots, \mathrm{otp}(\alpha_{k-1} \cap u_2), i\rangle : \alpha_0, \dots, \alpha_{k-1} \in u_2$ with no repetition, $\alpha_{k-1} = F_i(\{\alpha_0, \dots, \alpha_{k-2}\})\ \}$ (this implies (3)).

As the sets involved are all finite, and there are at most μ functions F_i, the range of G is contained in μ. (In fact, these conditions are somewhat more than is needed.)

Suppose we are given $w, \bar{u}$ as in the statement of Pr. That is, $w \in [\lambda]^n$, $\bar{u} = \langle u_v : v \in [w]^k \rangle$, $v \in [w]^k \to v \subseteq u_v \in [\lambda]^{<\aleph_0}$, and $G \restriction \{(u_v, \mathrm{cl}_1(u_v))\} : v \in [w]^k\}$ is constant. By our choice of F, there is some $v_* \in [w]^{k-1}$ such that $w \setminus v_* \subseteq F(v_*)$. Choose any two distinct elements $\alpha \neq \beta \in w \setminus v_*$. Let $v_\alpha = v_* \cup \{\alpha\}$, and let $v_\beta = v \cup \{\beta\}$. Observe that since $\langle F_i(v_*) : i < \mu \rangle$ lists $F(v_*) \supseteq w \setminus v_*$ without repetition, there is some $i = i(\alpha) < \mu$ such that $F_i(v_*) = \alpha$; fix this i for the next paragraph.

By our assumption, $G(u_{v_\alpha}, \mathrm{cl}_1(u_{v_\alpha})) = G(u_{v_\beta}, \mathrm{cl}_1(u_{v_\beta}))$. Let $f : u_{v_\alpha} \to u_{v_\beta}$ be a one-to-one order preserving map guaranteed by (1). Since by definition $v_\alpha \subseteq u_{v_\alpha}$, there exist distinct $\alpha_0, \dots, \alpha_{k-1} \in u_{v_\alpha}$ such that $F_i(\{\alpha_0, \dots, \alpha_{k-2}\}) = \alpha_{k-1}$, e.g. $\{\alpha_0, \dots, \alpha_{k-2}\} = v_*$ and $\alpha_{k-1} = \alpha$. This means $i \in \mathrm{cont}(u_{v_\alpha})$. Item (3) tells us that $F_i(f(\alpha_0), \dots, f(\alpha_{k-2})) = f(\alpha_{k-1})$. So again by definition of cont, $i \in \mathrm{cont}(u_{v_\beta})$. Note that $\{f(\alpha_0), \dots, f(\alpha_{k-2})\}$ need not be the set v_*. However, to compute $\mathrm{cl}_1(u_{v_\beta})$, for every $j \in \mathrm{cont}(u_{v_\beta})$ we apply F_j to *every* element of $[u_{v_\beta}]^{k-1}$, in particular to v_*. So $F_i(v_*) \in \mathrm{cl}_1(\mathrm{cont}(u_{v_\beta}))$, i.e. $\alpha \in u_{v_\beta}$.

Since α, β were arbitrary distinct elements of $w \setminus v_*$, we have shown that for any $\gamma \in w \setminus v_*$, $\gamma \in \mathrm{cl}_1(u_{v_\beta})$. Since $v_* \subseteq v_\beta \subseteq u_{v_\beta} \subseteq \mathrm{cl}_1(u_{v_\beta})$ by definition, we conclude $w \subseteq \mathrm{cl}_1(u_{v_\beta})$, which completes the proof. $\square$

Observation 2.6.

(1) If $\lambda_1 \geq \lambda_2 \geq \mu_2 \geq \mu_1$ *and* $\mathbf{i} \in \{0,1\}$ *then*

$$\mathrm{Pr^{\mathbf{i}}}_{n,k}(\lambda_1, \mu_1) \implies \mathrm{Pr^{\mathbf{i}}}_{n,k}(\lambda_2, \mu_2).$$

(2) $\mathrm{Pr}^0_{n,k}(\lambda, \mu) \implies \mathrm{Pr}^1_{n,k}(\lambda, \mu)$.

Proof. (1) If $\mathbf{i} = 1$, let $M_1 = (\lambda_1, R)$ and let $M_2 = M_1 \restriction \lambda_2 = (\lambda_2, R)$. If $G_1 : [\lambda_1]^{<\aleph_0} \times [\lambda_1]^{<\aleph_0} \to \mu_1$ is suitable, then $G_2 = G_1 \restriction [\lambda_2]^{<\aleph_0} \times [\lambda_2]^{<\aleph_0}$ is suitable, and has range $\subseteq \mu_1 \subseteq \mu_2$. (2) is immediate. $\square$

Now we turn to freeness and non-saturation.

Claim 2.7. *Suppose* $(\lambda, k, \mu) \to n$ *and* $k < n$. *Then for any* $F : [\lambda]^k \to [\lambda]^{<\mu}$, *e.g.* F *a strong set mapping, there is* $w \in [\lambda]^n$ *such that for all* $u \in [w]^k$, $w \not\subseteq F(u)$.

Proof. If F is a set mapping, this is immediate. If not, identify λ with $\lambda \setminus \{0\}$ and define $G : [\lambda]^k \to [\lambda]^{<\mu}$ by: $G(u) = (F(u) \setminus u) \ \cup \ \{0\}$ [so that $G(u) \neq \emptyset$]. (Alternately, first assume without loss of generality that $F(u) \supsetneq u$ for all $u \in [\lambda]^k$, then define $G(u) = F(u) \setminus u$.) Then G is a set mapping, so there is some $w \in [\lambda]^n$ such that for all $u \in [w]^k$, $w \cap G(u) = \emptyset$. Thus for each $u \in [w]^k$, if $w \cap F(u) \neq \emptyset$ then $w \cap F(u) \subseteq u$. Since $|u| = k$, this means $|w \cap F(u)| \leq k < n = |w|$ so $w \not\subseteq F(u)$ as desired. $\square$

We now prove Lemma 1.6, promised on p. 7 above.

Proof of Lemma 1.6. Let us define a model $N = (\lambda, R^N)$ by: $R^N = \{(\alpha_0, \ldots, \alpha_k) : \alpha_i < \lambda \text{ and } i_1 < i_2 \leq k \implies \alpha_{i_1} \neq \alpha_{i_2} \mod n \text{ and } i_1 < i_2 \leq k \implies (\exists \beta)(\alpha_{i_1} < n\beta \leq \alpha_{i_2} \vee \alpha_{i_2} < n\beta \leq \alpha_{i_1})\}$. By definition R^N is irreflexive, symmetric, and $(k+1)$-ary. Let us first show that if $w \in [\lambda]^{n+1}$ then w is not a complete R^N-hypergraph. If $|w| = n+1$, for some $\alpha_1 \neq \alpha_2 \in w$ we have $\alpha_1 = \alpha_2 \mod n$. Choose $u \subseteq w$ such that $\alpha_1 \in u$, $\alpha_2 \in u$, and $|u| = k+1$. Then by definition R cannot hold on $\{\alpha : \alpha \in u\}$, so w cannot be a complete R-hypergraph. So N is a submodel of some $N' \models T_{n,k}$ of cardinality λ.

Second, let us show that if $F : [\lambda]^k \to [\lambda]^{<\mu}$ is a given strong set mapping then for some $w \in \mathcal{P}$ we have $(\forall v \in [w]^k)(w \not\subseteq F(v))$. First let $F_1 : [\lambda]^k \to [\lambda]^{<\mu}$ be defined by: if $v = \{\alpha_0, \ldots, \alpha_{k-1}\} \in [\lambda]^k$ then

$$\begin{aligned} F_1(v) = \{\beta :\ & \text{for some } i_0, \ldots, i_{k-1} < n \text{ and } \gamma < \lambda \\ & \text{we have } (\beta < n\gamma + n) \wedge (\gamma < n\beta + n) \\ & \text{and } \gamma \in F(\{n\alpha_0 + i_0, \ldots, n\alpha_{k-1} + i_{k-1}\})\}. \end{aligned}$$

Why is $F_1 : [\lambda]^k \to [\lambda]^{<\mu}$? Recall n is fixed at the start of the proof. So given v, we first choose $i_0, \ldots, i_{k-1}$ (there are finitely many choices) and then choose those $\gamma \in F(\{n\alpha_0 + i_0, \ldots, n\alpha_{k-1} + i_{k-1}\}$ (there are $< \mu$ choices) which satisfy an additional criterion. So $F_1 : [\lambda]^k \to [\lambda]^{<\mu}$. As we have assumed $(\lambda, k, \mu) \to n$, by Claim 2.7 there is $w_1 = \{\alpha^*_i : i < n\}$ such that $\alpha^*_0 < \cdots < \alpha^*_{n-1} < \lambda$ and for all $v \in [n]^k$, $w_1 \not\subseteq F_1(\{\alpha^*_\ell : \ell \in v\})$.

Define $\beta_i = n\alpha^*_i + i$ for $i < n$ and let $w_2 = \{\beta_i : i < n\}$, and let us show w_2 is as required for F. First, trivially $|w_2| = n$, as $|w_1| = n$. Second, by definition of R^N above, w_2 is a complete R^N-hypergraph. Third, let us show that if $u_2 \in [w_2]^k$ and $w_2 \subseteq F(u_2)$ then we get a contradiction. Let $v \in [n]^k$ be such that $u_2 = \{\beta_i : i \in v\}$. Then $u_1 := \{\alpha^*_i : i \in v\} \in [w_1]^k$ and $w_1 \subseteq F_1(u_1)$. [This is because for each α^*_i, $i < n$ we presently have $\beta_i \in F(u)$ so there is an element

$$\gamma \in F(\{n\alpha^*_i + i : i \in v\}) = F(\beta_0, \dots, \beta_{k-1})$$

such that $(\alpha^*_i < n\gamma + n) \wedge (\gamma < n\alpha^*_i + n)$, namely $\gamma = \beta_i$, which belongs to $F(\beta_0, \dots, \beta_{k-1})$ since F is a strong set mapping.] This contradicts the choice of w_1. We have shown that for all $u \in [w_2]^k$, $w_2 \not\subseteq F(u)$, so w_2 is as required which completes the proof.
Lemma 1.6. □

Convention 2.8. *In the remainder of the paper,*

*(1) "*Pr*" used without a superscript means* Pr^0.
(2) All theories are complete and countable unless otherwise stated.

3. Separation of variables and optimal ultrafilters

In this section we explain the advances from [22] and [26], Theorems 3.F and 3.G below, which frame the rest of the proof. The gold standard for saturation is the following class of ultrafilters, called good, introduced by Keisler [11]. Keisler proved that good ultrafilters exist, assuming GCH, and Kunen eliminated the assumption of GCH [15].

Definition 3.1. *A filter $\mathcal{D}$ on λ is called* good *if every monotonic $f : [\lambda]^{<\aleph_0} \rightarrow \mathcal{D}$ has a multiplicative refinement. In other words, if f satisfies $u \subseteq v$ implies $f(v) \subseteq f(u)$ then there is $g : [\lambda]^{<\aleph_0} \rightarrow \mathcal{D}$ such that $g(u) \subseteq f(u)$ for all finite u and $g(u \cup v) = g(u) \cap g(v)$ for all finite u, v.*

Fact 3.2 (Keisler [12])**.** *If $\mathcal{D}$ is a regular ultrafilter on λ, then $\mathcal{D}$ is good if and only if for every complete countable theory T and any $M \models T$, $M^\lambda/\mathcal{D}$ is λ^+-saturated.*

We will informally say "$\mathcal{D}$ is good for T" to mean that for all[4] $M \models T$, $M^\lambda/\mathcal{D}$ is λ^+-saturated. Thus, $\mathcal{D}$ is good precisely when it is good for every (complete, countable) T. For more on this correpondence, see [26] §2.

In this paper a main object is to show certain regular ultrafilters are good for some of the $T_{k+1,k}$ while not for others. Our first point of leverage for seeing gradations in goodness will be from [22]. Towards this, let us set notation for Boolean algebras arising as the completion of the Boolean algebra generated by α (usually, 2^λ) independent partitions of size μ.

Definition 3.3. *For an infinite cardinal μ and an ordinal α,*

(1) Let $\mathrm{FI}_\mu(\alpha)$ denote the set of partial functions from α to μ with finite domain.
(2) $\mathfrak{B}^0 = \mathfrak{B}^0_{\alpha,\mu}$ is the Boolean algebra generated by:
$\{\mathbf{x}_f : f \in \mathrm{FI}_\mu(\alpha)\}$ freely subject to the conditions that
(a) $\mathbf{x}_{f_1} \leq \mathbf{x}_{f_2}$ when $f_1 \subseteq f_2 \in \mathrm{FI}_\mu(\alpha)$
(b) $\mathbf{x}_f \cap \mathbf{x}_{f'} \neq 0$ iff f, f' are compatible functions.

[4]When $\mathcal{D}$ is regular, if $M \equiv N$ in a countable signature then $M^\lambda/\mathcal{D}$ is λ^+-saturated if and only if $N^\lambda/\mathcal{D}$ is λ^+-saturated, Keisler [12] Corollary 2.1a.

(3) $\mathfrak{B}^1_{\alpha,\mu}$ *is the completion of* $\mathfrak{B}^0_{\alpha,\mu}$.
(4) *When* $\mathfrak{B}$ *is a Boolean algebra,* $\mathfrak{B}^+$ *denotes* $\mathfrak{B} \setminus \{0\}$.

Convention 3.4. *We will assume that giving* $\mathfrak{B}$ *determines* α, μ, *and a set of generators* $\langle \mathbf{x}_f : f \in \mathrm{FI}_\mu(\alpha) \rangle$.

Fact 3.5. *The existence of* $\mathfrak{B}^0_{2^\lambda,\mu}$ *thus its completion is by Engelking-Karlowicz* [5]. *See also Fichtenholz and Kantorovich*[7]*, Hausdorff* [8]*, or Shelah* [31] *Appendix, Theorem 1.5.*

The next definition was used in Theorem 3.F, i.e. [22] Theorem 6.13. It allows us to involve arbitrary ultrafilters $\mathcal{D}_*$ on complete Boolean algebras in the construction of regular ultrafilters $\mathcal{D}$.

Definition 3.6 (Regular ultrafilters built from tuples, from [22] Theorem 6.13)**.** *Suppose* $\mathcal{D}$ *is a regular ultrafilter on* I, $|I| = \lambda$. *We say that* $\mathcal{D}$ *is built from* $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ *when:*

(1) $\mathcal{D}_0$ *is a regular,* $|I|^+$*-excellent filter on* I
(*for the purposes of this paper, it is sufficient to use regular and good*)
(2) $\mathfrak{B}$ *is a Boolean algebra*
(3) $\mathcal{D}_*$ *is an ultrafilter on* $\mathfrak{B}$
(4) *there exists a surjective homomorphism* $\mathbf{j} : \mathcal{P}(I) \to \mathfrak{B}$ *such that:*
   (a) $\mathcal{D}_0 = \mathbf{j}^{-1}(\{1_\mathfrak{B}\})$
   (b) $\mathcal{D} = \{A \subseteq I : \mathbf{j}(A) \in \mathcal{D}_*\}$.

It was verified in [22] Theorem 8.1 that whenever $\mu \leq \lambda$ and $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$ there exists a regular good $\mathcal{D}_0$ on λ and a surjective homorphism $\mathbf{j} : \mathcal{P}(I) \to \mathfrak{B}$ such that $\mathcal{D}_0 = \mathbf{j}^{-1}(1)$. Thus, Definition 3.6 is meaningful.

Suppose now that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$, witnessed by $\mathbf{j}$. Consider a complete countable T and $M \models T$. Suppose $N \preceq M^\lambda/\mathcal{D}$, $|N| = \lambda$ and $p \in \mathbf{S}(N)$, where $p = \langle \varphi_\alpha(x, a_\alpha) : \alpha < \lambda \rangle$. (As ultraproducts commute with reducts, we may assume without loss of generality that $T = T^{eq}$ and so that each a_α is a singleton.) For each finite $u \subseteq \lambda$, the Łos map Ł sends $u \mapsto B_u$ where

$$B_u := \{t \in I : M \models (\exists x) \bigwedge_{\alpha \in u} \{R(x, a_\alpha[t])\}.$$

Let $\mathbf{b}_u = \mathbf{j}(B_u)$. The key model-theoretic property of the sequence $\langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ in $\mathfrak{B}$ is captured by the following definition.

Definition 3.7. (Possibility patterns [22] Definition 6.1) *Let* $\mathfrak{B}$ *be a Boolean algebra, normally complete, and* $\bar{\varphi} = \langle \varphi_\alpha : \alpha < \lambda \rangle$ *a sequence of formulas. Say that* $\overline{\mathbf{b}}$ *is a* $(\lambda, \mathfrak{B}, T, \bar{\varphi})$*-possibility when:*

(1) $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ *is a sequence of elements of* $\mathfrak{B}^+$
(2) *if* $v \subseteq u \in [\lambda]^{<\aleph_0}$ *then* $\mathbf{b}_u \subseteq \mathbf{b}_v$
(3) *if* $u_* \in [\lambda]^{<\aleph_0}$ *and* $\mathbf{c} \in \mathfrak{B}^+$ *satisfies*

$$(u \subseteq u_* \implies ((\mathbf{c} \leq \mathbf{b}_u) \ \vee \ (\mathbf{c} \leq 1 - \mathbf{b}_u))) \wedge \left(\alpha \in u_* \implies \mathbf{c} \leq \mathbf{b}_{\{\alpha\}}\right)$$

then we can find a model $M \models T$ *and* $a_\alpha \in M$ *for* $\alpha \in u_*$ *such that for every* $u \subseteq u_*$,

$$M \models (\exists x) \bigwedge_{\alpha \in u} \varphi_\alpha(x; a_\alpha) \ \textit{iff} \ \mathbf{c} \leq \mathbf{b}_u.$$

If Δ is any set of formulas, we say $\overline{\mathbf{b}}$ is a $(\lambda, \mathfrak{B}, T, \Delta)$-possibility if it is a $(\lambda, \mathfrak{B}, T, \bar{\varphi})$-possibility for some sequence $\bar{\varphi}$ of formulas from Δ.

In some sense, 3.7 says that the "Venn diagram" of the elements of $\bar{\mathbf{b}}$ accurately reflects the intersection patterns of the given sequence of formulas in the monster model. We will often keep track of a full type $p \in \mathbf{S}(N)$, but recall that it suffices to deal with φ-types for each φ because saturation of ultrapowers reduces to saturation of φ-types, [18] Theorem 12.

Definition 3.8. (Moral ultrafilters on Boolean algebras, [22] Definition 6.3) *We say that an ultrafilter $\mathcal{D}_*$ on the Boolean algebra $\mathfrak{B}$ is $(\lambda, \mathfrak{B}, T, \Delta)$-moral when for every $(\lambda, \mathfrak{B}, T, \Delta)$-possibility $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ which is a sequence of elements of $\mathcal{D}_*$, there is a multiplicative $\mathcal{D}_*$-refinement $\overline{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0} \rangle$, i.e.*

(1) $u_1, u_2 \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_{u_1} \cap \mathbf{b}'_{u_2} = \mathbf{b}'_{u_1 \cup u_2}$
(2) $u \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_u \subseteq \mathbf{b}_u$
(3) $u \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_u \in \mathcal{D}_*$.

We write $(\lambda, \mathfrak{B}, T)$-moral in the case where Δ is all formulas of the language.

The following key theorem of [22] connects "morality" of $\mathcal{D}_*$ to goodness of $\mathcal{D}$ in the natural way.

Theorem 3.F ("Separation of variables", Malliaris and Shelah [22] Theorem 5.11)**.** *Suppose that $\mathcal{D}$ is a regular ultrafilter on I built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$. Then the following are equivalent:*

(A) $\mathcal{D}_*$ *is* $(|I|, \mathfrak{B}, T)$*-moral.*
(B) $\mathcal{D}$ *is good for* T*.*

Theorem 3.F helps with analyzing the intermediate classes in Keisler's order, as shown in [22]. It also focuses the regular ultrafilter construction problems essential to Keisler's order on to the problem of constructing ultrafilters $\mathcal{D}_*$ on complete Boolean algebras, where one has a priori much more freedom and is not bound by regularity. Much recent work has focused on building such $\mathcal{D}_*$. In the paper [26], which is foundational for the present argument, we built a powerful family of so-called *optimal* ultrafilters over any suitable tuple of cardinals $(\lambda, \mu, \theta, \sigma)$, along with their simpler avatars the *perfect* ultrafilters. In the present paper, we use $\theta = \sigma = \aleph_0$ so the criterion of "suitable" reduces to requiring that $\lambda > \mu \geq \aleph_0$. Given the transparency of the theories involved, we have written the present proof to use only the more easily quotable definition of "perfect."

Definition 3.9. *Let $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ be a sequence of elements of $\mathfrak{B} = \mathfrak{B}^1_{\alpha,\mu}$. We say X is a support of $\overline{\mathbf{b}}$ in $\mathfrak{B}$ when $X \subseteq \{\mathbf{x}_f : f \in \mathrm{FI}_\mu(\alpha)\}$ and for each $u \in [\lambda]^{<\aleph_0}$ there is a maximal antichain of $\mathfrak{B}$ consisting of elements of X all of which are either $\leq \mathbf{b}_u$ or $\leq 1 - \mathbf{b}_u$. When a support $\mathrm{supp}(\bar{\mathbf{b}})$ is given, write*

$$\mathfrak{B}^+_{\mathrm{supp}(\bar{\mathbf{b}}),\mu} \text{ to mean } \mathfrak{B}^+_{\alpha_*,\mu}$$

where $\alpha_* \leq \alpha$ is minimal such that $\bigcup\{\mathrm{dom}(f) : \mathbf{x}_f \in \mathrm{supp}(\overline{\mathbf{b}})\} \subseteq \alpha_*$.

Definition 3.10 (Perfect ultrafilters, [26] Definition 9.15[5])**.** *We say that an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$ is* (λ, μ)-perfect *when (A) implies (B):*

[5]Corrected in arxiv v2 to say explicitly that any large enough α will work, matching [26]. This had been the intention, so there is no change in the proof.

(A) $\langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ *is a monotonic sequence of elements of* $\mathcal{D}_*$ *and* $\operatorname{supp}(\bar{\mathbf{b}})$ *is a support for* $\bar{\mathbf{b}}$ *of cardinality* $\leq \lambda$*, see* 3.9*, such that for every* $\alpha < 2^\lambda$ *with* $\bigcup\{\operatorname{dom}(f) : \mathbf{x}_f \in \operatorname{supp}(\bar{\mathbf{b}})\} \subseteq \alpha$*, there exists a multiplicative sequence*

$$\langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0} \rangle$$

of elements of $\mathfrak{B}^+$ *such that*

(a) $\mathbf{b}'_u \leq \mathbf{b}_u$ *for all* $u \in [\lambda]^{<\aleph_0}$*,*

(b) for every $\mathbf{c} \in \mathfrak{B}^+_{\alpha,\mu} \cap \mathcal{D}_*$*, no intersection of finitely many members of* $\{\mathbf{b}'_{\{i\}} \cup (1 - \mathbf{b}_{\{i\}}) : i < \lambda\}$ *is disjoint to* $\mathbf{c}$.

(B) there is a multiplicative sequence $\bar{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0} \rangle$ *of elements of* $\mathcal{D}_*$ *which refines* $\bar{\mathbf{b}}$.

Definition 3.11. *Suppose* $\mathcal{D}$ *built from* $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ *where* $\mathcal{D}_0$ *is a regular filter on* I*,* $|I| = \lambda$*,* $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$ *and* $\mathcal{D}_*$ *is* (λ,μ)*-perfect. In this case we say* $\mathcal{D}$ *is* (λ,μ)-perfected.

Theorem 3.G ([26] Theorem 9.18 and Conclusion 9.20)**.** *For any infinite* $\lambda > \mu$*, there exists a regular,* (λ,μ)*-perfect ultrafilter on* $\mathfrak{B}^1_{2^\lambda,\mu}$*. Moreover, there exists a* (λ,μ)*-perfected, thus regular, ultrafilter on* λ *which is not good for any non-simple theory, in fact, not* μ^{++}*-good for any non-simple theory.*

4. The saturation condition

In this section we prove that whenever n, k, λ, μ are such that the property $\Pr_{n,k}(\lambda,\mu)$ from 1.5 above holds, then any (λ,μ) perfect(ed) ultrafilter will be able to handle the theory $T_{n,k}$.

Theorem 4.1. *Suppse we are given* $k < n$*,* $\mu < \lambda$*,* $\mathcal{D}$*, and* T*, where:*

(1) λ, μ, n, k *are such that* $\Pr_{n,k}(\lambda,\mu)$ *holds, from 2.5.*

(2) $T = T_{n,k}$.

(3) $\mathcal{D}$ *is a* (λ,μ)*-perfected ultrafilter on* I*,* $|I| = \lambda$.

Then $\mathcal{D}$ *is good for* T*, i.e. for any* $M \models T$*,* $M^I/\mathcal{D}$ *is* λ^+*-saturated.*

Proof. To begin let us fix several objects.

- The assumption on $\mathcal{D}$ means we may fix $\mathcal{D}_0$, $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$, $\mathbf{j}$ and a (λ,μ)-perfect ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$ such that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ via $\mathbf{j}$.
- The fact that $\mathcal{D}$ is regular means we may choose any $M \models T_{n,k}$ as the index model. For convenience, suppose $|M| > \lambda$.
- Fix a lifting from $M^I/\mathcal{D}$ to M^I, so that for each $a \in M^I/\mathcal{D}$ and each index $t \in I$ the projection $a[t]$ is well defined. If $\bar{c} = \langle c_i : i < m \rangle \in {}^m(M^I/\mathcal{D})$ then we use $\bar{c}[t]$ to denote $\langle c_i[t] : i < m \rangle$.
- Fix a partial type $p = p(x)$ over $A \subseteq M^I/\mathcal{D}$, $|A| \leq \lambda$ which we will try to realize. Without loss of generality, p is nonalgebraic and $M^I/\mathcal{D} \restriction A \prec M^I/\mathcal{D}$. Then, by our choice of theory, it suffices to consider $p \in \mathbf{S}_\Delta(A)$, $|A| \leq \lambda$ where $\Delta = \{R(x, x_1, \ldots, x_k), \neg R(x, x_1, \ldots, x_k)\}$.

With these objects in hand let us proceed with the analysis.

(4.1) Let $\langle a_i : i < \lambda \rangle$ list the elements of A without repetition.

$$\text{Let } \langle v_\beta : \beta < \lambda \rangle \text{ enumerate } [\lambda]^k \text{ without repetition.} \tag{4.2}$$

We will generally use $u, v, w \subseteq \lambda$ for sets of i's and $s \subseteq \lambda$ for sets of β's. When $w \in {}^n\lambda$ is a finite sequence or a finite set (which, for this purpose, we consider as a sequence, in increasing order) let $\bar{a}_w$ mean $\langle a_i : i \in w \rangle$. Then for some function $\mathbf{t} : \lambda \to \{0, 1\}$, (4.2) induces an enumeration of p as

$$p = \langle R(x, \bar{a}_{v_\beta})^{\mathbf{t}(\beta)} : \beta < \lambda \rangle. \tag{4.3}$$

Recall that here $\varphi^0 = \neg\varphi$, $\varphi^1 = \varphi$. For each $s \in \Omega := [\lambda]^{<\aleph_0}$, we will denote the set of indices for vertices appearing in $\{R(x, \bar{a}_{v_\beta})^{\mathbf{t}(\beta)} : \beta \in s\}$ as follows:

$$\mathrm{vert}(s) = \bigcup \{v_\beta : \beta \in s\}. \tag{4.4}$$

In the other direction, let the index operator ind accept a finite set of indices for elements of A and return the relevant indices for formulas in the type:

$$\mathrm{ind}(u) = \{\beta < \lambda : v \in [u]^k \text{ and } v_\beta = v\}. \tag{4.5}$$

As we assumed the list (4.2) was without repetition, $\mathrm{ind} : [\lambda]^{<\aleph_0} \to [\lambda]^{<\aleph_0}$.

Now for each $s \in \Omega$, the Łos map Ł: $\Omega \to \mathcal{D}$ sends $s \mapsto B_s$ where

$$B_s := \{t \in I : M \models (\exists x) \bigwedge_{\beta \in s} R(x, \bar{a}_{v_\beta}[t])^{\mathbf{t}(\beta)}\}. \tag{4.6}$$

Define $\mathbf{b}_s = \mathbf{j}(B_s)$. This gives a possibility pattern for $T = T_{n,k}$ (Definition 3.7):

$$\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in \Omega \rangle. \tag{4.7}$$

With this setup, the strategy for the remainder of the proof will be to construct a sequence $\langle \mathbf{b}'_s : s \in \Omega \rangle$ which, along with $\bar{\mathbf{b}}$, satisfies the hypotheses of Definition 3.10(A). Then 3.10(B) will guarantee that $\bar{\mathbf{b}}$ has a multiplicative refinement in $\mathfrak{B}$ and thus, by separation of variables, that $\mathcal{D}$ is good for T. We will proceed as follows. First, we build an appropriate support for $\bar{\mathbf{b}}$. Second, we use this data to define associated equivalence relations. Third, we define the sequence $\bar{\mathbf{b}}'$. It will be immediate from the definition that this sequence is multiplicative and refines $\bar{\mathbf{b}}$ on singletons. Fourth, we show that the sequence $\bar{\mathbf{b}}'$ is not trivial, i.e. it satisfies 3.10(A)(b). Finally, we show that $\bar{\mathbf{b}}'$ is a refinement of $\bar{\mathbf{b}}$, and thus satisfies 3.10(A)(a).

Our first task is to choose an appropriate support for $\bar{\mathbf{b}}$ in the sense of 3.9. Following an idea from [23], whenever $i, j \in \lambda$ let

$$A_{a_i = a_j} := \{t \in I : a_i[t] = a_j[t]\} \text{ and let } \mathbf{a}_{a_i = a_j} := \mathbf{j}(A_{a_i = a_j}). \tag{4.8}$$

For each $i < \lambda$ let $\mathcal{F}_{\{i\}}$ be the set of all $f \in \mathrm{FI}_\mu(2^\lambda)$ such that for some $j \leq i$, both (4.9) and (4.10) hold:

$$\mathbf{x}_f \leq \mathbf{a}_{a_i = a_j}. \tag{4.9}$$

$$\text{for all } k < j, \;\; \mathbf{x}_f \cap \mathbf{a}_{a_i = a_k} = 0. \tag{4.10}$$

For each finite $u \subseteq \lambda$, define $\mathcal{F}_u$ to be $\bigcap \{\mathcal{F}_{\{i\}} : i \in u\}$. Note that each $\mathcal{F}_u$ is upward closed, i.e. $f \in \mathcal{F}_u$ and $g \supseteq f$ implies $g \in \mathcal{F}_u$. For each $s \in \Omega$, the benefit of working with elements of $\mathcal{F}_{\mathrm{vert}(s)}$ will be that we may consider the partial function $i \mapsto \rho_i(f)$ on $\mathrm{FI}_\mu(2^\lambda)$ where

$$\rho_i(f) = \min\{j \leq i : \mathbf{x}_f \leq \mathbf{a}_{a_i = a_j}\}. \tag{4.11}$$

The key point is that if $f \in \mathcal{F}_{\mathrm{vert}(s)}$ and $i \in \mathrm{vert}(s)$ and $\rho_i(f) = j$, then for no $f' \supseteq f$ does there exist $j' < j$ such that $\mathbf{x}_{f'} \leq \mathbf{a}_{a_i = a_{j'}}$. We will refer to this property by saying that for each $i \in s$, ρ_i is "s-accurate."

As a result, for each choice of $s \in \Omega$ and $f \in \mathcal{F}_{\mathrm{vert}(s)}$ we may naturally collect all the "active" indices by mapping

$$(s, f) \mapsto w^*_{s,f} := \{j \leq i : \text{ for some } i \in \mathrm{vert}(s),\ \rho_i(f) = j\} \cup \mathrm{vert}(s). \tag{4.12}$$

Recalling the one-step closure operator cl_1 from 2.4 above, it will also be useful to keep track of the slightly larger, but still finite, set:

$$(s, f) \mapsto w_{s,f} := \{j \leq i : \text{ for some } i \in \mathrm{cl}_1(w^*_{s,\zeta}),\ \rho_i(f) = j\} \cup \mathrm{cl}_1(w^*_{s,\zeta}). \tag{4.13}$$

Naming both $w^*_{s,f}$ and $w_{s,f}$ sets the stage for the application of G from 2.5 towards the end of the proof. The map (4.13) is really like a finite closure operator: for each $s \in \Omega$ and $f \in \mathcal{F}_{\mathrm{vert}(s)}$, we have that $\mathrm{vert}(s) \subseteq w_{s,\zeta} \in [\lambda]^{<\aleph_0}$, $f \in \mathcal{F}_{w_{s,f}}$, and $w_{\mathrm{ind}(w_{s,f}),f} = w_{s,f}$. Moreover, if $f \in \mathcal{F}_{\mathrm{vert}(s)}$ then $f \in \mathcal{F}_{w_{s,f}}$. Notice also that

$$\text{for any } s \in \Omega \text{ and any } \mathbf{c} \in \mathfrak{B}^+, \text{ there is } f \in \mathcal{F}_{\mathrm{vert}(s)} \text{ with } \mathbf{x}_f \leq \mathbf{c}. \tag{4.14}$$

Why? Recall that for $\mathbf{a}, \mathbf{c} \in \mathfrak{B}^+$, we say that $\mathbf{c}$ supports $\mathbf{a}$ when either $\mathbf{c} \leq \mathbf{a}$ or $\mathbf{c} \leq 1 - \mathbf{a}$. Without loss of generality, $\mathbf{c}$ supports $\mathbf{b}_s$. Since $\mathrm{vert}(s)$ is finite, it will suffice to prove that for a given $i \in \mathrm{vert}(s)$ we can find f such that $\mathbf{x}_f \leq \mathbf{c}$ and $f \in \mathcal{F}_{\{i\}}$. As the generators are dense in the completion, there is $f \in \mathrm{FI}_\mu(2^\lambda)$ with $\mathbf{x}_f \leq \mathbf{c}$, and (4.9) trivially holds of f in the case $j = i$. If (4.10) does not hold in the case $j = i$, there are $i_1 < i$ and $f_1 \supseteq f$ such that (4.9) holds of f_1 in the case $j = i_1$. Since the ordinals are well ordered, after iterating this for finitely many steps we find $j = i_k$ and $f_k \supseteq \cdots \supseteq f_1 \supseteq f$ for which (4.10) also holds. This proves (4.14).

We need one more ingredient to construct the support: the partitions should decide not only equality but also the formulas R on elements from $w_{s,\zeta}$. Towards this, for each $u \in [\lambda]^{k+1}$, write

$$\mathbf{a}_{R(\bar{a}_u)} = \mathbf{j}(\ \{t \in I : M \models R(\bar{a}_u)\}). \tag{4.15}$$

We may also say that $1 - \mathbf{a}_{R(x,\bar{a}_v)} = \mathbf{a}_{\neg R(x,\bar{a}_v)}$ and $1 - \mathbf{a}_{R(\bar{a}_u)} = \mathbf{a}_{\neg R(\bar{a}_u)}$, naturally defined. We may now state a definition. There is a component of support and a component of coherence across all $s \in \Omega$.

(4.16) $\quad \bar{f} = \langle\ \bar{f}_s = \langle (f_{s,\zeta}, w_{s,\zeta}) : \zeta < \mu \rangle\ : s \in \Omega \rangle$ is a *good support* for $\bar{\mathbf{b}}$ when:

(1) for each $s \in \Omega$,
   (a) for each $\zeta < \mu$, $f = f_{s,\zeta} \in \mathcal{F}_{\mathrm{vert}(s)}$.
   (b) for each $\zeta < \mu$, $w_{s,\zeta} = w_{s,f_{s,\zeta}}$, which is well defined by (a) and (4.13).
   (c) the sequence $\langle \mathbf{x}_{f_{s,\zeta}} : \zeta < \mu \rangle$ is a maximal antichain of $\mathfrak{B}$ supporting each element of the set[6]

$$\{\mathbf{b}_{s'} : s' \subseteq s\} \cup \{\mathbf{a}_{R(\bar{a}_u)} : u \in [w_{s,\zeta}]^{k+1}\}.$$

(2) for each $s, s' \in \Omega$ with $s' \subseteq s$, $\bar{f}_s$ refines $\bar{f}_{s'}$.
(3) for every finite $X \subseteq \bigcup\{\mathrm{dom}(f_{s,\zeta}) : s \in \Omega, \zeta < \mu\}$ and every $s \in \Omega$, there is $s_* \in \Omega$ such that $s \subseteq s_*$ and $\zeta < \mu \implies X \subseteq \mathrm{dom}(f_{s_*,\zeta})$.

[6]By condition (1)(a), $\{\mathbf{a}_{a_i = a_j} : i, j \in w_{s,\zeta}\}$ are implicitly also here.

One way of building a good partition is to miniaturize the argument from [26], as follows. First, we address (1)(a)+(c). For each $s \in \Omega$, we try to choose $f_{s,\zeta}$ by induction on $\zeta < \mu^+$ such that $0 \in \operatorname{dom}(f_{s,\zeta})$. Arriving to ζ, suppose we have some remaining unallocated $\mathbf{c} \in \mathfrak{B}^+$, i.e. a nonzero $\mathbf{c}$ disjoint to $\bigcup\{\mathbf{x}_{f_{s,\gamma}} : \gamma < \zeta\}$. Without loss of generality, $\mathbf{c}$ supports $\mathbf{b}_s$. By (4.14), we may choose $f \in \mathcal{F}_{\operatorname{vert}(s)}$ so that $\mathbf{x}_f \leq \mathbf{c}$. Condition (1)(c) asks that $\mathbf{x}_f$ also support each element of a finite set, so without loss of generality (by taking intersections) we may assume (c) is satisfied. This completes the inductive step. As no antichain of $\mathfrak{B}$ has cardinality greater than μ, the construction will stop at an ordinal $< \mu^+$, but as $0 \in \operatorname{dom}(f_{s,\zeta})$ for each ζ the ordinal is $\geq \mu$. Without loss of generality the sequence is indexed by μ. Then (1)(b) holds by (4.13).

To ensure conditions (2) and (3), we refine the partitions just obtained. Let $\langle s_\ell : \ell < \lambda \rangle$ list Ω. We update $\overline{f}_{s_\ell} = \langle f_{s_\ell,\zeta} : \zeta < \mu \rangle$ by induction on $\ell < \mu$ as follows. Arriving to ℓ, if $(\exists k < \ell)(s_\ell \subseteq s_k)$ then let $k(\ell) = \min\{k < \ell : s_\ell \subseteq s_k\}$ and let $\overline{f}_{s_\ell} = \overline{f}_{s_k}$. If there is no such j, we choose $\overline{f}_{s_\ell}$ such that it refines $\overline{f}_{s_k}$ (i.e. every $f_{s_\ell,\zeta}$ extends $f_{s_k,\zeta}$ for some $\zeta < \mu$) whenever $k < \ell$ and $s_k \subseteq s_\ell$. There are at most $2^{|s_\ell|} < \aleph_0$ such j so this can be done.

At the end of this process, if necessary, we may re-index the partitions so that they are of order type μ. By construction, for each $s \in \Omega$ and $\zeta < \mu$ the set $w_{s,\zeta}$ is well defined by (1)(b). This completes the construction of a good support for $\bar{\mathbf{b}}$.

(4.17) For the remainder of the proof, we fix a good support $\bar{f}$ for $\bar{\mathbf{b}}$.

(4.18) Fix $\mathcal{V} \subseteq 2^\lambda$, $|\mathcal{V}| \leq \lambda$ such that $\bigcup\{\operatorname{dom}(f_{u,\zeta}) : u \in \Omega, \zeta < \mu\} \subseteq \mathcal{V}$.

Finally, for each $s \in \Omega$ and each $\zeta < \mu$, define[7]

(4.19) $\mathcal{G}_{s,\zeta}$ to be the set of functions $g = g_{s,\zeta} : \operatorname{ind}(w_{s,\zeta}) \to \{0,1\}$ such that:

(a) if $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{b}_s$, then for all $\beta \in s$, $g_{s,\zeta}(\beta) = \mathbf{t}(\beta)$.
(b) if $\gamma \in \operatorname{ind}(w_{s,\zeta})$, $i \neq j \in v_\gamma$ and $\rho_i(f_{s,\zeta}) = \rho_j(f_{s,\zeta})$, then $g_{s,\zeta}(\gamma) = 0$.
(c) if $\beta \neq \gamma \in \operatorname{ind}(w_{s,\zeta})$ and $\{\rho_i(f_{s,\zeta}) : i \in v_\gamma\} = \{\rho_i(f_{s,\zeta}) : i \in v_\beta\}$ then $g_{s,\zeta}(\gamma) = g_{s,\zeta}(\beta)$.
(d) if $w \in [w_{s,\zeta}]^n$ and $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{a}_{R(\bar{a}_u)}$ for each $u \in [w]^{k+1}$, then $g_{s,\zeta} \restriction \operatorname{ind}(w)$ is not constantly 1.

Regarding condition (a), recall that by construction in (4.16)(1)(c) $\mathbf{x}_{f_{s,\zeta}}$ decides $\mathbf{b}_{s'}$ for all $s' \subseteq s$ and it likewise decides $\mathbf{a}_{R(\bar{a}_u)}$ for each $u \in [w_{s,\zeta}]^{k+1}$. To see that $\mathcal{G}_{s,\zeta} \neq \emptyset$ simply involves unwinding the definition. There are two cases. If $\mathbf{x}_{f_{s,\zeta}} \leq 1 - \mathbf{b}_s$ then $\mathcal{G}_{s,\zeta}$ contains the function which is constantly 0. If $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{b}_s$, then recalling (4.15) we have that if there is $u \in [w_{s,\zeta}]^{k+1}$ such that each $v \in [u]^k$ is v_β for some $\beta \in s$, then $\mathbf{x}_{f_{s,\zeta}} \leq 1 - \mathbf{a}_{R(\bar{a}_u)}$. Thus, we may set $g(\gamma) = 1$ if and only if $\gamma \in \operatorname{ind}(w^*_{s,\zeta})$ and $\{\rho_i(f_{s,\zeta}) : i \in v_\gamma\} = \{\rho_i(f_{s,\zeta}) : i \in v_\beta\}$ for some $\beta \in s$. (Call such a g "minimal," since the only edges it has are those required by (a) and

[7]Informally, elements of $\mathcal{G}_{s,\zeta}$ specify consistent R-types over the parameters with indices in $\operatorname{ind}(w_{s,\zeta})$. Edges only hold on distinct tuples since R is irreflexive. Given two tuples which "collapse" to the same values, either both or neither have an edge. The type extends $p \restriction s$ if possible, that is, if the Łos map allows it. In the case where $\mathbf{j}$ is the identity so the elements $\mathbf{x}_{f_{s,\zeta}}$ are subsets of I, the reader may think of $g^*_{s,\zeta}$ as coding an R-type over $\{a_i[t] : i \in w_{s,\zeta}\}$ which is consistent for any $t \in w_{s,\zeta}$. We will essentially arrive at this picture towards the end of the proof; we will find a set C such that (among other things) $\mathbf{j}(C) \subseteq \mathbf{x}_{f_{s,\zeta}}$, choose $t \in C$ and consider the type given by $g^*_{s,\zeta}$ at t.

then extended to colliding indices by (c).) Note that since each ρ_i is s-accurate, it is sufficient to give the behavior of g on the set $\{\rho_i(f_{s,\zeta}) : i \in w_{s,\zeta}\}$, as the condition of s-accurate and the definition (4.15) ensure that if $u, u' \in [w_{s,\zeta}]^{k+1}$ and $\{\rho_i(f_{s,\zeta}) : i \in u\} = \{\rho_i(f_{s,\zeta}) : i \in u'\}$ then $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{a}_{R(\bar{a}_u)}$ if and only if $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{a}_{R(\bar{a}_{u'})}$. So indeed $\mathcal{G}_{s,\zeta} \neq \emptyset$. For the remainder of the proof,

$$\text{for each } s \in \Omega \text{ and } \zeta < \mu, \text{ fix } g_{s,\zeta} \in \mathcal{G}_{s,\zeta} \text{ (w.l.o.g. minimal).} \tag{4.20}$$

$$\text{for each } s \in \Omega \text{ and } \zeta < \mu, \text{ let } g^*_{s,\zeta} \text{ be the restriction of } g_{s,\zeta} \text{ to } \mathrm{ind}(w^*_{s,\zeta}). \tag{4.21}$$

We will informally refer to these objects $g_{s,\zeta}$ as "floating types."

Our second task is to organize the data already obtained in terms of a family of equivalence relations. This will elide some of the background noise and so give us a cleaner picture of any barriers to realizing the type. By hypothesis (1) of the Theorem, $\mathrm{Pr}_{n,k}(\lambda, \mu)$ holds. Thus, identifying λ with the set of indices for elements of A as in (4.1), let us fix $G : [\lambda]^{<\aleph_0} \times [\lambda]^{<\aleph_0} \to \mu$ such that:

> for each $w \in [\lambda]^n$ and each sequence $\langle u_v : v \in [w]^k \rangle$ of finite subsets of λ such that $v \in [w]^k$ implies $v \subseteq u_v$ and $G \restriction \{(u_v, \mathrm{cl}_1(u_v)) : v \in [w]^k\}$ is constant, there is $v \in [w]^k$ such that $w \subseteq \mathrm{cl}_1(u_v)$.

Let E be the equivalence relation on $W = \Omega \times \mu \times \mu$ given by:

$$E((s,\zeta,\xi),(s',\zeta',\xi')) \text{ if and only if} \tag{4.22}$$

(a) $\zeta = \zeta'$ and $\xi = \xi'$.
(b) $\mathrm{otp}(s) = \mathrm{otp}(s')$, $\mathrm{otp}(\mathrm{ind}(w_{s,\zeta})) = \mathrm{otp}(\mathrm{ind}(w_{s',\zeta}))$ and the order preserving map from $\mathrm{ind}(w_{s,\zeta})$ onto $\mathrm{ind}(w_{s',\zeta})$ takes s to s' and $\mathrm{ind}(w^*_{s,\zeta})$ onto $\mathrm{ind}(w^*_{s',\zeta})$.
(c) $\mathrm{otp}(\mathrm{vert}(s)) = \mathrm{otp}(\mathrm{vert}(s'))$, $\mathrm{otp}(w_{s,\zeta}) = \mathrm{otp}(w_{s',\zeta})$, $\mathrm{otp}(w^*_{s,\zeta}) = \mathrm{otp}(w^*_{s',\zeta})$ and the order preserving map from $w_{s,\zeta}$ onto $w_{s',\zeta}$ takes $w^*_{s,\zeta}$ to $w^*_{s',\zeta}$, and $\mathrm{vert}(s)$ to $\mathrm{vert}(s')$.
(d) $\mathrm{otp}(\mathrm{dom}(f_{s,\zeta})) = \mathrm{otp}(\mathrm{dom}(f_{s',\zeta}))$.
(e) if $\gamma_s \in \mathrm{dom}(f_{s,\zeta})$, $\gamma_{s'} \in \mathrm{dom}(f_{s',\zeta})$ and $\mathrm{otp}(\gamma_s \cap \mathrm{dom}(f_{s,\zeta})) = \mathrm{otp}(\gamma_{s'} \cap \mathrm{dom}(f_{s',\zeta}))$ then $f_{s,\zeta}(\gamma_s) = f_{s',\zeta}(\gamma_{s'})$.
(f) $\langle g_{s,\zeta}(\beta) : \beta \in \mathrm{ind}(w_{s,\zeta})\rangle = \langle g_{s,\zeta}(\beta) : \beta \in \mathrm{ind}(w_{s,\zeta})\rangle$.
(g) $G(w^*_{s,\zeta}, \mathrm{cl}_1(w^*_{s,\zeta})) = G(w^*_{s',\zeta}, \mathrm{cl}_1(w^*_{s,\zeta})) = \xi$. Note that $\mathrm{cl}_1(w^*_{s,\zeta}) \subseteq w_{s,\zeta}$.

Since the sets and ordinals in question are all finite, but $\zeta < \mu$ may vary, it is easy to see that there are precisely μ equivalence classes of E. Choose an enumeration of these classes as

$$\bar{W} = \langle W_\epsilon : \epsilon < \mu \rangle, \text{ so } W = \textstyle\bigcup_\epsilon W_\epsilon. \tag{4.23}$$

Fix a representative function

$$h : \mu \to W \text{ such that } h(\epsilon) \in W_\epsilon. \tag{4.24}$$

In the rest of the proof, we will often denote the values of ζ, ξ at $h(\epsilon)$ by $\zeta_{h(\epsilon)}, \xi_{h(\epsilon)}$ respectively. The next definition will be central. For each $\beta < \lambda$, $\epsilon < \mu$ let us collect all elements of Ω which occur as part of an ϵ-template tuple (s, ζ, ξ) where $\beta \in s$ and $\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{b}_s$:

$$\mathcal{U}_{\beta,\epsilon} = \{s \colon (s, \zeta_{h(\epsilon)}, \xi_{h(\epsilon)}) \in W_\epsilon,\ \beta \in s,\ \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} \leq \mathbf{b}_s\}. \tag{4.25}$$

A useful property of these sets is the following: for each $\epsilon < \mu$,

$$\text{if } s \in \mathcal{U}_{\beta,\epsilon} \text{ and } s' \in \mathcal{U}_{\beta',\epsilon} \text{ then } G(w_{s,\zeta_{h(\epsilon)}}) = G(w_{s',\zeta_{h(\epsilon)}}) = \xi_{h(\epsilon)}. \tag{4.26}$$

This completes our construction of the equivalence relations. We now have the necessary scaffolding for the third task.

Our third task is to define the sequence $\bar{\mathbf{b}}'$. Recalling $\mathcal{V}$ from (4.18), fix $\alpha < 2^\lambda$ so that $\mathcal{V} \subseteq \alpha$. Without loss of generality, $\alpha \geq \lambda$. We now copy the functions $f_{s,\zeta}$ onto a new domain where new partitions will allow us to code additional information.[8] Let Code_m denote some fixed one-to-one m-fold coding function from λ^m to λ. Let tv denote the truth value of an expression (either 0 or 1).

$$\text{For each } s \in \Omega,\ \zeta < \mu \text{ define } f^* = f^*_{s,\zeta} \text{ as follows.} \tag{4.27}$$

(1) $\mathrm{dom}(f^*) \subseteq \alpha \cdot 2 + \lambda \cdot 5$ is finite, $\mathrm{range}(f^*) \subseteq \mu$, and f^* is determined by the remaining conditions.

(2) if $\gamma \in \mathrm{dom}(f_{s,\zeta})$ then
$$f^*(\alpha + \gamma) = \mathrm{Code}_2(f_{s,\zeta}(\gamma), \mathrm{otp}(\gamma \cap \mathrm{dom}(f_{s,\zeta}))).$$

(3) if $\gamma = \langle i, j \rangle \in \mathrm{range}(\mathrm{Code}_2(w_{s,\zeta} \times w_{s,\zeta}))$, then
$$f^*(\alpha \cdot 2 + \gamma) = \mathrm{tv}(\rho_i(f_{s,\zeta}) = \rho_j(f_{s,\zeta})).$$

(4) if $\gamma = \langle i_1, \dots, i_k \rangle \in \mathrm{range}(\mathrm{Code}_k(w_{s,\zeta} \times \cdots \times w_{s,\zeta}))$, then
$$f^*(\alpha \cdot 2 + \lambda + \gamma) = \mathrm{tv}(\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{b}_s).$$

(5) if $\gamma = \langle i_0, \dots, i_k \rangle \in \mathrm{range}(\mathrm{Code}_{k+1}(w_{s,\zeta} \times \cdots \times w_{s,\zeta}))$, then
$$f^*(\alpha \cdot 2 + \lambda \cdot 2 + \gamma) = \mathrm{tv}(\mathbf{x}_{f_{s,\zeta}} \leq \mathbf{a}_{R(\bar{a}_{\langle i_0, \dots, i_k \rangle})}).$$

(6) if $\gamma \in w_{s,\zeta}$, then
$$f^*(\alpha \cdot 2 + \lambda \cdot 3 + \gamma) = \mathrm{Code}_3(\mathrm{tv}(\gamma \in \mathrm{vert}(s)), \mathrm{otp}(\gamma \cap \mathrm{vert}(s)), \mathrm{otp}(\gamma \cap w_{s,\zeta})).$$

(7) if $\gamma \in \mathrm{ind}(w_{s,\zeta})$, then
$$f^*(\alpha \cdot 2 + \lambda \cdot 4 + \gamma) = \mathrm{Code}_4(\mathrm{tv}(\gamma \in s), \mathrm{otp}(\gamma \cap s), \mathrm{otp}(\gamma \cap \mathrm{ind}(w_{s,\zeta})), g_{s,\zeta}(\gamma)).$$

This completes the definition (4.27). Of course, this definition could be made more efficient and the domain smaller (say, by more judicious use of Code_m). Finally,

$$\text{let } \bar{\mathbf{c}} = \langle \mathbf{c}_\epsilon : \epsilon < \mu \rangle \text{ be given by } \mathbf{c}_\epsilon = \mathbf{x}_{\{(\alpha+\alpha+\lambda\cdot 5, \epsilon)\}}. \tag{4.28}$$

This new antichain will help us to divide the work in the next definition. Notice that any of its elements will have nonzero intersection with any of the elements from $\mathfrak{B}^+_{\alpha+\alpha+\lambda\cdot 5}$.

We have all the ingredients to define $\bar{\mathbf{b}}'$. For each $\beta < \lambda$, let

$$\mathbf{b}'_{\{\beta\}} = \Big(\bigcup \{ \mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} \ :\ \epsilon < \mu,\ s \in \mathcal{U}_{\beta,\epsilon} \} \Big) \cap \mathbf{b}_{\{\beta\}}. \tag{4.29}$$

Let us justify that (4.29) is not zero: for each $\epsilon < \mu$ such that $\mathcal{U}_{\beta,\epsilon} \neq \emptyset$, and for each $s \in \mathcal{U}_{\beta,\epsilon}$,

$$\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\{\beta\}} \ >\ 0.$$

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[8] Compare the usual construction of good ultrafilters.

This is because domains of the functions corresponding to $\mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}}$, $\mathbf{c}_\epsilon$ and $\mathbf{x}_{f^*_{s,\zeta_{h(\epsilon)}}}$ are mutually disjoint, and adding $\mathbf{b}_{\{\beta\}}$ is allowed by the definition of $\mathcal{U}_{\beta,\epsilon}$. (Recall that by monotonicity, $\beta \in s$ implies $\mathbf{b}_s \leq \mathbf{b}_{\{\beta\}}$.) For each $s \in \Omega \setminus \emptyset$, define

$$\mathbf{b}'_s = \bigcap \{\mathbf{b}'_{\{\beta\}} : \beta \in s\}. \tag{4.30}$$

Let $\mathbf{b}'_\emptyset = 1_{\mathfrak{B}}$. This completes the definition of the sequence $\bar{\mathbf{b}}'$:

$$\bar{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in \Omega \rangle. \tag{4.31}$$

By construction, $\bar{\mathbf{b}}'$ is multiplicative, and $\mathbf{b}'_s \leq \mathbf{b}_s$ when $|s| = 1$.

Our fourth task is to prove that the sequence $\bar{\mathbf{b}}'$ defined in (4.31) satisfies Definition 3.10(A)(b) along with $\bar{\mathbf{b}}$ and the choice of support $\bar{f}$ determined earlier in the proof (i.e. α_* of Definition 3.9 may be taken to be the α of the present proof). Compare this to the Step 8 Claim of [26] 6.2.

As the generators are dense in the completion, it will suffice to show that for any $f \in \mathrm{FI}_\mu(\alpha)$, any finite $\mathcal{I} \subseteq \lambda$, and any $\mathbf{a} \in \mathcal{D}_*$ such that $\mathrm{supp}(\mathbf{a}) \subseteq \alpha$,

$$\mathbf{a} \cap \bigcap \{\mathbf{b}'_{\{\beta\}} \cup (1 - \mathbf{b}_{\{\beta\}}) : \beta \in \mathcal{I}\} > 0. \tag{4.32}$$

Taking intersections if necessary, we may write $\mathcal{I}$ as the disjoint union of $\mathcal{I}_0$ and $\mathcal{I}_1$ where for each $\beta \in \mathcal{I}_0$, $\mathbf{a} \leq 1 - \mathbf{b}_{\{\beta\}}$ and for each $\beta \in \mathcal{I}_1$, $\mathbf{a} \leq \mathbf{b}_{\{\beta\}}$. Recalling that $\mathbf{b}'_s \leq \mathbf{b}_s$ when $|s| = 1$, we suppose that $\mathcal{I}_1$ is nonempty (otherwise we are done) and it will suffice to show that

$$\mathbf{a} \cap \bigcap \{\mathbf{b}'_{\{\beta\}} : \beta \in \mathcal{I}_1\} > 0. \tag{4.33}$$

As $\mathbf{b}_{\mathcal{I}_1} \in \mathcal{D}_*$, without loss of generality $\mathbf{a} \leq \mathbf{b}_{\mathcal{I}_1}$ and we can find $f \in \mathrm{FI}_\mu(\alpha)$ such that $\mathbf{x}_f \leq \mathbf{a}$. Recall $\mathcal{V}$ from (4.18). Write f as the disjoint union $f^{\mathrm{in}} \cup f^{\mathrm{out}}$ where $\mathrm{dom}(f^{\mathrm{in}}) \subseteq \mathcal{V}$ and $\mathrm{dom}(f^{\mathrm{out}}) \subseteq \alpha \setminus \mathcal{V}$. Necessarily $\mathbf{b}_{\mathcal{I}_1} \cap \mathbf{x}_{f^{\mathrm{in}}} > 0$. As $\bar{f}_{\mathcal{I}_1}$ gives rise to a partition, let $\zeta_* < \mu$ be such that

$$\mathbf{x}_{f_{\mathcal{I}_1,\zeta_*}} \cap \mathbf{x}_{f^{\mathrm{in}}} \cap \mathbf{b}_{\mathcal{I}_1} > 0. \tag{4.34}$$

Recall the function G which was given as a witness to Pr. Let $\xi_* = G(w_{\mathcal{I}_1,\zeta_*})$ and let $\epsilon < \mu$ be such that $(\mathcal{I}_1, \zeta_*, \xi_*) = (\mathcal{I}_1, \zeta_{h(\epsilon)}, \xi_{h(\epsilon)}) \in W_\epsilon$. Going forward, we will write $\zeta_{h(\epsilon)}$ instead of ζ_* for clarity. As we have $\mathbf{x}_f \leq \mathbf{b}_{\mathcal{I}_1}$, it follows from the definition (4.25) that

$$\mathcal{I}_1 \in \mathcal{U}_{\beta,\epsilon} \text{ for each } \beta \in \mathcal{I}_1. \tag{4.35}$$

Now let us verify that

$$0 < \mathbf{x}_{f^{\mathrm{out}}} \cap \mathbf{x}_{f^{\mathrm{in}}} \cap \left(\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{\mathcal{I}_1,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{\mathcal{I}_1,\zeta_{h(\epsilon)}}} \right) \cap \mathbf{b}_{\mathcal{I}_1}. \tag{4.36}$$

The reason is that conflicts can only arise when the domains of the relevant functions intersect. By construction,

$$\mathbf{c}_\epsilon, \ \mathbf{x}_{f^{\mathrm{out}}}, \ \mathbf{x}_{f^*_{\mathcal{I}_1,\zeta_{h(\epsilon)}}}, \ \mathbf{x}_{f_{\mathcal{I}_1,\zeta_{h(\epsilon)}}}$$

do not interfere with each other and the first three do not interfere with $\mathbf{x}_{f^{\mathrm{in}}}$ or with $\mathbf{b}_{\mathcal{I}_1}$. By (4.34) $\mathbf{x}_{f_{\mathcal{I}_1,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f^{\mathrm{in}}} \cap \mathbf{b}_{\mathcal{I}_1}$ is nonzero. Replacing $\mathbf{x}_f = \mathbf{x}_{f^{\mathrm{in}}} \cap \mathbf{x}_{f^{\mathrm{out}}}$ and quoting the definition of $\mathbf{b}'_{\mathcal{I}_1}$ in (4.29) and (4.30), we are done. This completes the proof of (4.32).

To complete the proof of Theorem 4.1, it remains to show that for each $s \in \Omega$, $\mathbf{b}'_s \leq \mathbf{b}_s$. This will suffice for 3.10(A)(a). The background template for our argument is [26] Claim 6.2, Step 10, item (5). Before beginning this proof, note that by our definition of the sequence $\bar{\mathbf{b}}'$, whenever $0 < \mathbf{c} \leq \mathbf{c}_\epsilon \cap \mathbf{b}'_{\{\beta\}}$, necessarily

$$\bigcup \{\mathbf{c} \cap \mathbf{x}_{f^*_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} : s \in \mathcal{U}_{\beta,\epsilon}\} > 0. \tag{4.37}$$

In particular, under this hypothesis, there is $s \in \mathcal{U}_{\beta,\epsilon}$ such that

$$\mathbf{c} \cap \mathbf{x}_{f^*_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} > 0 \quad \text{thus} \quad \mathbf{c} \cap \mathbf{x}_{f_{s,\zeta_{h(\epsilon)}}} \cap \mathbf{b}'_{\{\beta\}} > 0. \tag{4.38}$$

Now suppose for a contradiction that $\bar{\mathbf{b}}'$ is not a multiplicative refinement of $\bar{\mathbf{b}}$. Then for some finite $\mathcal{I} \subseteq \lambda$ and some $\mathbf{c}_0 \in \mathfrak{B}^+$,

$$\mathbf{c}_0 \;\leq\; \mathbf{b}'_{\mathcal{I}} \setminus \mathbf{b}_{\mathcal{I}} \;=\; \bigcap_{\beta \in \mathcal{I}} \mathbf{b}'_\beta \setminus \mathbf{b}_{\mathcal{I}}. \tag{4.39}$$

Without loss of generality, $\mathbf{c}_0 \leq \mathbf{c}_\epsilon$ for some $\epsilon < \mu$ and $\mathbf{c}_0 = \mathbf{x}_f$ for some $f \in \mathrm{FI}_\mu(2^\lambda)$. Enumerate $\mathcal{I}$ as $\langle \beta_i : i < |\mathcal{I}| \rangle$. Working in $\mathfrak{B}$, by induction on $i < |\mathcal{I}|$

(4.40) we choose functions f_i and sets s_{β_i} such that:

(i) $f_i \in \mathrm{FI}_\mu(2^\lambda)$
(ii) $j < i$ implies $f_j \subseteq f_i$
(iii) $s_{\beta_i} \in \mathcal{U}_{\beta_i,\epsilon}$
(iv) $f_i \supseteq f_{s_{\beta_i},\zeta_{h(\epsilon)}} \cup f^*_{s_\beta,\zeta_{h(\epsilon)}}$.

Let $f_{-1} = f$. Suppose we have defined f_j for $-1 \leq j < j+1 = i$, and we define f_i and s_{β_i} as follows. By hypothesis,

$$\mathbf{x}_{f_j} \leq \mathbf{b}'_{\mathcal{I}} \cap \mathbf{c}_\epsilon. \tag{4.41}$$

First note that by (4.41) and monotonicity of $\bar{\mathbf{b}}'$,

$$\mathbf{x}_{f_j} \leq \mathbf{b}'_{\{\beta_i\}} \cap \mathbf{c}_\epsilon. \tag{4.42}$$

Second, by (4.39), $\mathbf{c}_0 \leq \mathbf{c}_\epsilon \cap \mathbf{b}'_{\{\beta_i\}}$. Thus by (4.37), $\mathcal{U}_{\beta_i,\epsilon} \neq \emptyset$. Apply (4.38) to choose $s_{\beta_i} \in \mathcal{U}_{\beta_i,\epsilon}$ such that

$$\mathbf{x}_{f_j} \cap \mathbf{x}_{f_{s_{\beta_i},\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f^*_{s_{\beta_i},\zeta_{h(\epsilon)}}} > 0.$$

Combining this equation with (4.42),

$$\mathbf{x}_{f_j} \cap \mathbf{c}_\epsilon \cap \mathbf{b}'_{\{\beta_i\}} \cap \mathbf{x}_{f_{s_{\beta_i},\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f^*_{s_{\beta_i},\zeta_{h(\epsilon)}}} > 0.$$

Let $f_i = f_j \cup f_{s_{\beta_i},\zeta_{h(\epsilon)}} \cup f^*_{s_{\beta_i},\zeta_{h(\epsilon)}}$. This completes the induction. For future reference, let's fix two objects from this construction:

$$\text{Let } f_* := \bigcup_{i<|\mathcal{I}|} f_i. \tag{4.43}$$

$$\text{Let } \langle s_{\beta_i} : i < |\mathcal{I}| \rangle = \langle s_\beta : \beta < \beta_* \rangle \text{ be as just inductively defined.} \tag{4.44}$$

Note that by construction,

$$\text{for each } \beta \in \mathcal{I},\ \mathbf{x}_{f_*} \leq \mathbf{x}_{f_{s_\beta,\zeta_{h(\epsilon)}}}. \tag{4.45}$$

Consider the set of indices for 'active' elements:

$$W = \bigcup \{w_{s_\beta,\zeta_{h(\epsilon)}} : \beta \in \mathcal{I}\}. \tag{4.46}$$

$$W \supseteq W^* = \bigcup\{w^*_{s_\beta,\zeta_{h(\epsilon)}} : \beta \in \mathcal{I}\}. \tag{4.47}$$

To finish the argument, we will move back to the index model. Informally, the point will be that $\mathbf{x}_{f_*}$ holds open a 'space' in the Boolean algebra which reflects a particular configuration at some index $t \in I$ (a configuration which we will show cannot happen). First, we shall be careful to choose an appropriate t, as follows. Since the theory $T_{n,k}$ is $\aleph_0$-categorical, let $\Gamma = \Gamma(W)$ be the finite set of formulas in the variables $\{x_i : i \in W\}$. For $v \subseteq W$, let $\varphi(\bar{x}_v)$ denote that the free variables of φ are among $\langle x_i : i \in v\rangle$, and as above let $\bar{a}_v$ denote $\langle a_i : i \in v\rangle$. For each $\varphi = \varphi(\bar{x}_v) \in \Gamma$, the Łos map gives

$$C_{\varphi(\bar{a}_v)} := \{t \in I : M \models \varphi(\bar{a}_v[t])\} \text{ and let } \mathbf{c}_{\varphi(\bar{a}_v)} = \mathbf{j}(C_{\varphi(\bar{a}_v)}).$$

Γ is finite, so we may assume, without loss of generality (by increasing f_* if necessary), that $\mathbf{x}_{f_*}$ supports (decides) each of the finitely many $\mathbf{c}_{\varphi(\bar{a}_v)}$. More precisely, we may assume Γ admits a partition into disjoint sets $\Gamma_0 \cup \Gamma_1$ where

$$\varphi(\bar{x}_v) \in \Gamma_0 \text{ if and only if } \operatorname{tv}\left(\mathbf{x}_{f_*} \leq \mathbf{c}_{\varphi(\bar{a}_v)}\right) = 0.$$

The "accurate" subset of I is the one defined by

$$C := \bigcap\{C_{\varphi(\bar{a}_v)} : \varphi(\bar{x}_v) \in \Gamma_1\} \cap \bigcap\{I \setminus C_{\varphi(\bar{a}_v)} : \varphi(\bar{x}_v) \in \Gamma_0\} \subseteq I.$$

Since $\mathbf{j}(C) \geq \mathbf{x}_{f_*} > 0$, necessarily C is nonempty.

(4.48) Fix some $t \in C$ (so $t \in I$) for the remainder of the proof.

Now consider the picture in the model M given by index t. The set of elements $\{a_i[t] : i \in W\}$ accurately reflects the picture given by $\mathbf{x}_{f_*}$ in the following ways. First, if $j \leq i \in W$, then $M \models a_i[t] = a_j[t]$ if and only if $\rho_i(f_*) = \rho_j(f_*)$. Second, for all $u \in [W]^{k+1}$, $M \models R(\bar{a}_u[t])$ if and only if $\mathbf{x}_{f_*} \leq \mathbf{a}_{R(\bar{a}_u)}$ in the sense of (4.15). Moreover, for each $\beta \in \mathcal{I}$, $\mathbf{x}_{f_*} \leq \mathbf{a}_{R(\bar{a}_u)}$ if and only if $\mathbf{x}_{f_{s_\beta,\zeta_{h(\epsilon)}}} \leq \mathbf{a}_{R(\bar{a}_u)}$.

At the given index t, the "floating types" of (4.19) have become actual partial types, which we now name. For each $\beta \in \mathcal{I}$, let

$$r_\beta(x) := \{R(x, \bar{a}_{v_\gamma}[t])^{g_{s_\beta,\zeta_{h(\epsilon)}}(\beta)} : \gamma \in \operatorname{ind}(w_{s_\beta,\zeta_{h(\epsilon)}})\}. \tag{4.49}$$

and its restriction

$$r^*_\beta(x) := \{R(x, \bar{a}_{v_\gamma}[t])^{g_{s_\beta,\zeta_{h(\epsilon)}}(\beta)} : \gamma \in \operatorname{ind}(w^*_{s_\beta,\zeta_{h(\epsilon)}})\}. \tag{4.50}$$

Condition 4.19(b) ensures that each $r_\beta(x)$ is a complete, consistent R-type over $\{a_i[t] : i \in w_{s_\beta,\zeta_{h(\epsilon)}}\}$, and hence that each $r^*_\beta(x)$, is a complete, consistent R-type over $\{a_i[t] : i \in w^*_{s_\beta,\zeta_{h(\epsilon)}}\}$. Condition 4.19(a) ensures that $R(x, \bar{a}_{v_\beta})^{\mathbf{t}(\beta)} \in r^*_\beta(x) \subseteq r_\beta(x)$, because $\beta \in s_\beta \in \mathcal{U}_{\beta,\epsilon}$. However, $\bigcup\{r^*_\beta(x) : \beta \in \mathcal{I}\}$ is *not* a consistent partial type. This is because something even stronger is true:

$$\{R(x, \bar{a}_{v_\beta}[t])^{\mathbf{t}(\beta)} : \beta \in \mathcal{I}\} \text{ is not a consistent partial type.} \tag{4.51}$$

Why? $\mathbf{x}_{f_*} \cap \mathbf{b}_{\mathcal{I}} = 0$, so the formula $(\exists x) \bigwedge\{R(x, \bar{a}_{v_\beta}[t])^{\mathbf{t}(\beta)} : \beta \in \mathcal{I}\}$ belongs to Γ_0.

As we are working in $T_{n,k}$, the inconsistency of (4.51) can come from one of two sources: collisions or edges, which we rule out in turn.[9]

[9]To review our context: We have assumed for a contradiction that there exists some nonzero $\mathbf{c}_0$ in (4.39). As a consequence, we've found a finite set of formulas $\{R(x, \bar{a}_{v_\beta}[t])^{\mathbf{t}(\beta)} : \beta \in \mathcal{I}\}$ in M such that each is individually consistent but the whole set is inconsistent. As we are working in

Returning to the proof, the first possible problem is collision of parameters, i.e. perhaps there are $\beta \neq \gamma \in \mathcal{I}$ such that $\mathbf{t}(\beta) \neq \mathbf{t}(\gamma)$ but $\{a_i[t] : i \in v_\beta\} = \{a_j[t] : j \in v_\gamma\}$. [Note: for this part of the argument, we might as well work in the general context of the types r and $w_{s,\zeta}$; the r^* and $w^*_{s,\zeta}$ will be important presently.] By condition (4.40)(iv) in the inductive construction of f_*, we know that for each $\beta \in \mathcal{I}$, f_* extends an element of $\bar{f}_{s_\beta}$. Thus, for each $i \in w_{s_\beta,\zeta_{h(\epsilon)}}$, the 'minimum collision' functions $\rho_i(f_*)$ from (4.11) are well defined. Translating,

$$\{a_{\rho_i(f_*)}[t] : i \in v_\beta\} = \{a_i[t] : i \in v_\beta\} = \{a_j[t] : j \in v_\gamma\} = \{a_{\rho_j(f_*)}[t] : j \in v_\gamma\}.$$

Moreover, since the functions ρ_i were constructed to give a (definitive) minimal witness, we have that

$$\{\rho_i(f_*) : i \in v_\beta\} = \{\rho_j(f_*) : j \in v_\gamma\} \in [W]^k.$$

Call this set v. Let $\delta < \lambda$ be such that $v_\delta = v$ in the enumeration from (4.2). Recalling the definition of the $w_{s,\zeta}$ in (4.13), necessarily $v_\delta \in [w_{s_\beta,\zeta_{h(\epsilon)}}]^k$ and $v_\delta \in [w_{s_\gamma,\zeta_{h(\epsilon)}}]^k$, or in other words,

$$\delta \in \operatorname{ind}(w_{s_\beta,\zeta_{h(\epsilon)}}) \cap \operatorname{ind}(w_{s_\gamma,\zeta_{h(\epsilon)}}).$$

By definition of $g_{s,\zeta}$ in (4.19)(c), the collision in each case ensures that

$$g_{s_\beta,\zeta_{h(\epsilon)}}(\beta) = g_{s_\beta,\zeta_{h(\epsilon)}}(\delta) \text{ and likewise } g_{s_\gamma,\zeta_{h(\epsilon)}}(\delta) = g_{s_\gamma,\zeta_{h(\epsilon)}}(\gamma). \tag{4.52}$$

Recall that we had chosen $s_\beta \in \mathcal{U}_{\beta,\epsilon}$ and $s_\gamma \in \mathcal{U}_{\gamma,\epsilon}$ in (4.40)(iii), so condition (4.19)(a) gives that

$$g_{s_\beta,\zeta_{h(\epsilon)}}(\beta) = \mathbf{t}(\beta) \text{ and likewise } \mathbf{t}(\gamma) = g_{s_\gamma,\zeta_{h(\epsilon)}}(\gamma). \tag{4.53}$$

However, condition (4.40)(iv) in the inductive construction of f ensures that for each $\beta \in \mathcal{I}$, $f \supseteq f^*_{s_\beta,\zeta_{h(\epsilon)}}$. Since ϵ is fixed, by condition (4.27)(7),

$$g_{s_\beta,\zeta_{h(\epsilon)}}(\delta) = g_{s_\gamma,\zeta_{h(\epsilon)}}(\delta). \tag{4.54}$$

By (4.52), (4.54), and transitivity of equality,

$$g_{s_\beta,\zeta_{h(\epsilon)}}(\beta) = g_{s_\beta,\zeta_{h(\epsilon)}}(\gamma). \tag{4.55}$$

In the presence of our hypothesis that $\mathbf{t}(\beta) \neq \mathbf{t}(\gamma)$, equations (4.53) and (4.55) give an obvious contradiction. This contradiction shows that collision of parameters cannot be responsible for the inconsistency of the partial type.

The second possible problem is a background instance (or instances) of R on the parameters, i.e. perhaps there is $w \in [W_*]^n$ such that for all $u \in [w]^{k+1}$, $M \models R(\bar{a}_u[t])$, and for each $v \in [w]^k$, there is $\beta = \beta(v) \in \mathcal{I}$ such that $\mathbf{t}(\beta) = 1$ and $\{a_i[t] : i \in v_\beta\} = \{a_j[t] : j \in v\}$.

Recall our property Pr for the function G with range μ (fixed just before defining the equivalence relation E earlier in the proof) guarantees that: "for any $w \in [\lambda]^n$ and any $\langle u_v : v \in [w]^k\rangle$ such that $v \in [w]^k$ implies $v \subseteq u_v \in [\lambda]^{<\aleph_0}$,

---

$T_{n,k}$, the inconsistency of (4.51) can come from one of two sources: collisions or edges. Now, each of the $R(x, \bar{a}_{v_\beta}[t])^{\mathbf{t}(\beta)}$ is included in turn in a type $r^*_\beta(x)$ over the finite set $\{a_i[t] : i \in w^*_{s_\beta,\zeta_{h(\epsilon)}}\}$, and each $r^*_\beta(x)$ is included in a type $r_\beta(x)$, over the larger finite set $\{a_i[t] : i \in w_{s_\beta,\zeta_{h(\epsilon)}}\} \subseteq M$. Our plan is to first use consistency of each $r^*_\beta(x)$, and their various mutual coherence conditions as guaranteed by the ultrafilter, to rule out inconsistency from collisions. Second, we use consistency and the mutual coherence of the $r_\beta(x)$'s to rule out inconsistency from edges, by invoking G and recalling that $\mathrm{cl}_1(w^*_{s_\beta,\zeta_{h(\epsilon)}}) \subseteq w_{s_\beta,\zeta_{h(\epsilon)}}$ by (4.13).

if $G \restriction \langle (u_v, \mathrm{cl}_1(u_v)) : v \in [w]^k \rangle$ is constant, *then* for some $v \in [w]^k$ we have that $w \subseteq \mathrm{cl}_1(u_v)$." Apply this in the case where $u_v = w^*_{s_{\beta(v)},\zeta_{h(\epsilon)}}$ and $\mathrm{cl}_1(u_v) = \mathrm{cl}_1(w^*_{s_{\beta(v)},\zeta_{h(\epsilon)}}) \subseteq w_{s_{\beta(v)},\zeta_{h(\epsilon)}}$. Because ϵ is fixed, the value $\xi = \xi_{h(\epsilon)}$ of G on these sets is constant. Thus, there is some $\beta_* \in \mathcal{I}$ such that $w \subseteq \mathrm{cl}_1(w^*_{\beta_*,\zeta}) \subseteq w_{\beta_*,\zeta}$. In other words, the relevant near-complete hypergraph is already contained in the base set of one of our consistent partial types.

Now the argument is similar to that of the "collision" problem treated above. Fix for awhile $v \in [w]^k$ and $\beta = \beta(v)$. Let $\delta < \lambda$ be such that $v = v_\delta$. Again, for each $i \in w_{s_\beta,\zeta_{h(\epsilon)}}$ the functions $\rho_i(f_*)$ are well defined and entail that

$$\{\rho_i(f_*) : i \in v_\beta\} = \{\rho_j(f_*) : j \in v\}.$$

Thus, by (4.19)(c),

$$g_{s_\beta,\zeta_{h(\epsilon)}}(\beta) = g_{s_\beta,\zeta_{h(\epsilon)}}(\delta).$$

Since $w \subseteq w_{s_{\beta_*},\zeta_{h(\epsilon)}}$ and $v \in [w]^k$, we have also that $\delta \in \mathrm{dom}(g_{s_{\beta_*},\zeta_{h(\epsilon)}})$. Again by (4.40)(iv) and (4.27)(7), we have that

$$g_{s_{\beta_*},\zeta_{h(\epsilon)}}(\delta) = g_{s_\beta,\zeta_{h(\epsilon)}}(\delta) = \mathbf{t}(\beta) = 1.$$

As $v \in [w]^k$ was arbitrary, this shows that $r_{\beta_*}(x)$ includes $\{R(x, \bar{a}_v) : v \in [w]^k\}$. In light of our assumption that $M \models R(\bar{a}_u[t])$ for all $u \in [w]^{k+1}$, this contradicts r_{β_*} being a consistent partial type. This shows that an occurrence of R on the parameters cannot be responsible for inconsistency.

We have ruled out the only two possible causes of inconsistency for (4.51). This contradiction proves that the situation of (4.39) never arises. This completes the proof that $\bar{\mathbf{b}}'$ is a multiplicative refinement of $\bar{\mathbf{b}}$.

This completes the proof of Theorem 4.1. $\square$

Conclusion 4.2. *Suppose that for some ordinal α and integers ℓ, k,*

(1) $\ell < k$,
(2) $\mu = \aleph_\alpha$, $\lambda = \aleph_{\alpha+\ell}$
or just: $(\lambda, k-1, \mu^+) \not\to k$.
(3) $T = T_{n,k}$.

Then there is a regular (λ, μ)-perfect ultrafilter on λ which is good for T but not μ^{++}-good for any non-low or non-simple theory.

Proof. Theorem 3.G gives a (λ, μ)-perfected ultrafilter which is not μ^{++}-good for any non-simple or non-low theory. As for saturation, we know by Kuratowski-Sierpinski that $(\aleph_{(\alpha+1)+(k-2)}, k-1, \aleph_{\alpha+1}) \not\to k$, so when $\lambda = \aleph_{\alpha+(k-1)}$ and $\mu^+ = \aleph_{\alpha+1}$ we have that $(\lambda, k-1, \mu^+) \not\to k$. So by Lemma 2.5, $\mathrm{Pr}_{n,k}(\lambda, \mu)$ holds for these cardinals, therefore so the hypotheses of Theorem 4.1 are satisfied. $\square$

5. The non-saturation condition

In this section we prove the complementary result to Theorem 4.1, by connecting non-saturation of $T_{k+1,k}$ to existence of large free sets in set mappings.

Claim 5.1. *Suppose that:*

(1) for some ordinal α and integers $2 \leq k < \ell$, $\mu = \aleph_\alpha$, $\lambda = \aleph_{\alpha+\ell}$,
or just: $(\lambda, k, \mu^+) \to k+1$

(2) $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$
(3) $\mathcal{D}_*$ *is an ultrafilter on* $\mathfrak{B}$
(4) $T = T_{k+1,k}$

Then $\mathcal{D}_*$ *is not* (λ, T)*-moral.*

Remark 5.2. *Note that there is no mention of optimality or perfection of the ultrafilter. The only factor is the distance of* λ *and* μ *as reflected in the Boolean algebra* $\mathfrak{B}$ *(or what amounts to the size of a maximal antichain at the "transfer point" in Theorem 3.F). To justify item (1), recall that* $(\aleph_{(\alpha+1)+k}, k, \aleph_{(\alpha+1)}) \rightarrow k+1$ *by Kuratowski-Sierpinski, so if* $\ell = k+1$, $\aleph_{\alpha+\ell} = \aleph_{\alpha+(k+1)}$ *and* $\mu^+ = \aleph_{\alpha+1}$ *and we have* $\rightarrow$ *as desired.*

Proof. Our strategy will be to build a sequence $\bar{\mathbf{b}}$ of elements of $\mathfrak{B}^+$ and prove that it is a possibility pattern for T but does not have a multiplicative refinement. We continue with much of the notation and terminology of the previous section.

By Theorem 1.B above (and monotonicity), for $k \leq m = \ell - 1$, $(\aleph_{\alpha+m+1}, k, \aleph_{\alpha+1}) \rightarrow k+1$, so we can apply Claim 1.6 to (λ, k, μ^+). [Notice that μ^+ here replaces μ there.] Thus, we may fix a model M of $T_{k+1,k}$ with λ distinguished elements $\bar{b} = \langle b_\alpha : \alpha < \lambda \rangle$ with the following property. Let

$$\mathcal{P} = \{w \in [\lambda]^{k+1} : M \models R(\bar{b}_v)\}$$

noting that by choice of T, $\mathcal{P} \subsetneq [\lambda]^{k+1}$. The property is that whenever $F : [\lambda]^k \rightarrow [\lambda]^{\leq \mu}$ is a strong set mapping, for some $w \in \mathcal{P}$ we have

$$(\forall v \in [w]^k)(w \not\subseteq F(v)).$$

Without loss of generality we may extend M to be λ^+-saturated. For the remainder of the proof, fix a choice of ordinals $\langle \alpha_w : w \in \mathcal{P} \rangle$ with no repetitions, where each $\alpha_w < 2^\lambda$. Choose also for each $w \in \mathcal{P}$ a corresponding function $g_w \in \mathrm{FI}_\mu(\alpha_*)$ such that $\mathrm{dom}(g_w) = \{\alpha_w\}$ and $\mathbf{x}_{g_w} = \emptyset \mod \mathcal{D}_*$.

Overview in a special case. Before giving the construction in the generality of the Boolean algebra $\mathfrak{B}$, we describe for the reader the picture in the special case where we consider an ultrapower $N = M^I/\mathcal{D}$ where $\mathcal{D}$ is built from a regular filter $\mathcal{D}_0$ and $\mathfrak{B}$ is identified with some independent family $\mathcal{F} \subseteq {}^I\mu$ of cardinality 2^λ. What we would like to do is choose a set A of size λ in the ultrapower which is an empty graph in N, i.e. for all $u \in [A]^{k+1}$, $N \models \neg R(\bar{a}_u)$. As a result, the type $p(x) = \{R(x, \bar{a}_v) : v \in [A]^{k+1}\}$ will be a consistent partial type in N. However, by judicious choice of the parameter set A, we will be able to show that p cannot be realized. To do this we need to ensure that edges appear on the projections of A to the index models, but not too many and not too often.

We begin with the idea that for each $i < \lambda$, a_i is the equivalence class in N of the sequence which is constantly equal to b_i. We then essentially doctor this sequence by winnowing $\mathcal{P}$, i.e. erasing some of the edges. Formally, of course, at each index t we choose a sequence $\langle b'_i[t] : i < \lambda \rangle$ of distinct elements of M (using the fact that M is universal for models of T of size $\leq \lambda$) such that for all $w \subseteq \lambda$, if $M \models R(\bar{b}'_w)$ then $M \models R(\bar{b}_w)$, but not necessarily the inverse. We will then set $a_i = \langle b'_i[t] : t \in I \rangle / \mathcal{D}$ for each $i < \lambda$. How to winnow edges? Following the notation of the proof of 4.1, fix an enumeration of $[\lambda]^k$ as $\langle v_\beta : \beta < \lambda \rangle$ without repetition, so the eventual type will be enumerated by $\{R(x, \bar{a}_{v_\beta}) : \beta < \lambda\}$. Let $\Omega = [\lambda]^{<\aleph_0}$. For each $s \in \Omega$, let the 'critical set' $\mathrm{cs}(s)$ be the set of $w \in \mathcal{P}$ such that each $v \in [w]^k$ is v_β for some $\beta \in w$.

(Note that this is generally weaker than saying that $w \subseteq \mathrm{vert}(s)$.) The rule is that for each $t \in I$, and each $w \in \mathcal{P}$, we leave an edge on $\{b'_i : i \in w\}$ if and only if $t \in \mathbf{x}_{g_w}$. By the choice of g_w, no edge will persist in the ultrapower, so $\langle a_i : i < \lambda \rangle$ is an empty graph in N. It remains to prove the type is not realized. Before giving this argument, we carry out the construction just described in the generality of the Boolean algebra. (The type just described easily converts to a possibility pattern using the Łos map as in (4.6) p. 15, so we may conclude this argument using the more general proof.)

General proof. Let $\langle v_\alpha : \alpha < \lambda \rangle$ list $[\lambda]^k$ without repetition. For $s \subseteq \lambda$, let $\mathrm{vert}(s) = \bigcup \{v_\beta : \beta \in s\} \in [\lambda]^{<\aleph_0}$ collect the indices for all relevant vertices. Let $\Omega = [\lambda]^{<\aleph_0}$. For each $s \in \Omega$, let

$$\mathbf{b}_s = 1_{\mathfrak{B}} - \bigcup \{\mathbf{x}_{g_w} : w \in \mathcal{P} \text{ and } [w]^k \subseteq \{v_\beta : \beta \in s\}\}.$$

Essentially, we omit the formal representative of any bad configuration once our type fragment s includes indices for all of the edges (in the type) connecting to it.

Let us show that $\langle \mathbf{b}_s : s \in \Omega \rangle$ is a possibility pattern for $T_{k+1,k}$. Fix for awhile $s \in \Omega$ and $\mathbf{c} \in \mathfrak{B}^+$. Decreasing $\mathbf{c}$ if necessary, we may assume that for any $w \in \mathcal{P} \cap [\mathrm{vert}(s)]^{k+1}$ either $\mathbf{c} \leq \mathbf{x}_{g_w}$ or $\mathbf{c} \leq 1 - \mathbf{x}_{g_w}$. It follows that for any $s' \subseteq s$ either $\mathbf{c} \leq \mathbf{b}_{s'}$ or $\mathbf{b} \leq 1 - \mathbf{b}_{s'}$.

To satisfy Definition 3.7, we now need to choose parameters $b'_i \in M$ for $i \in \mathrm{vert}(s)$ such that: $\bar{b}' = \langle b'_i : i \in \mathrm{vert}(s) \rangle$ is without repetition and for any $s' \subseteq s$,

$$M \models (\exists x) \bigwedge_{\beta \in s'} \varphi(x; \bar{b}'_{v_\beta}) \text{ iff } \mathbf{c} \leq \mathbf{b}_{s'}.$$

We can do this by choosing our parameters so that for any $i_0, \dots, i_{k-1} \in \mathrm{vert}(s)$ we have $\langle b'_{i_\ell} : \ell < k \rangle \in R^M$ if and only if: $|\{i_\ell : \ell < k\}| = k$ [i.e. they are distinct] and $\{i_\ell : \ell < k\} \in \mathcal{P}$ and $\mathbf{c} \leq \mathbf{x}_{g_{\{i_\ell : \ell < k\}}}$. Note that there *is* such a sequence of parameters in the monster model (forgetting edges on the $\bar{b}$ as described above) so it suffices to show such a sequence works. If $\mathbf{c} \leq \mathbf{b}_{s'}$, then by definition of $\mathbf{b}_{s'}$, there is no $w \in \mathcal{P}$ such that $[w]^k \subseteq \{v_\beta : \beta \in s'\}$ and $\mathbf{c} \leq \mathbf{x}_{g_w}$. So there are never enough edges on the parameters to produce an inconsistency in the set

$$\{R(x; \bar{b}'_{v_\beta}) : \beta \in s'\}.$$

If $\mathbf{c} \cap \mathbf{b}_{s'} = 0_{\mathfrak{B}}$, then because $\mathbf{c} \in \mathfrak{B} \setminus \{0_{\mathfrak{B}}\}$, it must be that $\mathbf{b}_{s'} \neq 1_{\mathfrak{B}}$. By definition of the sequence $\bar{\mathbf{b}}$, there is $w \in \mathcal{P}$ with $[w]^k \subseteq \{v_\beta : \beta \in s'\}$ and (since $\mathbf{c}$ decides all relevant edges) $\mathbf{c} \leq \mathbf{x}_{g_w}$. Then $M \models R(\bar{b}'_w)$. Recalling that

$$\{R(x; \bar{b}'_{v_\beta}) : \beta \in s'\} \supseteq \{R(x; \bar{b}'_v) : v \in [w]^k\}$$

the left hand side cannot be consistent. This completes the proof that $\bar{\mathbf{b}}$ is a possibility pattern.

No multiplicative refinement. Now let us assume for a contradiction that $\langle \mathbf{b}'_s : s \in \Omega \rangle$ is a multiplicative refinement of the possibility pattern just described. That is, $s_1, s_2 \in \Omega$ implies $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = \mathbf{b}'_{s_1 \cap s_2}$ and for each $s \in \Omega$, $\mathbf{b}'_s \leq \mathbf{a}_s$. As each $\mathbf{b}'_{\{\beta\}} \in \mathfrak{B}^+$, we may write $\mathbf{b}'_{\{\beta\}} = \bigcup \{\mathbf{x}_{h_{\beta,i}} : i < i(\beta) \leq \mu\}$ where $\langle h_{\beta,i} : i < i(\beta) \rangle$ is a set of pairwise inconsistent functions from $\mathrm{FI}_\mu(2^\lambda)$. Let $S_\beta = \bigcup \{\mathrm{dom}(h_{\beta,i}) : i < i(\beta)\}$, so $S_\beta \subseteq 2^\lambda$ has cardinality $\leq \mu$.

First, we show that for each $w \in \mathcal{P}$ the domain of g_w is detected by the supports of at least one of the the k-element subsets of w.

Subclaim 5.3. *If $w \in \mathcal{P} \subseteq [\lambda]^{k+1}$ then $\alpha_w \in \bigcup\{S_\beta : v_\beta \in [w]^k\}$.*

Proof. Let $x = \{\beta : v_\beta \in [w]^k\} \in [\lambda]^{\binom{m}{k}}$. Since $\overline{\mathbf{b}}'$ is multiplicative,

$$\mathbf{b}'_x = \bigcap\{\mathbf{b}'_\beta : \beta \in x\}$$

Let $f \in \mathrm{FI}_\mu(2^\lambda)$ be such that $\mathbf{x}_f \leq \mathbf{b}'_x$. Then $\mathbf{x}_f \leq \mathbf{b}'_{\{\beta\}}$ for each $\beta \in x$. Letting $g = f \restriction \bigcup\{S_\beta : v_\beta \in [w]^k\} = \bigcup\{S_\beta : \beta \in x\}$, we have that $v_\beta \in [w]^k \implies \mathbf{x}_g \leq \mathbf{b}'_{\{\beta\}}$. This implies that $\mathbf{x}_g \leq \mathbf{b}'_x \leq \mathbf{b}_x$ because $\bar{\mathbf{b}}'$ refines $\bar{\mathbf{b}}$. By definition,

$$\mathbf{b}_x = 1_{\mathfrak{B}} - \bigcup\{\mathbf{x}_{g_u} : u \in \mathcal{P} \text{ and } [u]^k \subseteq \{v_\beta : \beta \in x\}\}.$$

So as $[w]^k \subseteq \{v_\beta : \beta \in x\}$, necessarily $\mathbf{x}_g \cap \mathbf{x}_{g_w} = 0_{\mathfrak{B}}$. Since our Boolean algebra $\mathfrak{B}$ was generated freely, it must be that $\mathrm{dom}(g_w) \cap \mathrm{dom}(g) \neq \emptyset$, but $\mathrm{dom}(g_w) = \{\alpha_w\}$. This shows that $\alpha_w \in \bigcup\{S_\beta : v_\beta \in [w]^k\}$ as desired.
This proves Subclaim 5.3. □

We resume our proof by contradiction. Define a strong set mapping $F : [\lambda]^k \to [\lambda]^{\leq\mu}$ by: if $v \in [\lambda]^k$ let β be such that $v = v_\beta$, and let

$$F(v) = \bigcup\{w \in [\lambda]^m : w \in \mathcal{P} \text{ and } \alpha_w \in S_\beta\}.$$

Then $F(v)$ is well defined, $F(v) \subseteq \lambda$, and $|F(v)| \leq \mu$ for $v \in [\lambda]^k$. (Recall that $\langle \alpha_w : w \in \mathcal{P}\rangle$ is without repetition.) Now for all $w \in \mathcal{P} \subseteq [\lambda]^{k+1}$, there is $v = v_\beta \in [w]^k$ such that $\alpha_w \in S_\beta$. Thus $w \subseteq F(v)$. We have proved that for all $w \in \mathcal{P}$,

$$(\exists v \in [w]^k)(w \subseteq F(v)).$$

This is a contradiction, so the possibility pattern $\overline{\mathbf{b}}$ does not have a solution. Thus, $\mathcal{D}_*$ cannot be moral for $T_{k+1,k}$. This completes the proof of Claim 5.1. □

Conclusion 5.4. *Suppose that for some ordinal α and integers ℓ, k,*

(1) $2 \leq k < \ell$
(2) $T = T_{k+1,k}$
(3) $\mu = \aleph_\alpha$, $\lambda = \aleph_{\alpha+\ell}$
(4) $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$
(5) $\mathcal{D}_*$ *is* any *ultrafilter on* $\mathfrak{B}$
(6) $\mathcal{D}_1$ *is* any *regular ultrafilter on λ built from* $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$

Then $\mathcal{D}_1$ is not good for T. In particular, if $\mathcal{D}_1$ is a (λ,μ)-perfected ultrafilter on λ, then $\mathcal{D}_1$ is not good for T.

Proof. By Claim 5.1 and Theorem 3.F. Note that if we allow $\ell = k = 1$, $T_{k+1,k}$ is not simple so we can likewise avoid saturation of T. □

6. Infinitely many classes

We emphasize that all results in this section are in ZFC.

Theorem 6.1. *Suppose $\mu = \aleph_\alpha$ and $\lambda = \aleph_{\alpha+\ell}$ for α an ordinal and ℓ a nonzero integer. Let $\mathcal{D}$ be a (λ, μ)-perfected ultrafilter on λ. Then for any $2 \leq k < \omega$:*

(a) If $k < \ell$, then $\mathcal{D}$-ultrapowers of models of $T_{k+1,k}$ are not λ^+-saturated.
(b) If $\ell < k$, then $\mathcal{D}$-ultrapowers of models of $T_{k+1,k}$ are λ^+-saturated.

Proof. (1) Conclusion 5.4.
(2) Conclusion 4.2. □

In fact, by the proofs, more is true:

Conclusion 6.2. *Suppose we are given:*

(a) *for some ordinal α and integer ℓ, $\mu = \aleph_\alpha$, $\lambda = \aleph_{\alpha+\ell}$*
(b) *$\mathcal{D}_1$ is built from $(\mathcal{D}_0, \mathfrak{B}_{2^\lambda,\mu}, \mathcal{D})$*
(c) *$T = T_{k+1,k}$*

Then:

(1) if $k < \ell$, $\mathcal{D}_1$ is not (λ^+, T)-good.
(2) if $\ell < k$ and in addition $\mathcal{D}$ is (λ, μ)-perfect, $\mathcal{D}_1$ is (λ^+, T)-good.

Theorem 6.3. *For any $k_* > 2$ and ordinal α there is a regular ultrafilter $\mathcal{D}$ on $\aleph_{\alpha+k_*}$ such that*

(1) if $k_* < k_2$ then $\mathcal{D}$ is good for T_{k_2,k_2+1}
(2) if $k_1 < k_*$ then $\mathcal{D}$ is not good for T_{k_1,k_1+1}.

Proof. By Theorem 6.1 and §3 Theorem 3.G. □

We now recall the definition of Keisler's order. For a current account of what is known, see [24] and for further intuition, see the introductory sections of [21]. Note that this allows us to compare any two theories, regardless of language.

Definition 6.4. (Keisler's order, Keisler 1967 [12]) *Let T_1, T_2 be complete countable theories. We write $T_1 \trianglelefteq T_2$ if: for any $\lambda \geq \aleph_0$, any $M_1 \models T_1$, any $M_2 \models T_2$ and any regular ultrafilter on λ,*

$$\text{if } {M_2}^\lambda/\mathcal{D} \text{ is } \lambda^+\text{-saturated then } {M_1}^\lambda/\mathcal{D} \text{ is } \lambda^+\text{-saturated.}$$

Here "regular" entails that the relation $\trianglelefteq$ is independent of the choice of M_1, M_2.

Corollary 6.5. *Let $\trianglelefteq$ mean in Keisler's order. Then:*

(1) If $2 \leq k_1$ and $k_1 + 1 < k_2$ then

$$T_{k_1,k_1+1} \not\trianglelefteq T_{k_2,k_2+1}.$$

(2) Keisler's (partial) order contains either an infinite descending chain or an infinite antichain within the simple unstable theories.

Proof. (1) is immediate by 6.3 and (2) follows by Ramsey's theorem. □

Note that Keisler's order is a partial order on equivalence classes of theories, and the following theorem proves *existence* of an infinite descending chain in this partial order already within the simple unstable rank one theories there may indeed be additional structure.

Theorem 6.6. *There is an infinite descending sequence of simple rank 1 theories in Keisler's order. More precisely, there are simple theories $\{T^*_n : n < \omega\}$ with trivial forking such that, writing*

- *$\mathcal{T}_A$ for the class of theories without fcp*
- *$\mathcal{T}_B$ for the class of stable theories with fcp*
- *$\mathcal{T}_C$ for the minimum unstable class, i.e. the Keisler-equivalence class of the random graph*
- *$\mathcal{T}_{max}$ for the Keisler-maximal class, i.e. the Keisler-equivalence class of linear order (or SOP_2)*
- *and $\mathcal{T}_n$ for the Keisler-equivalence class of T^*_n*

for all $m < n < \omega$ we have:

$$\mathcal{T}_A \lhd \mathcal{T}_B \lhd \mathcal{T}_C \lhd \qquad \cdots\cdots\cdots \mathcal{T}_n \lhd \mathcal{T}_m \lhd \cdots \lhd \mathcal{T}_2 \lhd \mathcal{T}_1 \lhd \mathcal{T}_0 \lhd \qquad \mathcal{T}_{max}.$$

Proof. The structure of the order on $\mathcal{T}_A$, $\mathcal{T}_B$, $\mathcal{T}_C$, $\mathcal{T}_{max}$ was known, see [21] §4. To obtain the infinite descending chain, let T^*_n be the disjoint union of the theories $T_{k,k+1}$ for $k > 2n + 2$. Here "disjoint union" is understood naturally, for instance, the theory of the model M formed by taking the disjoint union of models $M_k \models T_{k,k+1}$ in disjoint signatures. Clearly, $k' > k$ implies $T^*_{k'} \trianglelefteq T^*_k$ and $\lhd$ is by Theorem 6.3. This completes the proof. □

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