

THE UNIVERSITY OF CHICAGO

EQUIVARIANT LOCAL COEFFICIENTS AND THE $RO(G)$ -GRADED
COHOMOLOGY OF CLASSIFYING SPACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
MEGAN ELIZABETH SHULMAN

CHICAGO, ILLINOIS

JUNE 2010

To Mom and Dad

TABLE OF CONTENTS

LIST OF FIGURES	v
ABSTRACT	vi
ACKNOWLEDGEMENTS	vii
CHAPTER 1. INTRODUCTION	1
1.1 Overview	1
1.2 Other aspects of equivariant cohomology	2
CHAPTER 2. BACKGROUND ON MACKEY FUNCTORS AND BREDON COHOMOLOGY	3
2.1 Bredon cohomology and extension to $RO(G)$ -grading	3
2.2 A review of Mackey functors	7
2.2.1 Connection to coefficient systems	8
2.2.2 Box products and maps out of box products	9
2.2.3 Explicit formula for the box product	11
2.2.4 Graded Mackey functors	14
2.2.5 Green functors and modules over Green functors	14
2.2.6 Building new Mackey functors	16
2.2.7 Free Mackey functors	17
2.3 Mackey functor valued cohomology theories	18
2.3.1 The equivariant stable category	19
2.3.2 Summary of important facts about homology and cohomology	20
2.4 Mackey functors for $G = C_p$	22
2.4.1 Diagrams for Mackey functors	22
2.4.2 Some important Mackey functors	23
2.4.3 Box Products	27
2.5 Bredon Cohomology for $G = C_p$	30
2.5.1 Cup product and graded commutativity	31
2.5.2 Reinterpreting the cup product	35
2.5.3 Burnside ring coefficients	35
2.5.4 The cohomology of a point for $p = 2$	36
2.5.5 The cohomology of a point for $p > 2$	40
CHAPTER 3. COHOMOLOGY OF C_p -CW-COMPLEXES WITH CELLS IN EVEN DIMENSIONS	47
3.1 A freeness theorem	47

3.2	Tools for Computing Multiplicative Structure	53
3.3	Cohomology of Complex Projective Spaces	63
CHAPTER 4. SPECTRAL SEQUENCES FOR LOCAL COEFFICIENTS		66
4.1	The nonequivariant situation	66
4.2	Equivariant generalizations	70
4.3	Some remarks on the Serre Spectral Sequence	76
4.4	Equivariant classifying spaces	78
CHAPTER 5. EXAMPLE: THE COHOMOLOGY OF $B_{C_p}O(2)$		80
5.1	Identifying the fibers of $f: B_{C_p}O(2) \rightarrow B_{C_p}\mathbb{Z}/2$	80
5.2	The local coefficient system $h_{C_p}^{V+t}(f, A)$	81
5.3	The local coefficient system $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$	83
5.4	The calculation of $H_{C_p}^*(B_{C_p}O(2); A)$	84
CHAPTER 6. MULTIPLICATIVE STRUCTURE OF THE SPECTRAL SE- QUENCE		87
6.1	Multiplicative structure	87
6.1.1	The nonequivariant spectral sequence	87
6.1.2	The equivariant spectral sequence	90
6.2	Sample nonequivariant computations	91
REFERENCES		95

LIST OF FIGURES

2.56	Multiplication table	30
2.57	$f \smile g$ and $g \smile f$	32
2.63	A plot of $\tilde{H}_{C_2}^*(S^0)$ in the plane	37
2.65	The locations of the generators $\epsilon, \iota, \iota^{-1}, \xi, \epsilon^{-1}\kappa,$ and $\epsilon^{-1}t_\rho(\iota^3)$	38
2.73	A plot of $\tilde{H}_{C_p}^*(S^0)$ in the plane	41
3.6	The equivariant dimensions of Schubert cells	52
3.17	The kernel and cokernel of ρ	59
3.19	The cokernel of t_ρ	60
3.21	The kernel of $\tilde{H}_{C_p}^*(S^V)(\bullet) \rightarrow \tilde{H}_{C_p}^*(S^0)(\bullet)$	61
6.10	$\text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^s(\mathbb{Z}, \mathbb{Z}[x]^t)$	92

ABSTRACT

This thesis consists of two main parts. In the second part, we recall how a description of local coefficients that Eilenberg introduced in the 1940s leads to spectral sequences for the computation of homology and cohomology with local coefficients. We then show how to construct new equivariant analogues of these spectral sequences for $RO(G)$ -graded Bredon homology and cohomology. Finally, we use these spectral sequences to complete a sample calculation, in which we use the equivariant Serre spectral sequence and the equivariant cohomology of complex projective spaces to compute the cohomology of the equivariant classifying space $B_{C_p}O(2)$.

However, to complete this sample computation, we need to know the cohomology of the complex projective space $\mathbb{C}P(\mathcal{U}_{\mathbb{C}}) = B_{C_p}SO(2)$. This calculation was done in [Lew88], but relies on a theorem whose proof as given was incorrect. We spend the first part of this thesis providing a correct proof and summarizing the results of [Lew88].

ACKNOWLEDGEMENTS

I would like to thank my advisor, Peter May, for many years' worth of discussions about all things equivariant; John Greenlees, for several very helpful discussions about the multiplicative structure on cohomology; and Vigeik Angeltveit and Anna Marie Bohmann for proofreading drafts of this work. I would also like to thank Michael Shulman for many helpful conversations about category theory and Day tensor products, and Gaunce Lewis for writing so much about Mackey functors.

CHAPTER 1

INTRODUCTION

1.1 Overview

From a theoretical point of view, there are many reasons to think that $RO(G)$ -graded Bredon cohomology, and in particular Mackey functor valued $RO(G)$ -graded Bredon cohomology, is the correct equivariant analogue of ordinary cohomology. Recent results, such as [HHR09] and [AGH09], provide additional reasons to want to be able to compute using this theory. Unfortunately, computations in $RO(G)$ -graded Bredon cohomology are extremely difficult, and even the few nontrivial calculations in the literature are not well known.

There are two immediate problems when approaching computations in $RO(G)$ -graded Mackey functor valued Bredon cohomology. First, the reservoir of known low-level computations on which to build further results is extremely small; as an example, [HHR09] had to compute the cohomology of a point for $G = C_{2^n}$, the cyclic group of order 2^n , because this was not previously known. Second, even if there were such a reservoir of computations, many familiar nonequivariant tools do not yet have tractable equivariant analogues. This thesis represents a first step towards remedying both hurdles.

We begin in Chapter 2 by recalling some of the basic definitions, which are unfamiliar to many algebraic topologists, and discussing the computation by Stong and Lewis of the cohomology of a point when $G = C_p$, a cyclic group of prime order. We then summarize the results of L. Gaunce Lewis in [Lew88] concerning the equivariant cohomology of complex projective spaces. The central result of [Lew88] rests on a lemma whose proof as given is incorrect; a correction is included in Chapter 3.

After establishing these basic calculations, we turn in Chapter 4 to more sophisticated tools for computing $RO(G)$ -graded cohomology. Moerdijk and Svensson in [MS93] developed a Serre spectral sequence in integer-graded Bredon cohomology, and Kronholm extended this to a spectral sequence for $RO(G)$ -graded Bredon cohomology in [Kro09]. However, this Serre spectral sequence has not yet been used for any computations, in large part because it is impossible to ignore local coefficients when working equivariantly with Bredon (co)homology: in contrast to the nonequivariant situation, almost no interesting examples reduce to trivial local coefficients. It is thus necessary to develop some tools for working with homology and cohomology with local coefficients.

To this end, we recall a simple universal coefficient spectral sequence which appears in [CE56, p. 355], but which, to the best of our knowledge, has not previously been applied in conjunction with the Serre spectral sequence. Part of the reason is that the definition of local coefficients that appears in the construction of the Serre spectral sequence is not tautologically the same as the definition that gives the cited universal coefficient spectral sequence.

The connection comes from an old result of Eilenberg [Eil47], popularized by Whitehead [Whi78, VI.3.4] and, more recently, by Hatcher [Hat02, App3.H]. It identifies the local coefficients that appear in the context of fibrations with a more elementary definition in terms of the chains of the universal cover of the base space. The identification makes working with local coefficients much more feasible. We shall first say in Section 4.1 how this goes nonequivariantly, and illustrate with an example. We then explain the equivariant generalization in Section 4.2. The later parts of this thesis will review some necessary background on equivariant classifying spaces and finally go through a sample calculation in Chapter 5. This calculation computes characteristic classes in $RO(G)$ -graded Mackey functor valued cohomology of equivariant real 2-plane bundles when G is cyclic of prime order.

Finally, we conclude by discussing the multiplicative structure on this spectral sequence in Chapter 6, and go through some nonequivariant sample calculations in Section 6.2.

1.2 Other aspects of equivariant cohomology

We have mentioned the lack of known computations in $RO(G)$ -graded Bredon cohomology, but perhaps a few words about the reasons are in order.

To begin, although it is clear that G -spectra and thus equivariant cohomology theories should be graded on a richer structure than the integers, it is possible that the real G -representations are not quite the “true” indexing set. As we discuss later in this thesis, the Mackey functor valued Bredon cohomology $\underline{H}_G^*(X)$ of a G -space X should be considered free if it is a direct sum of suspended copies $\Sigma^\alpha \underline{H}_G^*(pt)$ of the cohomology of a point. However, as remarked in [FL04], there are nontrivially distinct sets $\{\alpha_i\}$ and $\{\beta_i\}$ of virtual representations such that $\bigoplus_i \Sigma^{\alpha_i} \underline{H}_G^*(pt)$ and $\bigoplus_i \Sigma^{\beta_i} \underline{H}_G^*(pt)$ are isomorphic free modules. This makes it harder to determine the “correct” $RO(G)$ dimensions of the generators of a free $\underline{H}_G^*(pt)$ -module.

Additionally, there are several possible definitions of a G -CW-complex. If we allow the spheres and disks of these complexes to have nontrivial actions of G —by allowing them to be the unit spheres and disks of representations of G , for example—then there is no equivariant Whitehead theorem: a G -map between G -CW-complexes need not be G -homotopic to a cellular map. Thus, if we define our cohomology theories using an appropriate equivariant analogue of cellular cohomology, a non-cellular G -map need not induce a map on cellular cohomology. In particular, given two different G -CW-structures on a G -space X , the identity map of X may not induce a self-map of the corresponding cellular cohomologies.

Some of the numerous other subtleties along these lines will become evident during the course of this thesis.

Throughout this thesis, k will refer to a fixed ground ring. G will be a finite group throughout, although a few results may apply more generally to discrete groups or compact Lie groups. Reduced cohomology theories are denoted with \tilde{H} , unreduced theories with H .

CHAPTER 2

BACKGROUND ON MACKEY FUNCTORS AND BREDON COHOMOLOGY

2.1 Bredon cohomology and extension to $RO(G)$ -grading

A nonequivariant (reduced) cohomology theory $\tilde{H}^*(-)$ is defined to be a sequence of contravariant functors from the homotopy category of based spaces to $\mathbb{k}\text{-mod}$, the category of \mathbb{k} -modules, together with suspension natural isomorphisms $\Sigma : \tilde{H}^n \rightarrow \tilde{H}^{n+1}$, satisfying the Eilenberg-Steenrod axioms. If it satisfies the dimension axiom as well, we say that the theory is ordinary. If X is an unbased space, then its unreduced cohomology is given by $H^*(X) = \tilde{H}^*(X_+)$, the reduced homology of X with a disjoint basepoint adjoined.

Similarly, for a fixed group of equivariance G , an **equivariant (reduced) cohomology theory** indexed on the integers is a sequence of contravariant functors from the G -homotopy category of based G -spaces to $\mathbb{k}\text{-mod}$, satisfying the obvious generalizations of the Eilenberg-Steenrod axioms: weak equivalence, exactness, additivity, and suspension. Note that a basepoint of a G -space X is required to be in the G -fixed points X^G . Also, a G -homotopy of G -maps $X \rightarrow Y$ consists of a G -map $X \times I \rightarrow Y$, where the interval I is given the trivial G action and the product has the diagonal G action. An equivariant cohomology theory is **ordinary** if it satisfies the dimension axiom: $\tilde{H}_G^n(G/K_+) = 0$ for all nonzero integers $n \in \mathbb{Z}$, $n \neq 0$, and all subgroups $K < G$.

There are some complications, of course. The above axioms assume that the collection of functors making up a cohomology theory, as in the nonequivariant situation, is indexed by \mathbb{Z} . However, equivariant theories are more naturally given by a collection of functors indexed on the free abelian group generated by the irreducible representations of G . For historical reasons, such a theory is called **$RO(G)$ -graded**, and we will continue this convention. However, keep in mind that, with the usual definition, the underlying abelian group of $RO(G)$ consists of *equivalence classes* of formal sums of representations. We cannot pay attention only to equivalence classes: the only way to make signs work out is to remember the isomorphisms between isomorphic representations. So, for the purposes of this thesis, we will define $RO(G)$ to be the free abelian group on irreducible representations of G . Concretely, this means that an $RO(G)$ -graded theory consists of a functor $\tilde{H}_G^{\sum_i a_i \lambda_i} : \text{ho-}G\text{-Top}_* \rightarrow \mathbb{k}\text{-mod}$ for each formal sum $\sum_i a_i \lambda_i$ of G -representations, where $\{\lambda_i\}_i$ runs over the irreducible representations of G and each a_i is a possibly negative integer. There are some important bookkeeping details which ensure that this is well-defined. We will largely ignore those issues in this thesis; see [May96] or [MS06] for details.

At any rate, after replacing n by $\alpha \in RO(G)$, the functors of an equivariant cohomology theory satisfy the weak equivalence, exactness, and additivity axioms, together with the following modified suspension axiom.

Axiom 2.1. Equivariant suspension: For each $\alpha \in RO(G)$ and actual representation V , there is a natural isomorphism

$$\Sigma^V : \tilde{H}_G^\alpha(X) \longrightarrow \tilde{H}_G^{\alpha+V}(\Sigma^V X) = \tilde{H}_G^{\alpha+V}(S^V \wedge X),$$

where S^V is the one-point compactification of the representation V . The suspension isomorphisms for different representations are compatible; for representations V and W , $\Sigma^{V+W} \cong \Sigma^V \Sigma^W$.

Somewhat surprisingly, an $RO(G)$ -graded theory is ordinary if it satisfies the dimension axiom for the integer-graded part of the theory: that is, $\tilde{H}_G^n(G/K_+) = 0$ for nonzero integers, but the dimension axiom says nothing about $\tilde{H}_G^\alpha(G/K_+)$ for $\alpha \notin \mathbb{Z}$. These cohomology groups in “off-integer” dimensions are determined by the other axioms, and are generally nonzero. As a result, the $RO(G)$ -graded cohomology of a point is quite complicated. This provides the first of many hurdles to computing with equivariant cohomology.

The ordinary equivariant cohomology theories are usually called **Bredon cohomology theories**; ordinary theories graded on the integers were introduced by Bredon in [Bre67]. There are corresponding notions of ordinary homology theories, satisfying the expected modifications of the above axioms.

The integer-graded Bredon cohomology theories have a very concrete description. The Bredon cohomology $\tilde{H}_G^*(X)$ can be thought of as compiling data about the nonequivariant cohomology rings $\tilde{H}^*(X^K)$ of the K -fixed points, for each subgroup K of G . They do this using the language of coefficient systems.

Definition 2.2. Fix a (finite) group of equivariance G . The **orbit category** \mathcal{O}_G has objects the orbits G/K , for K a subgroup of G ; morphisms are G -maps of G -sets.

There is a map $G/K \longrightarrow G/J$ if and only if K is subconjugate to J in G . If this is the case, then a G -map $\alpha : G/K \longrightarrow G/J$ is determined by what it does to the identity coset eK ; $\alpha(eK) = g_\alpha J$, for some g_α such that $g_\alpha^{-1} K g_\alpha \subset J$. It follows that the automorphism group of G/K is the **Weyl group** $W_G K = (N_G K)/K$, a quotient of the normalizer of K .

Definition 2.3. A **coefficient system** is a contravariant functor $\mathcal{O}_G^{\text{op}} \longrightarrow \mathbb{k}\text{-mod}$.

As the name suggests, the coefficients in an integer-graded ordinary equivariant cohomology theory are coefficient systems. One way to define Bredon cohomology on G -CW-complexes is as follows.

Definition 2.4. A **G -CW-complex with trivial cells** X is a colimit of n -skeleta X_n , formed as follows. X_0 is a discrete G -set; X_{n+1} is formed from X_n by attaching cells of the form $G/K \times D^{n+1}$ along boundary G -maps $G/K \times S^n \longrightarrow X_n$. Here D^{n+1} and S^n have the trivial G action, and K denotes a subgroup of G ; different cells may have different K .

A G -CW-complex with trivial cells X may be thought of as an ordinary CW-complex equipped with a cellular G action $G \times X \rightarrow X$ such that, for each n -cell D^n and $g \in G$, the action map $\varphi_g: x \mapsto gx$ either fixes D^n pointwise or gives a homeomorphism from D^n to a distinct second n -cell.

We will later generalize this notion to G -CW-complexes where the cells are allowed to have nontrivial G actions.

Example 2.5. Let $G = C_2$, the cyclic group with two elements, acting on $X = S^2$ by rotation by 180° . We can give this a G -CW structure as follows.

- $X_0 = C_2/C_2 \times S^0$, i.e. S^0 with the trivial G action.
- X_1 is given by attaching the single cell $C_2/\{e\} \times D^1$ along the map

$$C_2/\{e\} \times S^0 \longrightarrow S^0: (g, x) \mapsto x.$$

X_1 is thus G -homeomorphic to the unit circle in the complex plane \mathbb{C} , with C_2 acting by complex conjugation.

- Finally, $X = X_2$ is given by attaching the single two-cell $C_2/\{e\} \times D^2$ along the obvious map $C_2/\{e\} \times S^1 \rightarrow X_1$.

Now recall that nonequivariant cellular cohomology of a CW-complex X (which agrees with singular cohomology) is defined as the cohomology of the **cellular chain complex**. This has

$$C_n^{\text{cell}}(X) = H_n(X_n, X_{n-1}; \mathbb{k}).$$

We can use this to define a cellular chains coefficient system in the equivariant situation.

Definition 2.6. The n^{th} **cellular chains coefficient system** for a G -space X is the functor $\underline{C}_n^{\text{cell}}: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ given on objects by

$$G/K \mapsto C_n^{\text{cell}}(X^K).$$

A map $\alpha: G/K_1 \rightarrow G/K_2$ given by $\alpha: eK_1 \mapsto g_\alpha K_2$ defines a map

$$X^{K_2} \rightarrow X^{K_1}: x \mapsto g_\alpha x.$$

We use this to define the required map $C_n^{\text{cell}}(X^{K_2}) \rightarrow C_n^{\text{cell}}(X^{K_1})$ on morphisms.

Given a coefficient system M , we can then look at the set of natural transformations $\text{Hom}_{\mathcal{O}_G}(\underline{C}_n^{\text{cell}}(X), M)$ for each n . This has the structure of a \mathbb{k} -module. Further, $C_*^{\text{cell}}(X^K)$ is a chain complex. This gives $\underline{C}_*^{\text{cell}}(X)$ the structure of a chain complex in the category of coefficient systems, and makes $\text{Hom}(\underline{C}_*^{\text{cell}}(X), M)$ a (co)chain complex in the category of \mathbb{k} -modules.

Definition 2.7. Given a group G , a G -CW-complex X , and a coefficient system $M: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, the unreduced **Bredon cohomology** $H_G^*(X; M)$ of X with coefficients in M is the cohomology of the chain complex of \mathbb{k} -modules

$$\text{Hom}_{\mathcal{O}_G}(\underline{C}_*^{\text{cell}}(X), M).$$

As usual, the reduced Bredon cohomology of a based space X with basepoint x_0 is the cohomology of the pair $H_G^*(X, x_0; M)$.

It can be checked that each theory $\tilde{H}_G^*(-; M)$ is ordinary: for each $K < G$,

$$\tilde{H}_G^n(G/K_+; M) = \begin{cases} 0 & n \in \mathbb{Z}, n \neq 0 \\ M(G/K) & n = 0. \end{cases}$$

Remark 2.8. The notation $\text{Hom}_{\mathcal{O}_G}$ is used because natural transformations are the “hom of functors.” There is also a notion of a tensor product of functors $\otimes_{\mathcal{O}_G}$, defined using a coend; the Bredon *homology* with coefficients in some covariant coefficient system $N: \mathcal{O}_G \rightarrow \mathbb{k}\text{-mod}$ is the homology of the chain complex with n^{th} term

$$\underline{C}_n^{\text{cell}}(X) \otimes_{\mathcal{O}_G} N = \int^{G/K \in \mathcal{O}_G} \underline{C}_n^{\text{cell}}(X)(G/K) \otimes N(G/K);$$

see [May96].

However, as mentioned above, equivariant cohomology is more naturally graded on $RO(G)^1$. This is necessary in order to have Poincaré duality, for example. We will also see later in this thesis that the cohomology of complex projective spaces and Grassmannians is free (in the appropriate sense) as a module over the cohomology of a point for $RO(G)$ -graded cohomology, but not for integer-graded cohomology.

The following theorem describes when the integer-graded theory can be extended to an $RO(G)$ -graded theory. Mackey functors will be defined in Section 2.2.

Theorem 2.9 (Lewis, May, McClure). *For a group G and $M: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, the ordinary integer-graded cohomology theory $\tilde{H}_G^*(-; M)$ extends to an $RO(G)$ -grading if and only if M is the underlying contravariant coefficient system of a Mackey functor. Likewise, if $N: \mathcal{O}_G \rightarrow \mathbb{k}\text{-mod}$ is a covariant coefficient system, Bredon homology $H_*^G(-; N)$ with coefficients in N extends to an $RO(G)$ grading if and only if N is the underlying coefficient system of a Mackey functor.*

From now on, we will turn our attention from coefficient systems to Mackey functors.

¹Recall our nonstandard definition of $RO(G)$ as the free abelian group generated by the irreducible representations of G .

2.2 A review of Mackey functors

Recall that G always denotes a finite group.

Definition 2.10. The **Burnside category** \mathcal{B}_G is the full subcategory of the equivariant stable category on objects $\Sigma^\infty(\mathbf{b}_+)$, where \mathbf{b} is a finite G -set. Explicitly, the objects of \mathcal{B}_G are finite G -sets and the morphisms are the stable G -maps.

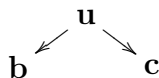
Definition 2.11. A **Mackey functor** is a contravariant additive functor from the Burnside category \mathcal{B}_G to the category $\mathbb{k}\text{-mod}$.

For obvious reasons, \mathcal{B}_G is sometimes called the “stable orbit category” and Mackey functors “stable coefficient systems.” The expansion from orbits to finite G -sets, which are disjoint unions of orbits, is largely for later convenience; since Mackey functors are additive functors, they are determined by their values on the orbits.

Although Definition 2.10 makes the connection between \mathcal{O}_G and \mathcal{B}_G clear, it is generally easier to work with the following combinatorial definition. It is shown in [May96] that the two definitions are equivalent.

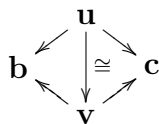
Definition 2.12. The category \mathcal{B}_G^+ is the category having:

- objects: the finite G -sets
- morphisms: equivalence classes of spans

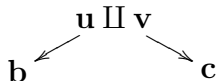


with composition given by pullbacks.

Two spans $\mathbf{b} \longleftarrow \mathbf{u} \longrightarrow \mathbf{c}$ and $\mathbf{b} \longleftarrow \mathbf{v} \longrightarrow \mathbf{c}$ are equivalent if there is a commutative diagram as follows:



Each hom set $\mathcal{B}_G^+(\mathbf{b}, \mathbf{c})$ has the structure of an abelian monoid. We may take the sum of $\mathbf{b} \longleftarrow \mathbf{u} \longrightarrow \mathbf{c}$ and $\mathbf{b} \longleftarrow \mathbf{v} \longrightarrow \mathbf{c}$ to be the span



We may thus apply the Grothendieck construction to the hom sets of \mathcal{B}_G^+ to get an abelian group.

For convenience in the remainder of this thesis, we will additionally enrich \mathcal{B}_G over $\mathbb{k}\text{-mod}$, rather than abelian groups. Doing so does not change the Mackey functors $\mathcal{B}_G^{\text{op}} \longrightarrow \mathbb{k}\text{-mod}$, but it will make the notation for represented functors less cumbersome.

Definition 2.13. The **Burnside category** \mathcal{B}_G is the category enriched over $\mathbb{k}\text{-mod}$ having:

- objects: the objects of \mathcal{B}_G^+
- morphisms: $\mathcal{B}_G(\mathbf{b}, \mathbf{c})$ is the tensor product (over \mathbb{Z}) of the ground ring \mathbb{k} with the Grothendieck group of $\mathcal{B}_G^+(\mathbf{b}, \mathbf{c})$.

Mackey functors form a category \mathcal{M} , with morphisms given by natural transformations of functors $\mathcal{B}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$. Explicitly, this means that a map of Mackey functors $M \rightarrow N$ consists of compatible maps $M(\mathbf{b}) \rightarrow N(\mathbf{b})$ for every finite G -set \mathbf{b} . Since Mackey functors are additive functors, a Mackey functor M over a commutative ring \mathbb{k} is determined by its values on the orbits G/K , and likewise a map $M \rightarrow N$ of Mackey functors is determined by the maps $M(G/K) \rightarrow N(G/K)$.

The Burnside category \mathcal{B}_G has several nice properties which will be useful when simplifying the box product formula in Subsection 2.2.2. Both are self-evident as long as Definition 2.13 is used.

Lemma 2.14. \mathcal{B}_G is self-dual, via the functor D which is the identity on objects and takes a span to its mirror image. □

Additionally, the monoidal product \times makes \mathcal{B}_G a closed symmetric monoidal category—and the internal hom is *also* \times . Note, however, that although \times denotes the Cartesian product in \mathcal{B}_G , it is not the categorical product.

Lemma 2.15. Let D be the self-duality functor of Lemma 2.14. Then, for any $\mathbf{b} \in \mathcal{B}_G$, the functor $- \times \mathbf{b}$ is left adjoint to $\mathbf{b} \times -$:

$$\mathcal{B}_G(\mathbf{a} \times \mathbf{b}, \mathbf{c}) \cong \mathcal{B}_G(\mathbf{a}, \mathbf{b} \times \mathbf{c}),$$

naturally in \mathbf{a} and \mathbf{c} . It is also natural in \mathbf{b} if we correct variance issues by replacing one \mathbf{b} with $D\mathbf{b}$. □

2.2.1 Connection to coefficient systems

The (unstable) orbit category embeds contravariantly and covariantly in \mathcal{B}_G . Both embeddings are the identity on objects; a map of G -sets $G/L \xrightarrow{\alpha} G/K$ is sent to either

$$\begin{array}{ccc}
 & G/L & \\
 \alpha \swarrow & & \searrow \\
 G/K & & G/L
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 & G/L & \\
 & \searrow \alpha & \\
 G/L & & G/K
 \end{array}$$

as appropriate. Furthermore, every morphism in \mathcal{B}_G can be written as a composite of such morphisms:

$$\begin{array}{ccc}
 & \mathbf{u} & \\
 \swarrow & & \searrow \\
 \mathbf{b} & & \mathbf{c}
 \end{array}
 =
 \begin{array}{ccccc}
 & \mathbf{u} & & \mathbf{u} & \\
 \swarrow & & \searrow & \swarrow & \searrow \\
 \mathbf{b} & & \mathbf{u} & & \mathbf{c}
 \end{array}$$

Furthermore, a general span breaks up as the disjoint union of spans involving orbits; disjoint union is both the categorical product and the categorical coproduct on \mathcal{B}_G . Hence a Mackey functor M determines and is determined by a pair of contravariant and covariant coefficient systems which agree on objects and which satisfy certain compatibility diagrams encoding the composition in \mathcal{B}_G .

These embeddings $\mathcal{O}_G \rightarrow \mathcal{B}_G$ come up frequently, and so we fix some notation.

Definition 2.16. Let $f : \mathbf{b} \rightarrow \mathbf{c}$ be a map of finite G -sets. Then the two spans

$$\begin{array}{c} \mathbf{b} \\ \parallel \swarrow f \\ \mathbf{b} \quad \mathbf{c} \end{array} \quad \text{and} \quad \begin{array}{c} \mathbf{b} \\ f \swarrow \parallel \\ \mathbf{c} \quad \mathbf{b} \end{array}$$

are called the **restriction** and **transfer**, respectively, of f . If a Mackey functor M is implicit, denote its value on the restriction by $r_f : M(\mathbf{c}) \rightarrow M(\mathbf{b})$ and its value on the transfer by $t_f : M(\mathbf{b}) \rightarrow M(\mathbf{c})$.

The notation comes from the language of representation theory, one of the first places where Mackey functors were studied. Alternatively, if we view \mathcal{B}_G as a subcategory of the stable category using Definition 2.10, restrictions and transfers correspond exactly to the maps of those names on representation spheres.

We are now in a position to understand the statement of Theorem 2.9, which says that equivariant homology and cohomology theories can be extended to $RO(G)$ -gradings if and only if they have coefficient systems which extend to Mackey functors.

The subsequent subsections will address the issue of which Mackey functor ought to be taken as the “universal” coefficient Mackey functor for cohomology.

2.2.2 Box products and maps out of box products

The category \mathcal{M} of Mackey functors is symmetric monoidal, with tensor product \square given by the Day tensor product². Explicitly, \square is a left Kan extension. Given Mackey functors $M, N : \mathcal{B}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, we can form the external product

$$\begin{aligned} M \bar{\square} N : \mathcal{B}_G^{\text{op}} \times \mathcal{B}_G^{\text{op}} &\rightarrow \mathbb{k}\text{-mod} \\ (\mathbf{b}, \mathbf{c}) &\mapsto M(\mathbf{b}) \otimes N(\mathbf{c}), \end{aligned}$$

and $M \square N$ is the left Kan extension of $M \bar{\square} N$ along the Cartesian product functor $\times : \mathcal{B}_G \times \mathcal{B}_G \rightarrow \mathcal{B}_G$:

$$\begin{array}{ccc} \mathcal{B}_G \times \mathcal{B}_G & \xrightarrow{M \bar{\square} N} & \mathbb{k}\text{-mod} \\ \times \downarrow & \nearrow M \square N & \\ \mathcal{B}_G & & \end{array}$$

²In the same way, if graded \mathbb{k} -modules are thought of as functors from the discrete category \mathbb{Z} to $\mathbb{k}\text{-mod}$, the usual notion of a tensor product of graded \mathbb{k} -modules is exactly the Day product on the functor category $[\mathbb{Z}, \mathbb{k}\text{-mod}]$.

In other words, natural transformations from $M \square N$ to another Mackey functor P are the same as natural transformations from $M \bar{\square} N$ to $P \circ \times$:

$$[\mathcal{B}_G^{\text{op}}, \mathbb{k}\text{-mod}](M \square N, P) \cong [\mathcal{B}_G^{\text{op}} \times \mathcal{B}_G^{\text{op}}, \mathbb{k}\text{-mod}](M \bar{\square} N, P \circ \times).$$

It follows that a map $M \square N \rightarrow P$ is specified by giving maps

$$M(\mathbf{b}) \otimes N(\mathbf{c}) \rightarrow P(\mathbf{b} \times \mathbf{c}),$$

for all $\mathbf{b}, \mathbf{c} \in \mathcal{B}_G$, natural with respect to \mathbf{b} and \mathbf{c} . In fact, we can get away with less.³

Lemma 2.17. *A map $\theta: M \square N \rightarrow P$ determines and is determined by a collection of maps*

$$\theta_{\mathbf{b}}: M(\mathbf{b}) \otimes N(\mathbf{b}) \rightarrow P(\mathbf{b})$$

for $\mathbf{b} \in \mathcal{B}_G$, such that the following diagrams commute for each map of G -sets $f: \mathbf{b} \rightarrow \mathbf{c}$.⁴

$$\begin{array}{ccc} M(\mathbf{c}) \otimes N(\mathbf{c}) & \xrightarrow{\theta_{\mathbf{c}}} & P(\mathbf{c}) \\ r_f \otimes r_f \downarrow & & \downarrow r_f \\ M(\mathbf{b}) \otimes N(\mathbf{b}) & \xrightarrow{\theta_{\mathbf{b}}} & P(\mathbf{b}) \end{array} \quad \begin{array}{ccc} & M(\mathbf{b}) \otimes N(\mathbf{b}) & \xrightarrow{\theta_{\mathbf{b}}} & P(\mathbf{b}) \\ & \text{id} \otimes r_f \nearrow & & \downarrow t_f \\ M(\mathbf{b}) \otimes N(\mathbf{c}) & & & \\ & t_f \otimes \text{id} \searrow & & \\ & M(\mathbf{c}) \otimes N(\mathbf{c}) & \xrightarrow{\theta_{\mathbf{c}}} & P(\mathbf{c}) \end{array}$$

$$\begin{array}{ccc} & M(\mathbf{b}) \otimes N(\mathbf{b}) & \xrightarrow{\theta_{\mathbf{b}}} & P(\mathbf{b}) \\ & r_f \otimes \text{id} \nearrow & & \downarrow t_f \\ M(\mathbf{c}) \otimes N(\mathbf{b}) & & & \\ & \text{id} \otimes t_f \searrow & & \\ & M(\mathbf{c}) \otimes N(\mathbf{c}) & \xrightarrow{\theta_{\mathbf{c}}} & P(\mathbf{c}) \end{array}$$

Definition 2.18. The data in Lemma 2.17 is called a **Dress pairing**, after [Dre73]. When the commuting diagrams above are expressed in terms of equations and elements, they are called the **Frobenius relations**.

In particular, notice that the maps $\theta_{\mathbf{b}}$ are natural with respect to restrictions, but not generally with respect to transfers! However, when f is an isomorphism, it is straightforward to check that the transfer associated to f is equivalent, as a span in \mathcal{B}_G , to the restriction associated to f^{-1} . It then follows that the appropriate map $\theta_{\mathbf{b}}$ is natural with respect to t_f , and the commuting pentagons become redundant.

Proof. To go from a map $\theta: M \square N \rightarrow P$ to a Dress pairing, consider the maps

$$\theta_{\mathbf{b}}: M(\mathbf{b}) \otimes N(\mathbf{b}) \rightarrow P(\mathbf{b} \times \mathbf{b}) \xrightarrow{r_{\Delta}} P(\mathbf{b}),$$

³The fact that Dress pairings are the same as maps out of box products is stated in [Lew80] and [Lew88], but without any indication of a proof. The proof is formal, of course, but not entirely obvious.

⁴Recall the notation for restrictions and transfers from Definition 2.16.

where the first map comes from the Kan adjunction and r_Δ is the restriction associated to the diagonal map of G -sets $\mathbf{b} \rightarrow \mathbf{b} \times \mathbf{b}$. We then get the commuting diagrams above by applying P to commuting diagrams in \mathcal{B}_G .

For the other direction, given the maps $\theta_{\mathbf{b}}$ subject to the three commuting diagrams of the lemma, we obtain the maps making up $\theta : M \square N \rightarrow P$ via the composites

$$M(\mathbf{b}) \otimes N(\mathbf{c}) \xrightarrow{r_{\rho_1} \otimes r_{\rho_2}} M(\mathbf{b} \times \mathbf{c}) \otimes N(\mathbf{b} \times \mathbf{c}) \xrightarrow{\theta_{\mathbf{b} \times \mathbf{c}}} P(\mathbf{b} \times \mathbf{c}),$$

where r_{ρ_1} and r_{ρ_2} are the restrictions associated with the projections of $\mathbf{b} \times \mathbf{c}$ onto its first and second factors.

We need to show that these maps are natural with respect to maps in $\mathcal{B}_G \times \mathcal{B}_G$. Since every span can be written as the composite of a restriction and a transfer, it suffices to check naturality with respect to restrictions and transfers. Naturality with respect to restrictions follows immediately from the commuting square in the lemma. For transfers, given maps $f : \mathbf{b} \rightarrow \mathbf{b}'$ and $g : \mathbf{c} \rightarrow \mathbf{c}'$, the map $t_f \otimes t_g : M(\mathbf{b}) \otimes N(\mathbf{c}) \rightarrow M(\mathbf{b}') \otimes N(\mathbf{c}')$ factors as

$$M(\mathbf{b}) \otimes N(\mathbf{c}) \xrightarrow{t_f \otimes \text{id}} M(\mathbf{b}') \otimes N(\mathbf{c}) \xrightarrow{\text{id} \otimes t_g} M(\mathbf{b}') \otimes N(\mathbf{c}'),$$

so it suffices to consider naturality in the case $g = \text{id} : \mathbf{c} \rightarrow \mathbf{c}$. The desired result follows from considering the diagram

$$\begin{array}{ccccc} M(\mathbf{b}) \otimes N(\mathbf{c}) & \xrightarrow{r_{\rho_1} \otimes r_{\rho_2}} & M(\mathbf{b} \times \mathbf{c}) \otimes N(\mathbf{b} \times \mathbf{c}) & \xrightarrow{\theta_{\mathbf{b} \times \mathbf{c}}} & P(\mathbf{b} \times \mathbf{c}) \\ & \searrow^{r_{\rho_1} \otimes r_{\rho_2}} & \nearrow^{\text{id} \otimes r_f \times \text{id}} & & \downarrow t_f \times \text{id} \\ & & M(\mathbf{b} \times \mathbf{c}) \otimes N(\mathbf{b}' \times \mathbf{c}) & & \\ & \swarrow_{t_f \otimes \text{id}} & \searrow_{t_f \times \text{id} \otimes \text{id}} & & \\ M(\mathbf{b}') \otimes N(\mathbf{c}) & \xrightarrow{r_{\rho_1} \otimes r_{\rho_2}} & M(\mathbf{b}' \times \mathbf{c}) \otimes N(\mathbf{b}' \times \mathbf{c}) & \xrightarrow{\theta_{\mathbf{b}' \times \mathbf{c}}} & P(\mathbf{b}' \times \mathbf{c}) \end{array}$$

The pentagon commutes by definition of a Dress pairing; the upper triangle comes from applying N to a commuting triangle in \mathcal{B}_G , thus commutes; and the lower left quadrilateral comes from applying M and N to commuting diagrams in \mathcal{B}_G , hence commutes as well. It follows that the outer rectangle commutes, giving the desired naturality.

It is straightforward to check that, if the maps $\theta_{\mathbf{b}} : M(\mathbf{b}) \otimes N(\mathbf{b}) \rightarrow P(\mathbf{b})$ came from a natural transformation $M \square N \rightarrow P \circ \times$, then the construction described above returns the original maps $M(\mathbf{b}) \otimes N(\mathbf{c}) \rightarrow P(\mathbf{b} \times \mathbf{c})$, and vice versa. \square

2.2.3 Explicit formula for the box product

There is also a general formula for left Kan extensions in terms of a coend [ML98], a particular type of colimit. In our situation, it simplifies to give an explicit formula for \square in terms of coends; coends of the form appearing below can be thought of as “tensor products of functors.”

Definition 2.19. Let \mathcal{C} be a small category, and let $F: \mathcal{C}^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ be a contravariant and $G: \mathcal{C} \rightarrow \mathbb{k}\text{-mod}$ a covariant functor. The **coend** $\int^{\mathbf{x} \in \mathcal{C}} F(\mathbf{x}) \otimes G(\mathbf{x})$, sometimes written as the tensor product of functors $F \otimes_{\mathcal{C}} G$, is the coequalizer of the diagram of \mathbb{k} -modules

$$\bigoplus_{f: \mathbf{x} \rightarrow \mathbf{y}} F(\mathbf{y}) \otimes G(\mathbf{x}) \begin{array}{c} \xrightarrow{F(f) \otimes \text{id}} \\ \xrightarrow{\text{id} \otimes G(f)} \end{array} \bigoplus_{\mathbf{z} \in \mathcal{C}} F(\mathbf{z}) \otimes G(\mathbf{z}).$$

That is, for each pair of objects \mathbf{x}, \mathbf{y} of \mathcal{C} , and for each morphism $f: \mathbf{x} \rightarrow \mathbf{y}$ in \mathcal{C} , we have a component $F(\mathbf{y}) \otimes G(\mathbf{x})$ in the left-hand direct sum. The upper arrow on this component $F(\mathbf{y}) \otimes G(\mathbf{x})$ is the composite

$$F(\mathbf{y}) \otimes G(\mathbf{x}) \xrightarrow{F(f) \otimes \text{id}} F(\mathbf{x}) \otimes G(\mathbf{x}) \xrightarrow{\text{inclusion}} \bigoplus_{\mathbf{z} \in \mathcal{C}} F(\mathbf{z}) \otimes G(\mathbf{z}),$$

while the lower arrow is

$$F(\mathbf{y}) \otimes G(\mathbf{x}) \xrightarrow{\text{id} \otimes G(f)} F(\mathbf{y}) \otimes G(\mathbf{y}) \xrightarrow{\text{inclusion}} \bigoplus_{\mathbf{z} \in \mathcal{C}} F(\mathbf{z}) \otimes G(\mathbf{z}).$$

The coend $\int^{\mathbf{x} \in \mathcal{C}} F(\mathbf{x}) \otimes G(\mathbf{x})$ may thus be thought of as a quotient of $\bigoplus_{\mathbf{z} \in \mathcal{C}} F(\mathbf{z}) \otimes G(\mathbf{z})$ by appropriate equivalence relations induced by the morphisms of \mathcal{C} .

Lemma 2.20. *Let M, N be Mackey functors, and let D be the self-duality functor of Lemma 2.14. Then the value of $M \square N$ at \mathbf{c} is given by a coend*

$$(M \square N)(\mathbf{c}) \cong \int^{\mathbf{a} \in \mathcal{B}_G} M(\mathbf{a}) \otimes N(\mathbf{c} \times D\mathbf{a}).$$

For a morphism $\mathbf{b} \rightarrow \mathbf{c}$, $(M \square N)(\mathbf{c}) \rightarrow (M \square N)(\mathbf{b})$ is induced by $N(\mathbf{c} \times D\mathbf{a}) \rightarrow N(\mathbf{b} \times D\mathbf{a})$.

Proof. There is a general formula for left Kan extensions in terms of a coend [ML98]; in our situation, it gives

$$(M \square N)(\mathbf{c}) = \int^{(\mathbf{a}, \mathbf{b}) \in \mathcal{B}_G \times \mathcal{B}_G} M(\mathbf{a}) \otimes N(\mathbf{b}) \otimes \mathcal{B}_G(\mathbf{c}, \mathbf{a} \times \mathbf{b}).$$

That is, the value of $M \square N$ at \mathbf{c} is given by the tensor product of functors of $M \square N$ with $\mathcal{B}_G(\mathbf{c}, - \times -)$.

Since tensor products commute with colimits, we may rewrite the left Kan extension

formula as

$$\begin{aligned}
(M \square N)(\mathbf{c}) &= \int^{(\mathbf{a}, \mathbf{b})} M(\mathbf{a}) \otimes N(\mathbf{b}) \otimes \mathcal{B}_G(\mathbf{c}, \mathbf{a} \times \mathbf{b}) \\
&\cong \int^{\mathbf{a}} \int^{\mathbf{b}} M(\mathbf{a}) \otimes N(\mathbf{b}) \otimes \mathcal{B}_G(\mathbf{c}, \mathbf{a} \times \mathbf{b}) \\
&\cong \int^{\mathbf{a}} M(\mathbf{a}) \otimes \int^{\mathbf{b}} N(\mathbf{b}) \otimes \mathcal{B}_G(\mathbf{c}, \mathbf{a} \times \mathbf{b}) \\
&\cong \int^{\mathbf{a}} M(\mathbf{a}) \otimes \int^{\mathbf{b}} N(\mathbf{b}) \otimes \mathcal{B}_G(\mathbf{c} \times D\mathbf{a}, \mathbf{b}) \\
&\cong \int^{\mathbf{a}} M(\mathbf{a}) \otimes N(\mathbf{c} \times D\mathbf{a}).
\end{aligned}$$

The last isomorphism comes from the Yoneda lemma, which tells us that tensoring N with a representable functor $\mathcal{B}_G(\mathbf{c} \times D\mathbf{a}, -)$ is the same as evaluating $N(\mathbf{c} \times D\mathbf{a})$.

The naturality of coends gives the final statement. \square

Remark 2.21. It is immediate that the coend in the general formula for left Kan extensions satisfies the desired universal property. Note, however, that Lemma 2.17, which introduces Dress pairings, does not immediately follow from the coend in Lemma 2.20.

Using the fact that tensoring with a represented functor is the same as evaluating at the representing object, Lemma 2.20 shows that the represented Mackey functor $\mathcal{B}_G(-, G/G)$ is the unit for the \square monoidal structure. More generally,

$$(\mathcal{B}_G(-, \mathbf{b}) \square M)(\mathbf{c}) = M(\mathbf{b} \times \mathbf{c});$$

this example will be discussed further in Subsection 2.2.6. The functor represented by G/G is usually called the **Burnside ring Mackey functor**.

Definition 2.22. The **Burnside ring Mackey functor** ${}_G A$, abbreviated A when the group G is implicit, is the represented functor $\mathcal{B}_G(-, G/G)$ ⁵.

Lemma 2.23. $(\mathcal{B}_G, A, \square)$ is a symmetric monoidal category. \square

Remark 2.24. For any Mackey functor M , the canonical isomorphism $A \square M \rightarrow M$ is adjoint to the map $A \bar{\square} M \rightarrow M \circ \times$ which takes $A(\mathbf{b}) \otimes M(\mathbf{c}) \rightarrow M(\mathbf{b} \times \mathbf{c})$ via $f \otimes x \mapsto (f \times \text{id})^* x$, for a span f in $\mathcal{B}_G(\mathbf{b}, G/G)$.

⁵Although we have enriched \mathcal{B}_G over $\mathbb{k}\text{-mod}$, it is more standard to enrich it over abelian groups. In that case, the Burnside ring Mackey functor is given by $\mathbb{k} \otimes \mathcal{B}_G(-, G/G)$.

2.2.4 Graded Mackey functors

In what follows, it will often be convenient to consider the category \mathcal{M}^* of $RO(G)$ -**graded Mackey functors**. As expected, a graded Mackey functor, written M^* or M_* depending on context, consists of a Mackey functor M^α (or M_α) for each $\alpha \in RO(G)$. The Day tensor product then allows us to define a graded box product \square_* from the ungraded box product \square . It can be explicitly described by

$$(M^* \square_* N^*)^\alpha \cong \bigoplus_{\beta_1 + \beta_2 = \alpha} (M^{\beta_1} \square N^{\beta_2}).$$

\square_* makes \mathcal{M}^* into a symmetric monoidal category. The unit is the graded Mackey functor A^* which has $A^0 = A$, the Burnside ring Mackey functor, and $A^\alpha = 0$ for all $\alpha \neq 0$.

2.2.5 Green functors and modules over Green functors

The monoidal structure gives rise to the notion of monoids in \mathcal{M} . The monoids are Mackey functors T together with a “product map” $\mu: T \square T \rightarrow T$ and a unit map $\eta: A \rightarrow T$ satisfying the usual unit and associativity commuting diagrams⁶:

$$\begin{array}{ccc} A \square T & \xrightarrow{\eta \square \text{id}} & T \square T & \xleftarrow{\text{id} \square \eta} & T \square A & \text{and} & T \square T \square T & \xrightarrow{\mu \square \text{id}} & T \square T \\ & \searrow \cong & \downarrow \mu & & \swarrow \cong & & \text{id} \square \mu \downarrow & & \downarrow \mu \\ & & T & & & & T \square T & \xrightarrow{\mu} & T \end{array}$$

These monoids are known as **Mackey functor rings** or **Green functors**. The same definition, with T replaced by a graded Mackey functor T^* and \square replaced by \square_* , gives the notion of a **graded Green functor**.

Remark 2.25. Lemma 2.17 tells us that specifying a product map $T \square T \rightarrow T$ is the same as giving maps $T(\mathbf{b}) \otimes T(\mathbf{b}) \rightarrow T(\mathbf{b})$ for each G -set \mathbf{b} , satisfying the commuting square and two commuting pentagons describing naturality with respect to maps in \mathcal{B}_G . Further, by the Yoneda lemma, the unit map $\eta: A \rightarrow T$ is determined by the image of the identity map $e := \eta_\bullet(\text{id}) \in T(\bullet)$, where \bullet is the terminal G -set, $\bullet = G/G$. For each G -set \mathbf{b} , we can consider the image $e_{\mathbf{b}} := r_{\mathbf{b}}(e)$ of e under the restriction $r_{\mathbf{b}}$ associated to the projection $\mathbf{b} \rightarrow \bullet$. Then the Dress maps $T(\mathbf{b}) \otimes T(\mathbf{b}) \rightarrow T(\mathbf{b})$ make $T(\mathbf{b})$ into a ring with unit $e_{\mathbf{b}}$. This follows from two facts. First, the adjoint to the unit diagram for the Green functor T

⁶If $f: M_1 \rightarrow M_2$ and $g: N_1 \rightarrow N_2$, then $f \square g: M_1 \square N_1 \rightarrow M_2 \square N_2$ is the natural transformation with components $f_{\mathbf{b}} \otimes g_{\mathbf{c}}: M_1(\mathbf{b}) \otimes N_1(\mathbf{c}) \rightarrow M_2(\mathbf{b}) \otimes N_2(\mathbf{c})$. We may then use the naturality in all variables of the adjunction

$$[\mathcal{B}_G^{\text{op}}, \mathbb{k}\text{-mod}](M \square N, P) \cong [\mathcal{B}_G^{\text{op}} \times \mathcal{B}_G^{\text{op}}, \mathbb{k}\text{-mod}](M \square N, P \circ \times)$$

to obtain a map $f \square g: M_1 \square N_1 \rightarrow M_2 \square N_2$.

is the commuting triangle of functors $\mathcal{B}_G^{\text{op}} \times \mathcal{B}_G^{\text{op}} \longrightarrow \mathbb{k}\text{-mod}$

$$\begin{array}{ccc} A \square T & \xrightarrow{\eta \square \text{id}} & T \square T \\ & \searrow & \downarrow \bar{\mu} \\ & & T \circ \times \end{array}$$

Using Remark 2.24, we see that $A \square T \longrightarrow T \circ \times$ takes $\text{id} \square x \mapsto x$ for any $x \in T(\mathbf{b})$. It follows that $\eta_{\bullet}(\text{id}) \square x = e \square x \mapsto x$ as well. We may then consider the diagram below.

$$\begin{array}{ccc} T(\bullet) \otimes T(\mathbf{b}) & \xrightarrow{\bar{\mu}} & T(\mathbf{b}) \\ r_{\mathbf{b}} \otimes \text{id} \downarrow & & r_{\text{proj}} \downarrow \\ T(\mathbf{b}) \otimes T(\mathbf{b}) & \xrightarrow{\bar{\mu}} & T(\mathbf{b} \times \mathbf{b}) \xrightarrow{r_{\Delta}} T(\mathbf{b}) \end{array}$$

The square commutes because $T \square T \longrightarrow T \circ \times$ is a natural transformation; the triangle comes from applying T to a commuting diagram in \mathcal{B}_G . Since the bottom composite is the Dress pairing, it follows that $e_{\mathbf{b}}x = x$ under this pairing.

Remark 2.26. The unit isomorphism $A \square A \longrightarrow A$ makes the Burnside ring Mackey functor A into a Green functor. It gets its name from the fact that, when $\mathbb{k} = \mathbb{Z}$, and taking into consideration the levelwise ring structure coming from the Dress pairing, its value $A(G/K)$ at the orbit G/K is the classical Burnside ring $A(K)$. Likewise, A^* is a graded Green functor.

Once we have monoids in \mathcal{M} and \mathcal{M}^* , it is a short step to modules over the monoids. If T is a Green functor, then a T -module M is a Mackey functor together with an action map $\phi: T \square M \longrightarrow M$ satisfying the expected diagrams.

$$\begin{array}{ccc} A \square M & \xrightarrow{\eta \square \text{id}} & T \square M \\ & \searrow \cong & \downarrow \phi \\ & & M \end{array} \quad \text{and} \quad \begin{array}{ccc} T \square T \square M & \xrightarrow{\mu \square \text{id}} & T \square M \\ \text{id} \square \phi \downarrow & & \downarrow \phi \\ T \square M & \xrightarrow{\phi} & M \end{array}$$

By definition, every Mackey functor is a module over A , just as every abelian group is a \mathbb{Z} -module, and every graded Mackey functor is a module over A^* .

Finally, we say that a Green functor T is **commutative** if the multiplication $\mu: T \square T \longrightarrow T$ satisfies the expected diagram

$$\begin{array}{ccc} T \square T & \xrightarrow{\text{switch}} & T \square T \\ & \searrow \mu & \swarrow \mu \\ & & T \end{array}$$

When we talk about “commutative” *graded* Green functors, we will mean “graded commutative,” because cohomology theories give graded-commutative graded Green functors. Recall that a nonequivariant graded ring is graded commutative if, for any two elements x and y , $yx = (-1)^{\dim(x)\dim(y)}xy$. The relevance of $(-1)^{\dim(x)\dim(y)}$ here is that ± 1 are exactly the units of the ring \mathbb{Z} , the universal coefficients for ordinary nonequivariant cohomology. Following the analogous argument for equivariant cohomology, we see that changing the order of multiplication in a graded Green functor $\tilde{H}_G^*(X; M)$ must induce a “sign,” coming from the degree of a certain homotopy equivalence between representation spheres. This degree is necessarily a unit in the Burnside Green functor A^* . We will defer an explicit description of signs until later, and make the following definition. A graded Green functor T^* is **commutative** if the diagram below commutes for every $\alpha, \beta \in RO(G)$. Here $u(\alpha, \beta)$ is a unit in the Burnside Green functor A^* which depends only on α and β .

$$\begin{array}{ccc} T^\alpha \square T^\beta & \xrightarrow{\text{switch}} & T^\beta \square T^\alpha \\ \mu \downarrow & & \mu \downarrow \\ T^{\alpha+\beta} & \xrightarrow{u(\alpha, \beta)} & T^{\alpha+\beta} \end{array}$$

We will later discuss the signs $u(\alpha, \beta)$ in more detail for $G = C_p$.

Since A is the unit for \square , it is the analogue of \mathbb{Z} in abelian groups and the natural choice of coefficient Green functor for cohomology. In what follows, when coefficients are not specified, Burnside ring coefficients are intended.

2.2.6 Building new Mackey functors

Various ways of producing new Mackey functors from old, or from objects in other categories, will recur throughout the course of this thesis, and so deserve some special mention now.

First, \mathcal{M} is tensored over $\mathbb{k}\text{-mod}$; given a Mackey functor M and a \mathbb{k} -module X , the Mackey functor $M \otimes X$ has $(M \otimes X)(\mathbf{b}) = M(\mathbf{b}) \otimes X$ with the appropriate morphisms.

Definition 2.27. Given a Mackey functor M and $\mathbf{b} \in \mathcal{B}_G$, the **shifted** Mackey functor $M_{\mathbf{b}}$ is the Mackey functor given on objects by

$$M_{\mathbf{b}}(\mathbf{c}) = M(\mathbf{b} \times \mathbf{c}),$$

with the evident value on morphisms. Alternatively, keeping in mind the comment after Lemma 2.20, we could define the shift of M by \mathbf{b} to be the Mackey functor $M \square A_{\mathbf{b}}$.

Now consider a Mackey functor M . For each subgroup $K < G$, the automorphism group of the G -set G/K is the Weyl group $W_G K = N_G K / K$. This acts on the left on G/K , viewed as an object of the Burnside category via the covariant embedding of \mathcal{O}_G into \mathcal{B}_G . It follows that the \mathbb{k} -module $M(G/K)$ inherits the structure of a $\mathbb{k}[W_G K]$ -module. We therefore have a forgetful functor $\mathcal{M} \rightarrow \mathbb{k}[W_G K]\text{-mod}$ for each subgroup K . When $K = \{e\}$, the trivial subgroup, it turns out that this forgetful functor has both left and right

adjoints, which we will not describe explicitly here; explicit descriptions for the simplified case $G = C_p$ will be given later.

Definition 2.28. We write \mathcal{L} for the left adjoint to the forgetful functor $M \mapsto M(G/\{e\})$, and \mathcal{R} for the right adjoint.

Remark 2.29. For $K \neq \{e\}$, we can generalize the construction of the right adjoint \mathcal{R} . The generalized version will only become an adjoint after restricting to a certain subcategory of \mathcal{M} , but the construction is frequently useful. This is discussed in detail in unpublished works of Gaunce Lewis, such as [Lew80], where the right adjoint is called $J_{G/K}$.

2.2.7 Free Mackey functors

The question of which Mackey functors modules over a given Mackey functor ring deserve to be called “free” will be very important later, when we start considering whether various cohomology groups are free. Certainly any object called free should be projective, but not every projective is free.

In a functor category, it is usual to define the free objects to be coproducts of the representable functors. The Yoneda lemma confirms that these are projective.

Proposition 2.30. *Let \mathcal{C} be a category enriched over $\mathbb{k}\text{-mod}$. Then for any object $\mathbf{b} \in \mathcal{C}$, the represented functor $A_{\mathbf{b}} = \mathcal{C}(-, \mathbf{b})$ is a projective functor $\mathcal{C}^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$.*

Proof. By the Yoneda lemma, for any functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, a map $A_{\mathbf{b}} \rightarrow F$ is the same data as an element of $F(\mathbf{b})$. The correspondence is given by looking at the image of the identity map in $\mathcal{C}(\mathbf{b}, \mathbf{b})$.

To show that $A_{\mathbf{b}}$ is projective, we must show that any levelwise surjection $\phi: F \rightarrow A_{\mathbf{b}}$ admits a section $A_{\mathbf{b}} \rightarrow F$. But, since $F(\mathbf{b}) \rightarrow A_{\mathbf{b}}(\mathbf{b})$ is surjective, there is some $y \in F(\mathbf{b})$ with $\phi_{\mathbf{b}}(y) = \text{id}$. The map $A_{\mathbf{b}} \rightarrow F$ defined by taking $\text{id} \mapsto y$ is then the required section. \square

Example 2.31. The category of $\mathbb{k}[G]$ -modules for a ring \mathbb{k} and group G can be viewed as the category of functors from G , viewed as a one-object category, to $\mathbb{k}\text{-mod}$. Since G has only one object \bullet , there is only one type of free object, namely the free \mathbb{k} -module on $G(-, \bullet)$, i.e. $\mathbb{k}[G]$ itself. So free objects are direct sums $\bigoplus \mathbb{k}[G]$; we know from basic algebra that every projective $\mathbb{k}[G]$ -module is a direct summand of such a free object.

It follows that the Burnside ring Mackey functor $A = \mathcal{B}_G(-, G/G)$ is free, as it should be; but now there are other types of free objects as well, namely $\mathcal{B}_G(-, G/K)$ for each subgroup $K < G$. As already mentioned, the category \mathcal{B}_G is closed monoidal under the Cartesian product \times , with internal hom also given by \times . As a result, we can give the name $A_{G/K}$ to $\mathcal{B}_G(-, G/K) = \mathcal{B}_G(-, G/K \times G/G)$ with no conflict of notation with Subsection 2.2.6.

For us, since the objects of \mathcal{B}_G are finite G -sets, a coproduct of representable Mackey functors is again representable. We may thus make the following simplified definition.

Definition 2.32. A Mackey functor M is **free** if it is of the form

$$M = A_{\mathbf{b}} = \mathcal{B}_G(-, \mathbf{b})$$

for some $\mathbf{b} \in \mathcal{B}_G$.

The free objects in \mathcal{M} also tell us what the free objects in \mathcal{M}^* should be. Writing $\Sigma^\alpha M^*$ to mean the Mackey functor with $(\Sigma^\alpha M^*)^\beta := M^{\beta-\alpha}$, the free objects in \mathcal{M}^* are coproducts of graded Mackey functors of the form $\Sigma^\alpha A_{\mathbf{b}}^*$, for $\mathbf{b} \in \mathcal{B}_G$ and $\alpha \in RO(G)$.

Finally, we will consider the question of which Mackey functor modules over a Green functor T are free. The forgetful functor $T\text{-mod} \rightarrow \mathcal{M}$ has a left adjoint $M \mapsto T \square M$. We thus make the following definition.

Definition 2.33. Let T be a Green functor, and $T\text{-mod}$ the category of Mackey functor modules over T . A **free** T -module is one of the form $T \square A_{\mathbf{b}} \cong T_{\mathbf{b}}$, for some $\mathbf{b} \in \mathcal{B}_G$.

Replacing T by T^* gives the analogous notion of a free T^* -module.

2.3 Mackey functor valued cohomology theories

At this point, we have informally described an $RO(G)$ -graded theory $\tilde{H}_G^*(-; M)$ for each Mackey functor M . However, there is still some structure arising from the G action of which we are not yet taking full advantage. Specifically, for each subgroup $K < G$, we could consider the theory $\tilde{H}_K^*(-; M|_K)$. Here $M|_K: \mathcal{B}_K^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ is the Mackey functor with values on orbits given by $M|_K(K/L) := M(G \times_K K/L) \cong M(G/L)$. For a fixed α , these theories fit together to form a Mackey functor. This gives rise to a Mackey functor valued cohomology theory. We can view this as either a collection of functors $\tilde{H}_G^\alpha(-; M): \text{ho-}G\text{-Top}_* \rightarrow \mathcal{M}$, where as before \mathcal{M} is the category of Mackey functors, or as a single functor

$$\tilde{H}_G^*(-; M): \text{ho-}G\text{-Top}_* \rightarrow \mathcal{M}^*$$

to the category of graded Mackey functors. In fact, there is a cup product on \tilde{H}_G^* , meaning that we can view it as a functor to the category of graded Green functors, i.e. monoids in \mathcal{M}^* .

The extension from the $\mathbb{k}\text{-mod}$ valued theory \tilde{H}_G^* to the Mackey functor valued theory \tilde{H}_G^* works as follows. On orbits,

$$\tilde{H}_G^*(X; M)(G/K) := \tilde{H}_G^*(G_+ \wedge_K X; M) \cong \tilde{H}_G^*(G/K_+ \wedge X; M) \cong \tilde{H}_K^*(X; M|_K).$$

On morphisms of type

$$\begin{array}{ccc} & \mathbf{b} & \\ \parallel & \searrow & \\ \mathbf{b} & & \mathbf{c} \end{array}$$

the required map $\widetilde{H}_G^*(X)(\mathbf{c}) \longrightarrow \widetilde{H}_G^*(X)(\mathbf{b})$ is induced on cohomology by the map of spaces $\mathbf{b}_+ \wedge X \longrightarrow \mathbf{c}_+ \wedge X$. On morphisms of type

$$\begin{array}{ccc} & \mathbf{b} & \\ & \swarrow & \searrow \\ \mathbf{c} & & \mathbf{b} \end{array}$$

the map $\widetilde{H}_G^*(X)(\mathbf{b}) \longrightarrow \widetilde{H}_G^*(X)(\mathbf{c})$ is induced by an appropriate transfer map $\Sigma^V \mathbf{c}_+ \longrightarrow \Sigma^V \mathbf{b}_+$ and the suspension isomorphism on cohomology. Hence \widetilde{H}_G^* ties together information about the equivariant theories \widetilde{H}_K^* for all subgroups $K < G$. Also, it makes the dimension axiom look a little cleaner.

Axiom 2.34. Mackey functor valued dimension:

$$\widetilde{H}_G^\alpha(S^0; M) = \begin{cases} M & \alpha = 0 \\ 0 & \alpha \in \mathbb{Z}, \alpha \neq 0. \end{cases}$$

For $\alpha \notin \mathbb{Z}$, $\widetilde{H}_G^\alpha(S^0; M)$ is not specified by the dimension axiom but is uniquely determined by the cohomology axioms.

Note that we need only state the axiom for $S^0 = G/G_+$: by definition, the Mackey functor $\widetilde{H}_G^*(G/K_+; M)$ is the shifted Mackey functor $\widetilde{H}_G^*(S^0; M) \square A_{G/K}$.

2.3.1 The equivariant stable category

We could alternatively approach Mackey functor valued cohomology from a stable point of view. It is known that the homotopy category of genuine⁷ G -spectra is enriched over the category of Mackey functors. Specifically, given genuine G -spectra D and E , we can define a Mackey functor $[D, E]_G$ on objects by

$$[D, E]_G(\mathbf{b}) := [D \wedge \mathbf{b}_+, E]_G = [D \wedge \Sigma^\infty \mathbf{b}_+, E]_G.$$

When $D = \Sigma^\alpha S$, a suspension of the sphere spectrum, this construction gives the **homotopy Mackey functor**

$$\pi_\alpha^G(E) := [\Sigma^\alpha S, E]_G.$$

The following theorem, which appears in an appendix to [LM06], shows that the homotopy Mackey functor interacts nicely with the smash product on the equivariant stable category. Lewis and Mandell say the theorem was folklore long before their paper.

Theorem 2.35. *The functor π_*^G is a lax symmetric monoidal functor from the genuine equivariant stable category (with monoidal structure given by the smash product) to the category of $RO(G)$ -graded Mackey functors.*

⁷i.e. indexed on $RO(G)$

That is, for $\alpha, \beta \in RO(G)$ and for spectra D and E , there is an associative, symmetric, and unital natural transformation

$$\pi_\alpha^G(D) \square \pi_\beta^G(E) \longrightarrow \pi_{\alpha+\beta}^G(D \wedge E).$$

These fit together to form a natural transformation

$$\pi_*^G(D) \square_* \pi_*^G(E) \longrightarrow \pi_*^G(D \wedge E)$$

with the same properties.

For each Mackey functor M , there is an **Eilenberg-MacLane G -spectrum** HM which has the property that, as Mackey functors,

$$\pi_n^G(HM) \cong \begin{cases} M & n = 0 \\ 0 & n \in \mathbb{Z}, n \neq 0. \end{cases}$$

(See [LMM81] or [May96] for a proof.) HM represents $RO(G)$ -graded, Mackey functor valued Bredon cohomology, in the sense that

$$\underline{\tilde{H}}_G^\alpha(X; M) \cong \underline{[\Sigma^\infty X, \Sigma^\alpha HM]}_G$$

for a based space X .

Corresponding to these cohomology theories are homology theories, given by

$$\underline{H}_\alpha^G(X; M) \cong \underline{[\Sigma^\alpha S, X_+ \wedge HM]}_G.$$

2.3.2 Summary of important facts about homology and cohomology

Most of the arguments in this thesis will rely on the following small collection of facts about $RO(G)$ -graded, Mackey functor valued cohomology. They are collected here for easy reference later.

Axioms 2.36. *A reduced ordinary equivariant cohomology theory $\underline{\tilde{H}}_G^*$ indexed on $RO(G)$ consists of a collection of functors $\underline{\tilde{H}}_G^\alpha: ho-G-Top_* \longrightarrow \mathcal{M}$ satisfying the following axioms.*

1. **Weak equivalence:** *Weak G -equivalences induce isomorphisms on $\underline{\tilde{H}}_G^*$.*
2. **Exactness:** *if $i: A \longrightarrow X$ is a G -cofibration with cofiber X/A , then the sequence*

$$\underline{\tilde{H}}_G^\alpha(X/A) \longrightarrow \underline{\tilde{H}}_G^\alpha(X) \longrightarrow \underline{\tilde{H}}_G^\alpha(A)$$

is exact in the abelian category of Mackey functors for each $\alpha \in RO(G)$.

3. **Additivity:** *if $X = \bigvee_i X_i$ as based spaces, then the inclusions $X_i \longrightarrow X$ induce isomorphisms $\underline{\tilde{H}}_G^*(X) \longrightarrow \prod_i \underline{\tilde{H}}_G^*(X_i)$.*

4. **Suspension:** for each $\alpha \in RO(G)$ and representation V , there is a natural isomorphism

$$\Sigma^V : \underline{\tilde{H}}_G^\alpha(X) \longrightarrow \underline{\tilde{H}}_G^{\alpha+V}(\Sigma^V X) = \underline{\tilde{H}}_G^{\alpha+V}(S^V \wedge X).$$

5. **Dimension:** $\underline{\tilde{H}}_G^n(S^0) = 0$ for all integers $n \in \mathbb{Z}$, $n \neq 0$.

6. **Bookkeeping:** Σ^0 is the identity natural transformation, and the suspension isomorphism $\Sigma^V : \underline{\tilde{H}}_G^\alpha(X) \longrightarrow \underline{\tilde{H}}_G^{\alpha+V}(\Sigma^V X)$ is covariantly natural in V and contravariantly natural in X . Let V, W , and W' be G representations, and $f : S^W \longrightarrow S^{W'}$ the stable homotopy class of the map associated to a G -linear isometric isomorphism $W \longrightarrow W'$. Then $\Sigma^V \Sigma^W \cong \Sigma^{V \oplus W}$ and the following diagrams commute:

$$\begin{array}{ccc} \underline{\tilde{H}}_G^\alpha(X) & \xrightarrow{\Sigma^W} & \underline{\tilde{H}}_G^{\alpha+W}(\Sigma^W X) \\ \Sigma^{W'} \downarrow & & \downarrow \underline{\tilde{H}}_G^{\text{id}+f}(\text{id}) \\ \underline{\tilde{H}}_G^{\alpha+W'}(\Sigma^{W'} X) & \xrightarrow{(\Sigma^f \text{id})} & \underline{\tilde{H}}_G^{\alpha+W}(\Sigma^W X) \end{array} \quad \begin{array}{ccc} \underline{\tilde{H}}_G^\alpha(X) & \xrightarrow{\Sigma^V} & \underline{\tilde{H}}_G^{\alpha+V}(\Sigma^V X) \\ \Sigma^{V+W} \swarrow & & \searrow \Sigma^W \\ \underline{\tilde{H}}_G^{\alpha+V+W}(\Sigma^{V+W} X) & & \end{array}$$

Similar axioms hold for reduced homology $\underline{\tilde{H}}_*^G(-; M)$.

Note that, since $\underline{\tilde{H}}_G^*(S^0; M) \neq M$ as a graded Green functor, saying that something is a module over the cohomology of a point is very different from saying it is a module over the coefficient Mackey functor M .

Fact 2.37. *If M is a Green functor (Mackey functor ring), then unreduced cohomology $\underline{\tilde{H}}_G^*(-; M)$ takes values in graded commutative Green functors. In particular, for every space X and $\alpha, \beta \in RO(G)$, we have maps*

$$\underline{\tilde{H}}_G^\alpha(X_+; M) \square \underline{\tilde{H}}_G^\beta(X_+; M) \longrightarrow \underline{\tilde{H}}_G^{\alpha+\beta}(X_+; M).$$

Fact 2.38. *If M is a Green functor, each $\underline{\tilde{H}}_G^*(X_+; M)$ is an algebra over the cohomology $\underline{\tilde{H}}_G^*(S^0; M)$ of a point, in the sense that there is a map of Green functors*

$$\underline{\tilde{H}}_G^*(S^0; M) \square \underline{\tilde{H}}_G^*(X_+; M) \longrightarrow \underline{\tilde{H}}_G^*(X_+; M)$$

satisfying the expected diagrams.

Fact 2.39. *The homology theory $\underline{\tilde{H}}_*^G(-; A)$ has a Hurewicz map, i.e. a natural transformation $\underline{\pi}_*^G(-) \longrightarrow \underline{\tilde{H}}_*^G(-; A)$.*

For the following, see [LMSM86] or [Wir74].

Fact 2.40. *For any group G , the orbits G/K_+ satisfy equivariant Spanier-Whitehead duality: there is an isomorphism in reduced (co)homology*

$$\underline{\tilde{H}}_G^\alpha(G/K_+; A) \cong \underline{\tilde{H}}_{- \alpha}^G(G/K_+; A).$$

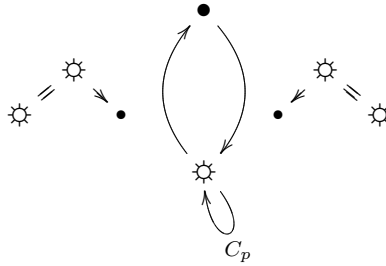
2.4 Mackey functors for $G = C_p$

When p is a prime, the cyclic group C_p has only two subgroups: $\{e\}$ and C_p itself. As a result, it is not difficult to give explicit descriptions of concrete Mackey functors when the group G of equivariance is C_p . [Lew88], and this thesis, are concerned with calculations in cohomology $\tilde{H}_{C_p}^*(-; A)$ for $G = C_p$, and so we will introduce here some explicit notation which will be useful later on.

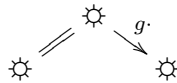
2.4.1 Diagrams for Mackey functors

Recall from Definition 2.11 that a Mackey functor M , as an additive functor $\mathcal{B}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, is determined by its values $M(G/K)$ on orbits. In particular, when $G = C_p$, Mackey functors are determined by their values at C_p/C_p and $C_p/\{e\}$. For convenience, we will write $C_p/C_p = \bullet$ (meant to suggest a single fixed point) and $C_p/\{e\} = \ast$ (meant to suggest all of C_p).

Rephrasing the definition, Mackey functors are additive \mathcal{B}_{C_p} -shaped diagrams in $\mathbb{k}\text{-mod}$. The adjective “additive” means that we can restrict our attention to the full subcategory of \mathcal{B}_{C_p} on objects \bullet and \ast . Considering the maps, i.e. spans, between these objects, we see that \mathcal{B}_{C_p} can be thought of as the category enriched over $\mathbb{k}\text{-mod}$ generated by the following diagram, modulo one relation.



Explicitly, the automorphism group of \bullet is a free \mathbb{k} -module on two generators, the identity span and the composite of the downward arrow followed by the upward one. $\mathcal{B}_{C_p}(\bullet, \ast)$ and $\mathcal{B}_{C_p}(\ast, \bullet)$ are each free on one generator. The automorphism \mathbb{k} -module of \ast is the free \mathbb{k} -module on generators



for each $g \in C_p$, and so has an action of C_p . The one relation comes from the fact, straightforward to check explicitly, that the composite of the upward followed by the downward arrow is equal to the sum of the p generators of $\mathcal{B}_{C_p}(\ast, \ast)$.

The consequence of all of this is that we have an explicit description of Mackey functors

for $G = C_p$. A Mackey functor $M: \mathcal{B}_{C_p}^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ consists of a diagram of the form

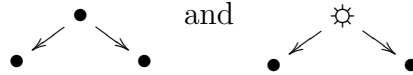
$$\begin{array}{c}
 M(\bullet) \\
 \begin{array}{c} \left. \begin{array}{c} \downarrow r_\rho \\ \uparrow t_\rho \end{array} \right\} \\
 M(\star) \\
 \downarrow C_p \\
 \text{triv}
 \end{array}
 \end{array}$$

where t_ρ and r_ρ are the restriction and transfer associated to the projection of G -sets $\rho: \star \rightarrow \bullet$; $M(\star)$ is a $\mathbb{k}[C_p]$ -module; $M(\bullet)$ is a \mathbb{k} -module which can be viewed as a $\mathbb{k}[C_p]$ -module with the trivial action; and all arrows are maps of $\mathbb{k}[C_p]$ -modules. The one relation discussed above shows that $r_\rho t_\rho x = \sum_{g \in C_p} gx = \text{trace}(x)$ for each $x \in M(\star)$. Conversely, any diagram satisfying these conditions ($M(\bullet)$ is a $\mathbb{k}[C_p]$ -module with the trivial action, $M(\star)$ is a $\mathbb{k}[C_p]$ -module, $r_\rho t_\rho = \text{trace}$) defines a Mackey functor.

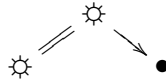
2.4.2 Some important Mackey functors

Since Mackey functors for $G = C_p$ are so simple, we can give explicit descriptions of all of the Mackey functors which will appear in our later calculations.

We have already seen the Burnside ring Mackey functor $A = \mathcal{B}_{C_p}(-, \bullet)$. $\mathcal{B}_{C_p}(\bullet, \bullet)$ is generated by the two spans



while $\mathcal{B}_{C_p}(\star, \bullet)$ is generated by the single span



It follows that $A(\bullet) \cong \mathbb{k}\mu \oplus \mathbb{k}\tau$, a free module on generators μ corresponding to the identity span and τ corresponding to the non-identity span above. $A(\star) \cong \mathbb{k}\iota$, a free module on the single generator ι , with the trivial C_p action. The arrows in A are given by precomposition with appropriate spans; it follows that $t_\rho(\iota) = \tau \in A(\bullet)$, $r_\rho(\mu) = \iota$, and $r_\rho(\tau) = r_\rho(t_\rho(\iota)) = p\iota$. Thus the Mackey functor A has diagram

$$\begin{array}{c}
 \mathbb{k} \oplus \mathbb{k} \\
 \begin{array}{c} \left(\begin{array}{c} \downarrow \begin{pmatrix} 1 \\ p \end{pmatrix} \\ \uparrow \begin{pmatrix} 0 & 1 \end{pmatrix} \end{array} \right) \\
 \mathbb{k} \\
 \downarrow \text{triv}
 \end{array}
 \end{array}$$

We can modify this construction slightly to produce what Lewis calls **twisted Burnside Mackey functors**. For any integer d , define ${}_dA$ to be the Mackey functor given by

$$\begin{array}{c} \mathbb{k} \oplus \mathbb{k} \\ \begin{array}{c} \left(\begin{array}{c} \downarrow \\ \left(\begin{array}{c} \uparrow \\ (0 \ 1) \end{array} \right) \end{array} \right) \\ \left(\begin{array}{c} d \\ p \end{array} \right) \end{array} \\ \mathbb{k} \\ \downarrow \\ \text{triv} \end{array}$$

Lemma 2.41. *There is an isomorphism ${}_{d_1}A \cong {}_{d_2}A$ if and only if there is a unit $u \in \mathbb{k}$ and an element $x \in \mathbb{k}$ such that $d_1 = ud_2 + px$.*

Proof. Suppose that $d_1 = ud_2 + px$ for some unit u and element x . Then the following is a map of Mackey functors.

$$\begin{array}{ccc} \mathbb{k} \oplus \mathbb{k} & \xrightarrow{\begin{pmatrix} u & x \\ 0 & 1 \end{pmatrix}} & \mathbb{k} \oplus \mathbb{k} \\ \begin{array}{c} \left(\begin{array}{c} \downarrow \\ \left(\begin{array}{c} \uparrow \\ (0 \ 1) \end{array} \right) \end{array} \right) \\ \left(\begin{array}{c} d_1 \\ p \end{array} \right) \end{array} & & \begin{array}{c} \left(\begin{array}{c} \downarrow \\ \left(\begin{array}{c} \uparrow \\ (0 \ 1) \end{array} \right) \end{array} \right) \\ \left(\begin{array}{c} d_2 \\ p \end{array} \right) \end{array} \\ \mathbb{k} & \xlongequal{\quad\quad\quad} & \mathbb{k} \\ \downarrow & & \downarrow \\ \text{triv} & & \text{triv} \end{array}$$

It is an isomorphism because the matrix X at the top of the diagram has determinant u .

For the other direction, observe that, in any isomorphism ${}_{d_1}A \cong {}_{d_2}A$, the bottom horizontal map must be either the identity or multiplication by a unit of \mathbb{k} ; we may assume without loss of generality that it is the identity. Since X must then satisfy $(0 \ 1)X = (0 \ 1)$, the bottom row of X is determined. Since the determinant of X must be invertible in \mathbb{k} , the upper left entry must be a unit u . It follows that, in any isomorphism ${}_{d_1}A \cong {}_{d_2}A$, we must have $d_2 = ud_1 + px$ for a unit u and $x \in \mathbb{k}$. \square

Next, for any \mathbb{k} -module C , we have the Mackey functor $\langle C \rangle$ given by

$$\begin{array}{c} C \\ \left(\begin{array}{c} \downarrow \\ \left(\begin{array}{c} \uparrow \\ 0 \end{array} \right) \end{array} \right) \\ 0 \\ \downarrow \\ \text{triv} \end{array}$$

where all maps are the zero map. This is the special case of the functor J_\bullet mentioned in Remark 2.29.

Although the twisted Burnside Mackey functors are generally distinct for different d , modulo the isomorphisms of Lemma 2.41, adding a copy of $\langle \mathbb{k} \rangle$ to two twisted Burnside Mackey functors almost always produces isomorphic Mackey functors; we will make this

precise in Lemma 2.42. This may be slightly surprising at first, but it is also necessary in order for $RO(G)$ -graded cohomology to be well defined.

Lemma 2.42. *For any integers d_1, d_2 prime to p , there is an isomorphism*

$$d_1 A \oplus \langle \mathbb{k} \rangle \cong d_2 A \oplus \langle \mathbb{k} \rangle.$$

Proof. Since the d_i are prime to p , we can find integers a_i, b_i such that $a_i d_i + b_i p = 1$. Then it can be checked that the following map of Mackey functors is an isomorphism.

$$\begin{array}{ccc}
 \mathbb{k} \oplus \mathbb{k} \oplus \mathbb{k} & \xrightarrow{\begin{pmatrix} d_1 a_2 & d_1 b_2 & (b_1 + b_2 - b_1 b_2 p) \\ 0 & 1 & 0 \\ p & -d_2 & -a_1 d_2 \end{pmatrix}} & \mathbb{k} \oplus \mathbb{k} \oplus \mathbb{k} \\
 \begin{pmatrix} d_1 \\ p \\ 0 \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} (0 \ 1 \ 0) & & \begin{pmatrix} d_2 \\ p \\ 0 \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} (0 \ 1 \ 0) \\
 \mathbb{k} & \xrightarrow{\quad \quad \quad} & \mathbb{k} \\
 \text{triv} \curvearrowright & & \text{triv} \curvearrowright
 \end{array}$$

The matrix X at the top of the diagram is constructed as follows: first, X should satisfy $X \begin{pmatrix} d_2 \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} d_1 \\ p \\ 0 \end{pmatrix}$ and $(0 \ 1 \ 0) X = (0 \ 1 \ 0)$, which determines the second row and the first two columns. The entries in the final column are then chosen to make the determinant equal to 1. \square

Remark 2.43. Lemma 2.42 holds more generally if we allow d_1, d_2 to be elements of \mathbb{k} such that there exist $a_i, b_i \in \mathbb{k}$ satisfying $a_i d_i + b_i p = 1$.

We are also in a position to describe the left and right adjoints \mathcal{L} and \mathcal{R} to the forgetful functor $M \mapsto M(\ast)$. If B is a $\mathbb{k}[C_p]$ -module, then $\mathcal{L}(B)$ and $\mathcal{R}(B)$ are given by

$$\mathcal{L}(B) = \text{trace} \begin{pmatrix} B/C_p \\ \uparrow \\ B \\ \downarrow \end{pmatrix} \text{proj} \qquad \mathcal{R}(B) = \text{inclusion} \begin{pmatrix} B^{C_p} \\ \uparrow \\ B \\ \downarrow \end{pmatrix} \text{trace}$$

The trace map takes $[x] \mapsto \sum_g gx$ in $\mathcal{L}(B)$ and $x \mapsto \sum_g gx$ in $\mathcal{R}(B)$. The cases $B = \mathbb{k}$ with either the trivial action or, when $p = 2$, the sign action, will come up frequently. For \mathbb{k} with the trivial action, we have

$$\mathcal{L}(\mathbb{k}) = \cdot p \begin{pmatrix} \mathbb{k} \\ \uparrow \\ \mathbb{k} \\ \downarrow \end{pmatrix} \text{id} \qquad \mathcal{R}(\mathbb{k}) = \text{id} \begin{pmatrix} \mathbb{k} \\ \uparrow \\ \mathbb{k} \\ \downarrow \end{pmatrix} \cdot p$$

If $p = 2$ and \mathbb{k} has no 2-torsion, let \mathbb{k}_- denote \mathbb{k} with the sign action. Then we have

$$\mathcal{L}(\mathbb{k}_-) = 0 \begin{array}{c} \mathbb{k}/2 \\ \downarrow \uparrow \text{proj} \\ \mathbb{k} \\ \downarrow \uparrow -1 \end{array} \quad \mathcal{R}(\mathbb{k}_-) = 0 \begin{array}{c} 0 \\ \downarrow \uparrow 0 \\ \mathbb{k} \\ \downarrow \uparrow -1 \end{array}$$

Here and in the rest of this thesis, if C is a \mathbb{k} -module, we write C/p for the \mathbb{k} -module $C/pC = C \otimes_{\mathbb{Z}} \mathbb{Z}/p$. Since the instances of \mathcal{L} and \mathcal{R} above come up so frequently, we will give them their own names:

$$R := \mathcal{R}(\mathbb{k}) \quad R_- := \mathcal{R}(\mathbb{k}_-) \quad L := \mathcal{L}(\mathbb{k}) \quad L_- := \mathcal{L}(\mathbb{k}_-)$$

Now that we have assigned names to the Mackey functors above, we state the following immediate corollaries of Lemma 2.41. This explains why we are usually only interested in ${}_d A$ when d is not a multiple of p .

Corollary 2.44. *If $d = px$ for some $x \in \mathbb{k}$, then ${}_d A$ splits up as the direct sum $\langle \mathbb{k} \rangle \oplus L$.*

Corollary 2.45. *If p is invertible in \mathbb{k} , then for any d , we have*

$${}_d A \cong A \cong \langle \mathbb{k} \rangle \oplus L.$$

Finally, for any Mackey functor M , we defined the shifted Mackey functor M_{\star} in Definition 2.27: $M_{\star}(\bullet) = M(\star)$ and $M_{\star}(\star) = M(\star \times \star) \cong \bigoplus_{g \in C_p} M(\star)$. The choice of representatives for the orbits in $\star \times \star$ affects the choice of basis in the identification $M_{\star}(\star) \cong \bigoplus_{g \in C_p} M(\star)$; we will make the choice which yields the diagram below.

$$\begin{array}{c} M(\star) \\ \text{diag} \left(\begin{array}{c} \downarrow \uparrow \\ \downarrow \uparrow \end{array} \right) \text{fold} \\ M(\star)^{\oplus p} \\ \downarrow \uparrow \text{perm} \end{array}$$

In particular, the represented Mackey functor A_{\star} has the following diagram.

$$\begin{array}{c} \mathbb{k} \\ \text{diag} \left(\begin{array}{c} \downarrow \uparrow \\ \downarrow \uparrow \end{array} \right) \text{fold} \\ \mathbb{k}^{\oplus p} \\ \downarrow \uparrow \text{perm} \end{array}$$

We know from Subsection 2.2.7 that a map of Mackey functors $A \longrightarrow M$ is determined by an element of $M(\bullet)$, and that a map $A_{\star} \longrightarrow M$ is determined by an element of $M(\star)$.

2.4.3 Box Products

We have already mentioned the box product \square , which gives the category \mathcal{M} of Mackey functors a monoidal structure. There are two ways of looking at this, in terms of the defining universal property and in terms of an explicit formula; both will be useful at various times.

We know from Lemma 2.20 that, for Mackey functors M and N ,

$$(M \square N)(\mathbf{c}) \cong \int^{\mathbf{a}} M(\mathbf{a}) \otimes N_{\mathbf{c}}(D\mathbf{a}) = M \otimes_{\mathcal{B}_{C_p}} (N_{\mathbf{c}} \circ D).$$

Since \mathcal{B}_{C_p} is generated by the objects \bullet and \star , we can simplify the above formula even further.⁸

Lemma 2.46. *Given Mackey functors M and N , their box product $M \square N$ is given by the diagram*

$$\begin{array}{c} [M(\bullet) \otimes N(\bullet) \oplus M(\star) \otimes N(\star)] / \sim \\ \begin{array}{c} \begin{array}{c} \downarrow r_\rho \quad \uparrow t_\rho \\ M(\star) \otimes N(\star) \\ \downarrow C_p \end{array} \end{array} \end{array}$$

Using r_ρ and t_ρ to also mean the restriction and transfer maps in M and N , the equivalence relation is generated by the relations

$$\begin{array}{ll} x \otimes t_\rho y \sim r_\rho x \otimes y & \text{for } x \in M(\bullet), y \in N(\star) \\ t_\rho w \otimes z \sim w \otimes r_\rho z & \text{for } w \in M(\star), z \in N(\bullet) \\ gw \otimes y \sim w \otimes g^{-1}y & \text{for } w \in M(\star), y \in N(\star), g \in C_p \end{array}$$

The action of C_p on $(M \square N)(\star)$ is the diagonal action on $M(\star) \otimes N(\star)$, i.e. $w \otimes y \longmapsto gw \otimes gy$. The transfer t_ρ of the box product is induced by the inclusion

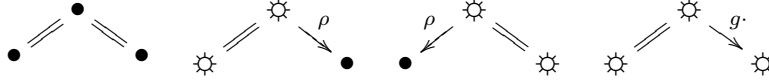
$$M(\star) \otimes N(\star) \longrightarrow M(\bullet) \otimes N(\bullet) \oplus M(\star) \otimes N(\star).$$

The restriction r_ρ of the box product is induced by $r_\rho \otimes r_\rho$ on $M(\bullet) \otimes N(\bullet)$ and by the trace on $M(\star) \otimes N(\star)$, $w \otimes y \longmapsto \sum_{g \in C_p} gw \otimes gy$.

Proof. As remarked above, $(M \square N)(\mathbf{c}) \cong M \otimes_{\mathcal{B}_{C_p}} (N_{\mathbf{c}} \circ D)$, which is given by the coequalizer of Definition 2.19. For this coequalizer, we may restrict our attention to the objects \bullet, \star

⁸Note that one of the relations is accidentally omitted in [Lew88].

and the generating spans



of \mathcal{B}_{C_p} . It is then an instructive exercise to arrive at the explicit formulas above. \square

Similarly, Lemma 2.17 gives us an explicit description of maps out of $M \square N$.

Proposition 2.47. *A map of Mackey functors $f: M \square N \longrightarrow P$ is determined by two maps*

$$\begin{aligned} M(\bullet) \otimes N(\bullet) &\xrightarrow{f_\bullet} P(\bullet) \\ M(\star) \otimes N(\star) &\xrightarrow{f_\star} P(\star) \end{aligned}$$

such that the following four formulas, called the Frobenius relations, hold for all $x \in M(\bullet)$, $x' \in M(\star)$, $y \in N(\bullet)$, and $y' \in N(\star)$.⁹

$$\begin{aligned} f_\star(r_\rho x \otimes r_\rho y) &= r_\rho f_\bullet(x \otimes y) \\ f_\bullet(t_\rho x' \otimes y) &= t_\rho f_\star(x' \otimes r_\rho y) \\ f_\bullet(x \otimes t_\rho y') &= t_\rho f_\star(r_\rho x \otimes y') \\ f_\star(gx' \otimes gy') &= g f_\star(x' \otimes y') \end{aligned}$$

Proof. By Lemma 2.17, a map $f: M \square N \longrightarrow P$ is given by two maps f_\bullet and f_\star , as above, such that the square and two pentagons in the definition of a Dress pairing commute for the projection map of C_p -sets $\rho: \star \longrightarrow \bullet$ and the multiplication map $g: \star \longrightarrow \star$, where g is a generator of C_p . Since the map g is an isomorphism, $t_g = (r_g)^{-1}$ as in the remark after Lemma 2.17, and so the diagrams of Lemma 2.17 reduce to the four formulas shown. \square

Example 2.48. For any Mackey functor M and $\mathbb{k}[C_p]$ -module B , we have

$$\mathcal{L}(B) \square M \cong \mathcal{L}(B \otimes M(\star)).$$

This comes from the fact that \square actually makes \mathcal{M} into a closed monoidal category. That is, there is an internal hom $\langle -, - \rangle$ such that

$$\mathcal{M}(M \square N, P) \cong \mathcal{M}(M, \langle N, P \rangle),$$

naturally in all variables. The claim then follows from the following facts: \square is a colimit; colimits commute with left adjoints; and $\langle N, P \rangle(\star) = \mathbb{k}\text{-mod}(N(\star), P(\star))$.

⁹Note that [Lew88] forgets to mention the map g when giving the compatibility relations.

Remark 2.49. In fact, for any finite group G , if we let $\star = G/\{e\}$, we have

$$(M \square N)(\star) \cong M(\star) \otimes N(\star)$$

and

$$\langle M, N \rangle(\star) \cong \mathbb{k}\text{-mod}(M(\star), N(\star));$$

this much is not special to C_p .

Remark 2.50. $\langle -, - \rangle$ is a right Kan extension, given by an end dual to the coend describing \square . The analogue to Lemma 2.20 shows that $\langle M, N \rangle(\mathbf{c}) \cong \text{Hom}_{\mathcal{B}_G}(M, N_{\mathbf{c}})$, i.e. natural transformations from M to $N_{\mathbf{c}}$.

Example 2.51. It is immediate from Lemma 2.46 that for any Mackey functor M and \mathbb{k} -module C ,

$$\langle C \rangle \square M \cong \langle C \otimes (M(\bullet)) / (\text{image } t_\rho) \rangle.$$

Example 2.52. We see by explicit computation that, for integers c and d relatively prime to p ,

$${}_d A \square {}_c A \cong {}_{cd} A.$$

To show this using Lemma 2.46, we need to identify

$$[({}_d A(\bullet) \otimes {}_c A(\bullet)) \oplus ({}_d A(\star) \otimes {}_c A(\star))] / \sim$$

with ${}_{cd} A(\bullet) \cong \mathbb{k} \oplus \mathbb{k}$. One approach is to choose the standard generators $\mu_d, \tau_d \in {}_d A(\bullet)$ and $\iota_d \in {}_d A(\star)$ for ${}_d A$, i.e. the ones which give the displayed diagram on page 24, and similarly for ${}_c A$ and ${}_{cd} A$. Then the map ${}_{cd} A(\bullet) \rightarrow ({}_d A \square {}_c A)(\bullet)$ sending $\mu_{cd} \mapsto \mu_d \otimes \mu_c$ and $\tau_{cd} \mapsto t_\rho(\iota_d \otimes \iota_c)$ gives the desired isomorphism.

Example 2.53. Box products with the Mackey functors $\mathcal{B}(B)$ are more complicated, since right adjoints do not generally commute with colimits. However, for $B = \mathbb{k}$, viewed as a $\mathbb{k}[G]$ -module with the trivial action, we can calculate $R \square {}_d A \cong R$ and $R \square R \cong R$ using Lemma 2.46. To see the first isomorphism, choose integers a_d, b_d such that $a_d d + b_d p = 1$, and define a new basis for ${}_d A(\bullet)$ by $\sigma_d = a_d \mu_d + b_d \tau_d$, $\kappa_d = p \mu_d - d \tau_d$. Ambiguously using 1 to denote the generator at each level of R , the isomorphism $R \rightarrow {}_d A \square R$ is given by sending $1 \mapsto \sigma_d \otimes 1$ at the \bullet level.

Example 2.54. Similarly, Lemma 2.46 tells us the following box products with R_- :

$$\begin{aligned} R_- \square R &\cong R_- \\ R_- \square R_- &\cong L \end{aligned}$$

In particular, even though $R_-(\bullet) = 0$, $(R_- \square R_-)(\bullet) \neq 0!$

Example 2.55. Using the description of M_{\star} , we have

$$A_{\star} \cong L_{\star} \cong R_{\star} \cong L_{-\star} \cong R_{-\star}.$$

	A_{\star}	dA	$\langle X_1 \rangle$	$\mathcal{L}(B_1)$	R	R_-
A_{\star}	$A_{\star \times \star}$	A_{\star}	$\langle 0 \rangle$	$\mathcal{L}(B_1^{\oplus p})$	A_{\star}	A_{\star}
${}_c A$	A_{\star}	cdA	$\langle X_1 \rangle$	$\mathcal{L}(B_1)$	R	R_-
$\langle X_2 \rangle$	$\langle 0 \rangle$	$\langle X_2 \rangle$	$\langle X_1 \otimes X_2 \rangle$	$\langle 0 \rangle$	$\langle X_2/p \rangle$	$\langle 0 \rangle$
$\mathcal{L}(B_2)$	$\mathcal{L}(B_2^{\oplus p})$	$\mathcal{L}(B_2)$	$\langle 0 \rangle$	$\mathcal{L}(B_1 \otimes B_2)$	$\mathcal{L}(B_2)$	$\mathcal{L}(B_1 \otimes \mathbb{k}_-)$
R	A_{\star}	R	$\langle X_1/p \rangle$	$\mathcal{L}(B_1)$	R	R_-
R_-	A_{\star}	R_-	$\langle 0 \rangle$	$\mathcal{L}(B_1 \otimes \mathbb{k}_-)$	R_-	L

Figure 2.56: Multiplication table for some relevant Mackey functors. Here, $c, d \in \mathbb{Z}$ are relatively prime to p ; X_1, X_2 are \mathbb{k} -modules; and B_1, B_2 are $\mathbb{k}[C_p]$ -modules. We continue to write X/p for $X \otimes_{\mathbb{Z}} \mathbb{Z}/p$.

Putting all of these examples together, we have the multiplication table in Figure 2.56 for the Mackey functors introduced in Subsection 2.4.2.

2.5 Bredon Cohomology for $G = C_p$

Using the definition of Mackey functor valued $RO(C_p)$ -graded Bredon cohomology $\tilde{H}_{C_p}^*(-; M)$ in terms of representing Eilenberg-MacLane spectra in the case $G = C_p$, we see that the α cohomology $\tilde{H}_{C_p}^{\alpha}(X; M)$ and homology $\tilde{H}_{C_p}^{\alpha}(X; M)$ of a C_p -space X is given by the diagrams

$$\begin{array}{ccc}
\tilde{H}_{C_p}^{\alpha}(X; M) & \text{and} & \tilde{H}_{C_p}^{\alpha}(X; M) \\
\text{proj}^* \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \text{transfer}^* & & \text{transfer}_* \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \text{proj}_* \\
\tilde{H}^{|\alpha|}(X; M(\star)) & & \tilde{H}_{|\alpha|}(X; M(\star)) \\
\downarrow & & \downarrow
\end{array}$$

The identification of $\tilde{H}_{C_p}^{\alpha}(X; M)(\star)$ with the nonequivariant cohomology group shown uses the adjunction $[C_{p+} \wedge X, HM]_{C_p} \cong [X, HM]$ together with the fact that the underlying nonequivariant spectrum of HM is a $H(M(\star))$. The C_p action comes from the action on the $\mathbb{k}[C_p]$ -module $M(\star)$.

Many of the results in the remainder of this section will implicitly use the structure of $RO(C_p)$, the free abelian group with generators the irreducible real representations of C_p . When $p = 2$, C_2 has exactly two irreducible representations: the one-dimensional trivial representation, and the one-dimensional sign representation.

For $p > 2$, there are no nontrivial one-dimensional representations of C_p . Every nontrivial irreducible representation is the underlying two-dimensional real representation of a one-dimensional complex representation of C_p . There are $p - 1$ nontrivial complex representations, given by choosing a generator $g \in C_p$ and specifying that $gz = e^{2\pi ik/p}z$ for some integer k . When we forget down to the underlying real representation, we also forget

about orientation, and so the representations given by $gz = e^{2\pi ik/p}z$ and $gz = e^{2\pi i(-k)/p}z$ are identified. It follows that the irreducible real representations of C_p are the one-dimensional trivial representation and the $\frac{p-1}{2}$ two-dimensional “rotation” representations described above.

2.5.1 Cup product and graded commutativity

The cup product structure on cohomology is a map

$$\underline{\tilde{H}}_{C_p}^*(X; M) \square \underline{\tilde{H}}_{C_p}^*(X; M) \xrightarrow{\smile} \underline{\tilde{H}}_{C_p}^*(X; M)$$

making $\underline{\tilde{H}}_{C_p}^*(X; M)$ into a graded commutative Green functor. As with nonequivariant cohomology, this cup product arises from the diagonal $X \rightarrow X \wedge X$ for a based C_p -space X . In Subsection 2.2.5, we deferred a discussion of signs until later, and so we will discuss them now for $G = C_p$.

From a represented point of view, the cup product pairing arises from the composite

$$\begin{aligned} [X, S^\alpha \wedge HM]_G \square [X, S^\beta \wedge HM]_G &\xrightarrow{\wedge} [X \wedge X, S^\alpha \wedge HM \wedge S^\beta \wedge HM]_G \longrightarrow \\ &\longrightarrow [X \wedge X, S^\alpha \wedge S^\beta \wedge HM \wedge HM]_G \longrightarrow [X, S^{\alpha+\beta} \wedge HM]_G. \end{aligned}$$

The arrow marked \wedge comes from taking the smash product of two maps; the middle arrow comes from switching the order of HM and S^β ; and the final arrow is induced by the diagonal $X \rightarrow X \wedge X$, the identification of $S^\alpha \wedge S^\beta$ with $S^{\alpha+\beta}$, and the map $HM \wedge HM \rightarrow HM$ arising from the Green functor structure of M .

We can then consider the commutativity of the cup product. Using γ to denote the switch map, this amounts to considering the commutativity of Figure 2.57, for given maps $f: X \rightarrow S^\alpha \wedge HM$ and $g: X \rightarrow S^\beta \wedge HM$. The three faces marked with \circlearrowleft commute by inspection. Provided that M is a commutative Green functor,

$$\begin{array}{ccc} HM \wedge HM & \xrightarrow{\gamma} & HM \wedge HM \\ & \searrow & \swarrow \\ & HM & \end{array}$$

commutes as well. We are left with the diagram

$$\begin{array}{ccc} S^\alpha \wedge S^\beta & \xrightarrow{\gamma} & S^\beta \wedge S^\alpha \\ \simeq \searrow & ? & \swarrow \simeq \\ & S^{\alpha+\beta} & \end{array}$$

In order to identify the vertical equivalences, we need to deal with some of the technicalities of the $RO(C_p)$ -grading discussed on page 3. If $\{\lambda_i\}$ are the irreducible representations of

$$\begin{array}{ccccc}
& & X & & \\
& \Delta \swarrow & & \searrow \Delta & \\
X \wedge X & \xrightarrow{\quad \gamma \quad} & X \wedge X & & \\
\downarrow f \wedge g & & \downarrow g \wedge f & & \\
(S^\alpha \wedge HM) \wedge (S^\beta \wedge HM) & \xrightarrow{\quad \gamma \quad} & (S^\beta \wedge HM) \wedge (S^\alpha \wedge HM) & & \\
\downarrow \text{id} \wedge \gamma \wedge \text{id} & & \downarrow \text{id} \wedge \gamma \wedge \text{id} & & \\
S^\alpha \wedge S^\beta \wedge HM \wedge HM & \xrightarrow{\quad \gamma \wedge \gamma \quad} & S^\beta \wedge S^\alpha \wedge HM \wedge HM & & \\
& \searrow & \swarrow & & \\
& S^{\alpha+\beta} \wedge HM & & &
\end{array}$$

Figure 2.57: $f \smile g$ (the left-hand composite) and $g \smile f$ for the representing maps of elements $f, g \in \widetilde{H}_{C_p}^*(X; M)$.

C_p , we can write

$$\begin{aligned}
\alpha &= \sum_i a_i \lambda_i, & S^\alpha &= \bigwedge_i S^{a_i \lambda_i} \\
\beta &= \sum_i b_i \lambda_i, & S^\beta &= \bigwedge_i S^{b_i \lambda_i} \\
\alpha + \beta &= \sum_i (a_i + b_i) \lambda_i, & S^{\alpha+\beta} &= \bigwedge_i S^{(a_i+b_i) \lambda_i}
\end{aligned}$$

Provided we are working in the stable category, these are well defined for all integer coefficients a_i and b_i . The maps $S^\alpha \wedge S^\beta \xrightarrow{\cong} S^{\alpha+\beta}$ and $S^\beta \wedge S^\alpha \xrightarrow{\cong} S^{\alpha+\beta}$ come from interchanging the factors corresponding to different representations λ_i and then comparing the maps $S^{a_i \lambda_i} \wedge S^{b_i \lambda_i} \rightarrow S^{(a_i+b_i) \lambda_i}$ and $S^{b_i \lambda_i} \wedge S^{a_i \lambda_i} \rightarrow S^{(a_i+b_i) \lambda_i}$ for each i .

We know that, in the stable category, $\underline{\pi}_{(a_i+b_i)\lambda_i}^{C_p}(S^{(a_i+b_i)\lambda_i}) \cong A$, the Burnside ring Green functor. If we canonically identify $S^{a_i \lambda_i} \wedge S^{b_i \lambda_i}$ with $S^{(a_i+b_i)\lambda_i}$, the switch map $S^{a_i \lambda_i} \wedge S^{b_i \lambda_i} \rightarrow S^{b_i \lambda_i} \wedge S^{a_i \lambda_i}$ represents some element of $A(\bullet)$, and in particular some unit of $A(\bullet)$, since it is a homotopy equivalence.

We are considering $G = C_p$. Let 1 denote the unit of the ring $A(\bullet) = A(C_p)$. If p is odd, then the only units in $A(\bullet)$ are ± 1 . Since the underlying nonequivariant degree of the switch map must be $(-1)^{a_i b_i |\lambda_i|^2}$, the equivariant degree is the same. Explicitly, it is -1 when a_i, b_i are both odd and λ_i is the one-dimensional trivial representation, and 1 otherwise.¹⁰ Since the nontrivial irreducible representations of C_p all have trivial fixed-point sets, we may phrase this as follows.

¹⁰All other irreducible representations have even dimension.

Proposition 2.58. *If p is an odd prime, then the switch map $S^\alpha \wedge S^\beta \longrightarrow S^\beta \wedge S^\alpha$ has degree $(-1)^{|\alpha|^{C_p}|\beta|^{C_p}}$. \square*

If $p = 2$, the signs are less trivial, and we must do some work to identify them. In the following, we will ambiguously use $\pi_*^{C_p}$ to denote both the stable and unstable homotopy groups, relying on context to distinguish them; note that the homotopy Mackey functor $\underline{\pi}_*^{C_p}$ is only defined in the stable context.

The following theorem of tom Dieck [tD75, page 20] will be helpful in identifying the homotopy class of the switch map.

Theorem 2.59 (tom Dieck). *For any group G , the map $\pi_0^G(\Sigma^\infty S^0) \longrightarrow \prod_{(K)} \mathbb{Z}$ sending $f: S^V \longrightarrow S^V$ to the tuple $(\deg f^K)$ is injective. Here $\deg f^K$ is the degree of the restriction $f^K: (S^V)^K \longrightarrow (S^V)^K$ to the K -fixed points, and the product runs over conjugacy classes of subgroups of G .*

In particular, when $G = C_2$, two maps $f, g: S^V \longrightarrow S^V$ are stably C_2 -homotopic if and only if they have the same nonequivariant degree and their restrictions f^{C_2}, g^{C_2} to the C_2 -fixed points have the same degree. This will be very useful for establishing the equivariant degree of self-maps of spheres.

For the following proposition, it may be helpful to recall the definition of the group structure on the equivariant homotopy groups $\pi_*^{C_2}$ for spaces. Suppose V is an honest real representation containing at least two copies of the real one-dimensional trivial representation. If $|V| = n + 2$, write the elements of V as $(n + 2)$ -tuples $(x_1, x_2, y_1, \dots, y_n)$, where C_2 acts trivially on x_1 and x_2 , and view S^V as the quotient of the unit cube $[-1, 1]^{n+2}$ by its boundary in V . Then, given a C_2 -space X and maps $f, g: S^V \longrightarrow X$, the inverse $-f$ of f is the continuous map given by the explicit formula

$$(-f)(x_1, x_2, y_1, \dots, y_n) = f(-x_1, x_2, y_1, \dots, y_n).$$

Similarly, $f + g$ is given by

$$(f + g)(x_1, x_2, y_1, \dots, y_n) = \begin{cases} f(2x_1 + 1, x_2, y_1, \dots, y_n) & x_1 \leq 0 \\ g(2x_1 - 1, x_2, y_1, \dots, y_n) & x_1 \geq 0 \end{cases}$$

We are ultimately interested in the stable homotopy groups $\pi_\alpha^{C_2}(\Sigma^\infty X)$ for a virtual representation α . To get a handle on these, we can choose an honest representation W such that $\alpha + W$ is an honest representation containing at least two copies of the trivial representation, and then consider $\pi_{\alpha+W}^{C_2}(\Sigma^W X)$; the stable homotopy group $\pi_\alpha^{C_2}(\Sigma^\infty X)$ is by definition $\text{colim}_W \pi_{\alpha+W}^{C_2}(\Sigma^W X)$.

As a final bit of notation, again let 1 be the multiplicative identity of the ring $A(\bullet)$, and set $\tau = t_\rho r_\rho(1)$. Then the units of $A(\bullet)$ are ± 1 and $\pm(1 - \tau)$; recall that the Frobenius relations tell us that $\tau^2 = p\tau = 2\tau$. Since the switch map is a representative of a unit in the Burnside ring Green functor, we need only identify which of these four units it represents.

Proposition 2.60. *Let ζ be the one-dimensional real sign representation of C_2 . If $\alpha = a_0 + a_1\zeta$ and $\beta = b_0 + b_1\zeta$, then the switch map $S^\alpha \wedge S^\beta \rightarrow S^\beta \wedge S^\alpha$ represents $(-1)^{a_0b_0}(1-\tau)^{a_1b_1}$ in $\pi_{\alpha+\beta}^G(\Sigma^\infty S^{\alpha+\beta}) \cong A(\bullet)$.*

Note that we are implicitly using the fact that ζ and the trivial representation are the only irreducible representations of C_2 .

Proof. It suffices to determine the equivariant degree of the switch maps on the spaces $S^1 \wedge S^1$ and $S^\zeta \wedge S^\zeta$; Proposition 2.60 will then follow.

First consider the switch map on $S^1 \wedge S^1 = S^2$. Since the C_2 -action is trivial, it is immediate that this map $S^2 \rightarrow S^2$ agrees with $-1 \in \pi_{2+W}^{C_2}(S^{2+W})$ for every W , and hence in the stable group $\pi_2^{C_2}(\Sigma^\infty S^2)$.

For the switch map on $S^\zeta \wedge S^\zeta \simeq S^{2\zeta}$, we first need explicit descriptions of representatives for 1 and τ in the stable homotopy groups of spaces. 1 is represented by the identity map. τ is the stable class given by the restriction associated to $\ast \rightarrow \bullet$ followed by the transfer associated to this map. This composite corresponds to the map on $S^{2\zeta+W}$ given by first collapsing to a point everything outside a small neighborhood of two antipodes not fixed by the action of C_2 , and then taking the equivariant fold map.

At this point, we can consider the degrees of the units of $A(\bullet)$, viewing them as self-maps of $S^{2\zeta+W}$ for W containing at least two copies of the trivial representation. Since τ has underlying nonequivariant degree 2 , $1 - \tau$ has nonequivariant degree -1 and $-(1 - \tau)$ has nonequivariant degree 1 . Restricted to the fixed-point set $S^2 \subset S^{2+2\zeta}$, $1 - \tau$ is the identity, which has degree 1 . On the other hand, using the definition of the group operation on $\pi_*^{C_2}$, we see that -1 has degree -1 on both the underlying nonequivariant sphere $S^{2+|W|}$ and the fixed-point sphere $S^{|W|C_2}$.

Finally, consider the stable class of the switch map $S^{2\zeta} \rightarrow S^{2\zeta}$. This map has underlying nonequivariant degree -1 , but, restricted to the fixed-point space of $S^{2\zeta+W}$, is the identity. Thus, by Theorem 2.59, the switch map must represent $1 - \tau$. \square

At this point we may return to the diagram of Figure 2.57. We have shown that $f \smile g$ and $g \smile f$ differ, as maps $X \rightarrow \Sigma^{\alpha+\beta} HM$, by composition with a switch map on $S^{\alpha+\beta}$ of known equivariant degree. This means that there are known units $u(\alpha, \beta) \in A(\bullet)$ making the diagram below commute.

$$\begin{array}{ccc} \tilde{H}_{C_p}^\alpha(X; M) \square \tilde{H}_{C_p}^\beta(X; M) & \xrightarrow{u(\alpha, \beta)\gamma} & \tilde{H}_{C_p}^\beta(X; M) \square \tilde{H}_{C_p}^\alpha(X; M) \\ & \searrow & \swarrow \\ & \tilde{H}_{C_p}^{\alpha+\beta}(X; M) & \end{array}$$

We will thus define a **graded commutative** Green functor to be a graded Green functor T^* whose multiplication makes all diagrams of the above form commute, for the same units $u(\alpha, \beta)$ established in Proposition 2.58 and Proposition 2.60.

2.5.2 Reinterpreting the cup product

The cup product structure is given by a map out of a box product, and so we know from Proposition 2.47 or Lemma 2.17 that it consists of ring maps

$$\begin{aligned} \tilde{H}_{C_p}^*(X; M) \otimes \tilde{H}_{C_p}^*(X; M) &\longrightarrow \tilde{H}_{C_p}^*(X; M) \\ \tilde{H}^{|\ast|}(X; M(\ast)) \otimes \tilde{H}^{|\ast|}(X; M(\ast)) &\longrightarrow \tilde{H}^{|\ast|}(X; M(\ast)) \end{aligned}$$

subject to some compatibility conditions. The second map is exactly the usual cup product on the nonequivariant cohomology of X , so finding the cup product structure on $\tilde{H}_{C_p}^*(X; M)$ amounts to finding it on $\tilde{H}_{C_p}^*(X; M)$.

The units of the Burnside ring Green functor which determine the graded commutativity of $\tilde{H}_{C_p}^*(X; M)$ necessarily determine the graded commutativity of the rings $\tilde{H}_{C_p}^*(X; M)$ and $\tilde{H}^{|\ast|}(X; M(\ast))$, as follows. If the switch map $S^\alpha \wedge S^\beta \longrightarrow S^\beta \wedge S^\alpha$ has degree $u(\alpha, \beta) \in A(\bullet)$, then, using the structure of $\tilde{H}_{C_p}^*(X; M)$ as a module over A , we have formulas

$$\begin{aligned} yx &= u(\alpha, \beta)xy & x &\in \tilde{H}_{C_p}^\alpha(X; M), \quad y \in \tilde{H}_{C_p}^\beta(X; M) \\ zw &= r_\rho(u(\alpha, \beta))wz & w &\in \tilde{H}_{C_p}^{|\alpha|}(X; M(\ast)), \quad z \in \tilde{H}_{C_p}^{|\beta|}(X; M(\ast)) \end{aligned}$$

Since $r_\rho(-1) = -1 \in A(\ast)$ and, when $p = 2$, $r_\rho(1 - \tau) = -1$ as well, the second formula is compatible with the usual graded commutativity of the nonequivariant cup product.

The nonequivariant cup product also determines certain parts of the multiplicative structure on $\tilde{H}_{C_p}^*(X; M) = \underline{\tilde{H}}_{C_p}^*(X; M)(\bullet)$. The Frobenius relations dictate products of the form $xt_\rho(w)$, for $x \in \tilde{H}_{C_p}^*(X; M)(\bullet)$ and $w \in \tilde{H}_{C_p}^*(X; M)(\ast)$; the fact that r_ρ is a ring homomorphism may provide some additional information as well.

2.5.3 Burnside ring coefficients

We now turn our attention to cohomology with coefficients in the Burnside Green functor A . Recall that our convention is to suppress A coefficients in the notation. Since $\tilde{H}_{C_p}^*(X)$ is a module over $\tilde{H}_{C_p}^*(S^0)$ for every C_p -space X , it is desirable to have an explicit description of $\tilde{H}_{C_p}^*(S^0)$. As previously remarked, the complexities of the $RO(C_p)$ -graded dimension axiom imply that $\tilde{H}_{C_p}^*(S^0)$ is *not* just the coefficient Green functor A .

The calculation of $\tilde{H}_{C_p}^*(S^0)$ is due to Stong when $p = 2$, and to Stong and Lewis for odd primes. This calculation makes extensive use of the cofiber sequences

$$\begin{array}{ccccc} C_{p+} & \longrightarrow & S(\lambda)_+ & \longrightarrow & \Sigma C_{p+} \\ & & S(\lambda)_+ & \longrightarrow & S^0 & \longrightarrow & S^\lambda \\ C_{2+} = S(\zeta)_+ & \longrightarrow & S^0 & \longrightarrow & S^\zeta \end{array}$$

where λ is a nontrivial irreducible real representation of C_p ; ζ is the nontrivial one-dimensional real representation of C_2 ; $S(V)$ is the unit sphere in the representation V ; and S^V is the one-point compactification of V . This calculation appears in the appendix to [Lew88], and will not be reproduced here; we will simply state the results without proof in Subsection 2.5.4 and Subsection 2.5.5.

Remark 2.61. The cohomology of a point for $G = C_p$ and any Mackey functor coefficients is known; see Chapters 8 and 9 of [FL04]. However, we will only discuss the result for A coefficients in this thesis.

2.5.4 The cohomology of a point for $p = 2$

We have already noted that the group C_2 has exactly one nontrivial irreducible real representation, namely the one-dimensional sign representation ζ . Thus every virtual representation class has a representative of the form $m + n\zeta$ for $m, n \in \mathbb{Z}$, where we use m to denote the real m -dimensional trivial representation. If $\alpha = m + n\zeta$, we can recover m and n from the dimensions $|\alpha| = m + n$ and $|\alpha^{C_2}| = m$. For compatibility with p odd, we will classify our virtual representations α by the ordered pair $(|\alpha^{C_2}|, |\alpha|)$.

Theorem 2.62 (Stong). *When $G = C_2$, the additive structure of $\tilde{H}_{C_2}^*(S^0)$ is as follows.*

$$\tilde{H}_{C_2}^\alpha(S^0) \cong \begin{cases} A & (|\alpha^{C_2}|, |\alpha|) = (0, 0) \\ R & (|\alpha^{C_2}|, |\alpha|) = (-2m, 0) \text{ for } m \geq 1 \\ R_- & (|\alpha^{C_2}|, |\alpha|) = (1 - 2m, 0) \text{ for } m \geq 0 \\ L & (|\alpha^{C_2}|, |\alpha|) = (2m, 0) \text{ for } m \geq 1 \\ L_- & (|\alpha^{C_2}|, |\alpha|) = (2m + 1, 0) \text{ for } m \geq 1 \\ \langle \mathbb{k} \rangle & (|\alpha^{C_2}|, |\alpha|) = (0, n) \text{ for } n \in \mathbb{Z} \\ \langle \mathbb{k}/2 \rangle & (|\alpha^{C_2}|, |\alpha|) = (-2m, n) \text{ for } m \geq 1, n \geq 1 \\ \langle \mathbb{k}/2 \rangle & (|\alpha^{C_2}|, |\alpha|) = (2m + 1, -n) \text{ for } m \geq 1, n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. See the appendix to [Lew88]. □

This is perhaps best visualized using Figure 2.63.

As outlined in Subsection 2.5.2, we will use the Dress pairing to describe the maps $\tilde{H}_{C_2}^\alpha(S^0) \square \tilde{H}_{C_2}^\beta(S^0) \longrightarrow \tilde{H}_{C_2}^{\alpha+\beta}$ making up the cup product structure on $\tilde{H}_{C_2}^*(S^0)$. This provides a very concrete picture of the cup product structure, although it has the disadvantage that it is difficult to tell from the Dress pairing when a map is an isomorphism.¹¹ We will begin by defining generators in the \mathbb{k} -modules $\tilde{H}_{C_2}^*(S^0)(\bullet)$ and $\tilde{H}_{C_2}^*(S^0)(\ast)$, and then discuss their products.

¹¹For most $\alpha, \beta \in RO(C_2)$, $\tilde{H}_{C_2}^{\alpha+\beta}(S^0)$ is either zero or isomorphic to $\tilde{H}_{C_2}^\alpha(S^0) \square \tilde{H}_{C_2}^\beta(S^0)$. When the latter is true, the cup product map $\tilde{H}_{C_2}^\alpha(S^0) \square \tilde{H}_{C_2}^\beta(S^0) \longrightarrow \tilde{H}_{C_2}^{\alpha+\beta}(S^0)$ is always an isomorphism.

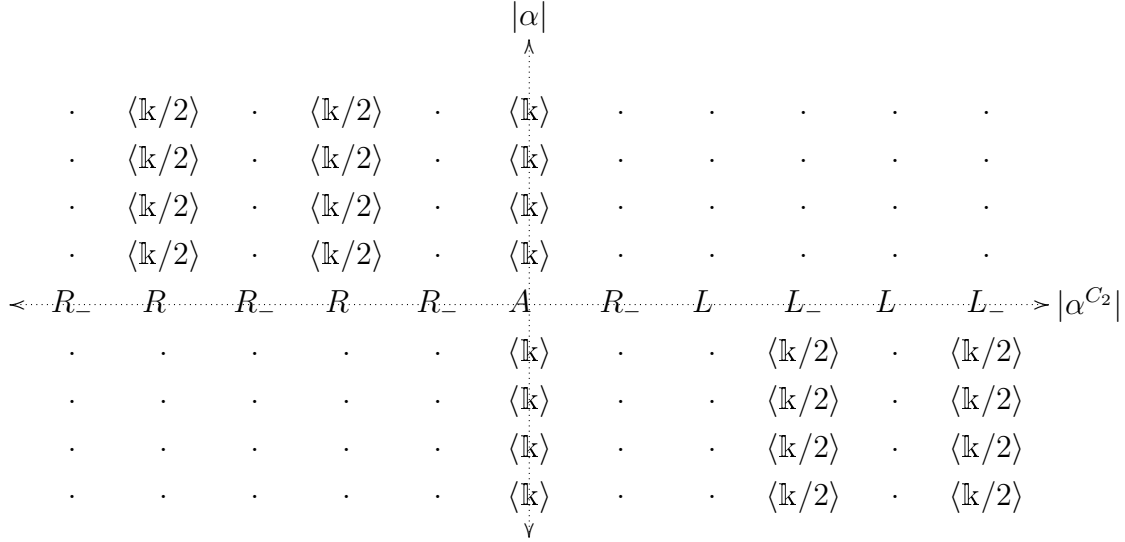


Figure 2.63: A plot of $\tilde{H}_{C_2}^*(S^0)$ in the plane. A dot represents the zero Mackey functor.

Consider first the Mackey functors which appear in $\tilde{H}_{C_2}^*(S^0)$: A , L , L_- , R , R_- , and $\langle C \rangle$. Once we choose an element $\mu \in A(\bullet)$ which restricts to a generator of $A(\ast) \cong \mathbb{k}$, $\{\mu, t_\rho r_\rho \mu\}$ gives a basis for $A(\bullet)$, and $r_\rho \mu$ is a generator of $A(\ast)$. Likewise, if a Mackey functor M is equal to L , L_- , or R_- , choosing a generator ι of $M(\ast)$ gives us the generator $t_\rho \iota$ of $M(\bullet)$. Finally, if $M = R$ or $M = \langle \mathbb{k} \rangle$ or $M = \langle \mathbb{k}/2 \rangle$, then choosing a generator ξ for $M(\bullet)$ gives the generator $r_\rho \xi$ of $M(\ast)$.

Recall that we use ζ to denote the real one-dimensional sign representation of C_2 . For ease of reference with Figure 2.63, note that the trivial representation has dimension $(1, 1)$ and ζ has dimension $(0, 1)$; so the virtual representation $1 - \zeta$ has dimension $(1, 0)$.

Definition 2.64. Define generators 1 , ι , ι^{-1} , ϵ , and ξ for certain of the Mackey functors $\tilde{H}_{C_2}^\alpha(S^0)$ as follows.

1. Let $1 \in \tilde{H}_{C_2}^0(S^0)(\bullet)$ be the image of the identity in $A(\bullet)$ under the unit map $A \longrightarrow \tilde{H}_{C_2}^*(S^0)$ of the graded Green functor $\tilde{H}_{C_2}^*(S^0)$.
2. The underlying nonequivariant space of S^ζ is S^1 . Recall that $\tilde{H}_{C_2}^\alpha(X)(\ast) \cong \tilde{H}^{|\alpha|}(X; \mathbb{k})$, the nonequivariant cohomology. As a result, fixing a nonequivariant identification of S^ζ with S^1 induces isomorphisms

$$\tilde{H}_{C_2}^0(S^0)(\ast) \cong \tilde{H}_{C_2}^1(S^1)(\ast) \longrightarrow \tilde{H}_{C_2}^1(S^\zeta)(\ast) \cong \tilde{H}_{C_2}^{1-\zeta}(S^0)(\ast)$$

and

$$\tilde{H}_{C_2}^0(S^0)(\ast) \cong \tilde{H}_{C_2}^\zeta(S^\zeta)(\ast) \longrightarrow \tilde{H}_{C_2}^\zeta(S^1)(\ast) \cong \tilde{H}_{C_2}^{\zeta-1}(S^0)(\ast).$$

Suggestively, we give the names $\iota \in \tilde{H}_{C_2}^{1-\zeta}(S^0)(\ast)$ and $\iota^{-1} \in \tilde{H}_{C_2}^{\zeta-1}(S^0)(\ast)$ to the images of $r_\rho 1$ under these composites.

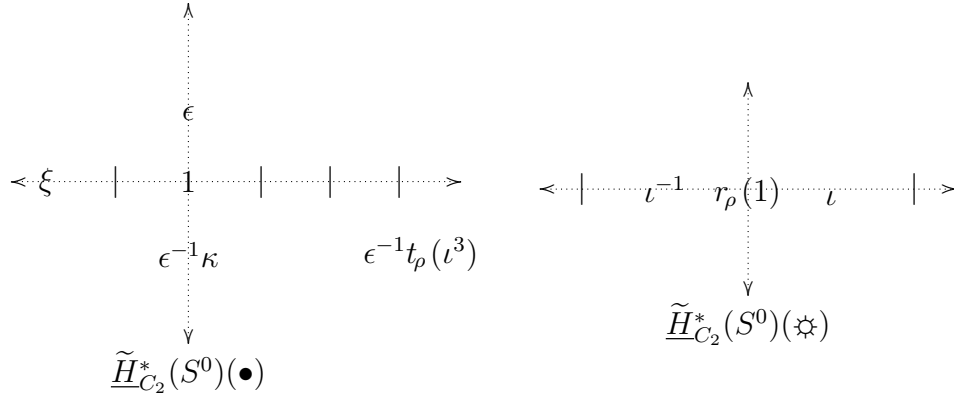


Figure 2.65: The locations of the generators ϵ , ι , ι^{-1} , ξ , $\epsilon^{-1}\kappa$, and $\epsilon^{-1}t_\rho(\iota^3)$.

3. Using the suspension isomorphism, the inclusion $S^0 \hookrightarrow S^\zeta$ induces a map

$$A \cong \tilde{H}_{C_2}^0(S^0) \cong \tilde{H}_{C_2}^\zeta(S^\zeta) \longrightarrow \tilde{H}_{C_2}^\zeta(S^0) \cong \langle \mathbb{k} \rangle.$$

Let ϵ be the image of $1 \in \tilde{H}_{C_2}^0(S^0)(\bullet)$ under this map.

4. Since r_ρ in R is an isomorphism, there is a unique element ξ in $\tilde{H}_{C_2}^{2\zeta-2}(S^0)(\bullet)$ whose image under r_ρ is $(\iota^{-1})^2$, i.e. the image of $\iota^{-1} \otimes \iota^{-1}$ under the appropriate Dress pairing.

For the final classes of generators, we need some lemmas. We have chosen the basis $\{1, t_\rho r_\rho 1\}$ of $\tilde{H}_{C_2}^0(S^0)(\bullet)$. However, employing the same trick as in the proof of Lemma 2.42, if we define $\kappa = 2 - t_\rho r_\rho 1$, then $\{1, \kappa\}$ is also a basis. It has the advantage that κ generates the kernel of r_ρ .

Lemma 2.66 (Lewis). *For each $m \geq 1$, there is a unique element $\epsilon^{-m}\kappa \in \tilde{H}_{C_2}^{-m\zeta}(S^0)(\bullet)$ such that, under the Dress pairing maps, $\epsilon^m(\epsilon^{-m}\kappa) = \kappa$. Moreover, $\epsilon^{-m}\kappa$ is a generator of $\tilde{H}_{C_2}^{-m\zeta}(S^0)(\bullet)$ for each m .*

Similarly, the elements $t_\rho(\iota^{2n+1})$ for $n \geq 1$ are divisible by ϵ .

Lemma 2.67 (Lewis). *For each $m \geq 1$ and $n \geq 1$, there is a unique element $\epsilon^{-m}t_\rho(\iota^{2n+1})$ in dimension $(2n+1, -m)$ such that, under the Dress pairing maps,*

$$\epsilon^m(\epsilon^{-m}t_\rho(\iota^{2n+1})) = t_\rho(\iota^{2n+1}).$$

Moreover, each $\epsilon^{-m}t_\rho(\iota^{2n+1})$ is a generator of $\tilde{H}_{C_2}^{(2n+1)(1-\zeta)-m\zeta}(S^0)(\bullet)$.

When dealing with the multiplicative structure of $\tilde{H}_{C_2}^*(S^0)$, it is often helpful to recall the Frobenius relations, i.e. the equations given in Proposition 2.47 and coming from the

commuting diagrams in Lemma 2.17. For example, looking at the levelwise ring structure of $A = \underline{\tilde{H}}_{C_2}^0(S^0)$, these tell us that

$$(t_\rho r_\rho 1)^2 = t_\rho ((r_\rho 1)(r_\rho t_\rho r_\rho 1)) = 2(t_\rho r_\rho 1),$$

using $G = C_2$. It follows that $\kappa^2 = 2\kappa$ as well.

Theorem 2.68 (Lewis). *As an algebra over $A(\ast) = \mathbb{k}$, $\underline{\tilde{H}}_{C_2}^*(S^0)(\ast) \cong \mathbb{k}[\iota, \iota^{-1}]$, the polynomial algebra on generators ι in dimension $(1, 0)$ and ι^{-1} in dimension $(-1, 0)$, subject to the relation $\iota \cdot \iota^{-1} = r_\rho(1)$, the unit of $\underline{\tilde{H}}_{C_2}^*(S^0)$.*

As an algebra over $A(\bullet) = A(C_2)$, the classical Burnside ring, $\underline{\tilde{H}}_{C_2}^(S^0)(\bullet)$ is graded commutative, as discussed in Proposition 2.60 and Subsection 2.5.2. It is generated by the elements*

$$\begin{aligned} \epsilon &\in \underline{\tilde{H}}_{C_2}^\zeta(S^0)(\bullet) \cong \langle \mathbb{k} \rangle(\bullet) \\ \xi &\in \underline{\tilde{H}}_{C_2}^{2\zeta-2}(S^0)(\bullet) \cong R(\bullet) \\ \epsilon^{-m}\kappa &\in \underline{\tilde{H}}_{C_2}^{-m\zeta}(S^0)(\bullet) \cong \langle \mathbb{k} \rangle(\bullet) && \text{for each } m \geq 1 \\ t_\rho(\iota^m) &\in \underline{\tilde{H}}_{C_2}^{m(1-\zeta)}(S^0)(\bullet) && \text{for each } m \geq 2 \\ \epsilon^{-m}t_\rho(\iota^{2n+1}) &\in \underline{\tilde{H}}_{C_2}^{(2n+1)(1-\zeta)-m\zeta}(S^0)(\bullet) \cong \langle \mathbb{k}/2 \rangle(\bullet) && \text{for each } m, n \geq 1 \end{aligned}$$

subject to the following relations. First, the underlying graded $A(\bullet)$ -module of $\underline{\tilde{H}}_{C_2}^*(S^0)(\bullet)$ is as described in Theorem 2.62; this forces the vanishing of some products. Further, the Frobenius relation $t_\rho(x) \cdot y = t_\rho(x \cdot r_\rho(y))$ dictates relations involving products with $t_\rho(r_\rho(1))$ and the generators $t_\rho(\iota^m)$. Finally, we have the following relations:

$$\begin{aligned} \epsilon(\epsilon^{-1}\kappa) &= \kappa \\ \epsilon(\epsilon^{-1}t_\rho(\iota^{2n+1})) &= t_\rho(\iota^{2n+1}) \end{aligned}$$

The generators of $\underline{\tilde{H}}_{C_2}^*(S^0)(\bullet)$ and $\underline{\tilde{H}}_{C_2}^*(S^0)(\ast)$ are related by $r_\rho \xi = \iota^{-2}$.

Proof. See the appendix to [Lew88]. □

Remark 2.69. We know $\underline{\tilde{H}}_{C_2}^*(S^0)$ is graded commutative. Using the notation of Proposition 2.60, $(1 - \tau)$ acts trivially on each \mathbb{k} -module $\underline{\tilde{H}}_{C_2}^{m\zeta}(S^0)(\bullet) \cong \langle \mathbb{k} \rangle$, so

$$\epsilon^{m_1+m_2}(\epsilon^{-m_1}\kappa)(\epsilon^{-m_2}\kappa) = \epsilon^{m_1}(\epsilon^{-m_1}\kappa)\epsilon^{m_2}(\epsilon^{-m_2}\kappa) = \kappa^2 = 2\kappa$$

and thus

$$(\epsilon^{-m_1}\kappa)(\epsilon^{-m_2}\kappa) = 2\epsilon^{-(m_1+m_2)}\kappa.$$

Similar reasoning, combined with the Frobenius relations, shows that

$$(\epsilon^{-m_1} t_\rho(\iota^{2n+1}))(\epsilon^{-m_2} \kappa) = 0$$

and

$$\xi(\epsilon^{-m} t_\rho(\iota^{2n+1})) = \epsilon^{-m} t_\rho(\iota^{2n-1}) \text{ for } m \geq 0, n \geq 2.$$

Instead of describing the underlying additive structure of $\widetilde{H}_{C_2}^*(S^0)$ and using this to simplify the description of the multiplicative structure, we could alternatively present $\widetilde{H}_{C_2}^*(S^0)$ solely in terms of generators and relations by explicitly listing all relations forced by the vanishing of a target, e.g. $t_\rho(\iota) = 0$; $2\xi\epsilon = 0$; and, using the Frobenius relations,

$$t_\rho(\iota^{m_1})t_\rho(\iota^{m_2}) = \begin{cases} 0 & \text{if } m_1 \text{ or } m_2 \text{ is odd;} \\ 2t_\rho(\iota^{m_1+m_2}) & \text{if both are even.} \end{cases}$$

See [Lew88], Theorem 4.3, for a complete presentation from this point of view.

2.5.5 The cohomology of a point for $p > 2$

As we did in Subsection 2.5.4, we will give the additive and multiplicative structure of $\widetilde{H}_{C_p}^*(S^0)$ without proof. Proofs for all claims in this subsection can be found in [Lew88].

Recall from the beginning of this section that C_p for $p > 2$ has $\frac{p+1}{2}$ irreducible representations: the one-dimensional trivial representation, and $\frac{p-1}{2}$ two-dimensional representations $\lambda_1, \dots, \lambda_{(p-1)/2}$ on which C_p acts by rotation. For concreteness, we make the following definition.

Definition 2.70. Fix a generator g of C_p . For each $1 \leq k \leq \frac{p-1}{2}$, let λ_k be the underlying real representation of the complex one-dimensional representation given by $gz = e^{2\pi ik/p}z$.

It turns out that the isomorphism type of the Mackey functor $\widetilde{H}_{C_p}^\alpha(S^0)$ depends only on the ordered pair of dimensions $(|\alpha^{C_p}|, |\alpha|)$ and a constant $d(\alpha) \in \mathbb{F}_p^\times / \{\pm 1\}$, defined below. In fact, if $|\alpha|$ and $|\alpha^{C_p}|$ are not both zero, the isomorphism type depends only on the ordered pair $(|\alpha^{C_p}|, |\alpha|)$. As a result, we will continue to display $\widetilde{H}_{C_p}^*(S^0)$ pictorially on a two-dimensional grid with axes corresponding to $|\alpha^{C_p}|$ and $|\alpha|$. Since the irreducible representations have dimensions $(1, 1)$ and $(0, 2)$, the difference between $|\alpha^G|$ and $|\alpha|$ is even for every virtual representation α of C_p ; thus not all ordered pairs give the dimensions of a virtual representation.

When α has dimension $(0, 0)$, $\widetilde{H}_{C_p}^*(S^0) \cong_{d(\alpha)} A$ for the constant $d(\alpha)$ alluded to above. Recall from Lemma 2.41 that, for integers d , the isomorphism type of ${}_d A$ depends only on the image of d in $\mathbb{F}_p^\times / \{\pm 1\}$ ¹²; so it makes sense to talk about ${}_d A$ when $d \in \mathbb{F}_p^\times / \{\pm 1\}$. We define the function $d : RO(G) \rightarrow \mathbb{F}_p^\times / \{\pm 1\}$ as follows.

¹²If \mathbb{k} has more units than just ± 1 , it may in fact depend only on a quotient of this.

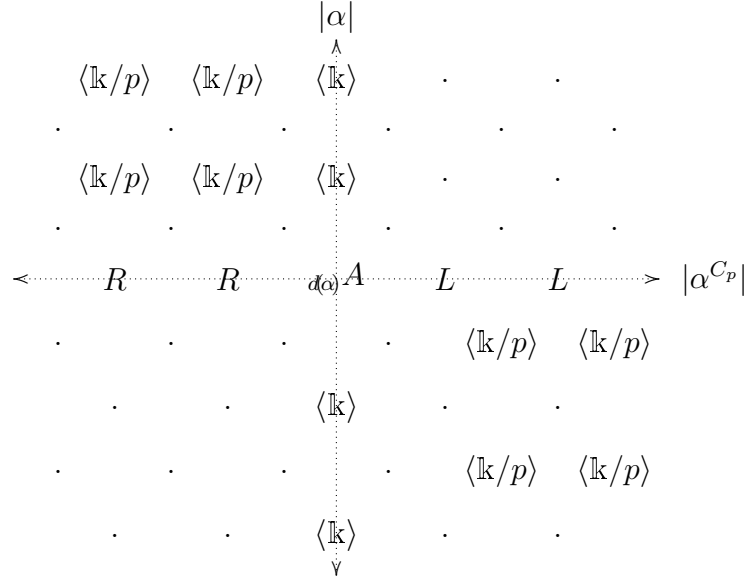


Figure 2.73: A plot of $\tilde{H}_{C_p}^*(S^0)$ in the plane. A dot represents the zero Mackey functor. A blank indicates that no virtual representation has the given dimension.

Definition 2.71. Suppose that $\alpha = a_0 + \sum_j a_j \lambda_j$, for $a_0, a_j \in \mathbb{Z}$. Then we set

$$d(\alpha) = 1^{a_1} 2^{a_2} \dots \left(\frac{p-1}{2} \right)^{a_{(p-1)/2}}.$$

This is the function which appears in Theorem 2.72.

Theorem 2.72 (Lewis, Stong). *When $G = C_p$, the additive structure of $\tilde{H}_{C_p}^*(S^0)$ is as follows.*

$$\tilde{H}_{C_p}^\alpha(S^0) \cong \begin{cases} d(\alpha)A & (|\alpha^{C_p}|, |\alpha|) = (0, 0) \\ R & (|\alpha^{C_p}|, |\alpha|) = (-2m, 0) \text{ for } m \geq 1 \\ L & (|\alpha^{C_p}|, |\alpha|) = (2m, 0) \text{ for } m \geq 1 \\ \langle \mathbb{k} \rangle & (|\alpha^{C_p}|, |\alpha|) = (0, 2n) \text{ for } n \in \mathbb{Z}, n \neq 0 \\ \langle \mathbb{k}/p \rangle & (|\alpha^{C_p}|, |\alpha|) = (-2m, 2n) \text{ for } m \geq 1, n \geq 1 \\ \langle \mathbb{k}/p \rangle & (|\alpha^{C_p}|, |\alpha|) = (2m+1, -2n+1) \text{ for } m \geq 1, n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. See the appendix to [Lew88]. □

This is depicted in Figure 2.73.

We will finish this section by describing the cup product structure on $\widetilde{H}_{C_p}^*(S^0)$. As with $p = 2$, our approach will be to present $\widetilde{H}_{C_p}^*(S^0)(\bullet)$ and $\widetilde{H}_{C_p}^*(S^0)(\ast)$ in terms of generators and relations. Unfortunately, the presence of the twists $d(\alpha)$ for $p > 2$ significantly complicates this approach: choosing a generator μ_α of ${}_{d(\alpha)}A(\bullet)$ means choosing a $d(\alpha) \in \mathbb{k}$ rather than in $\mathbb{F}_p^\times/\{\pm 1\}$, because $r_p(\mu_\alpha)$ must be $d(\alpha)$ times the generator of ${}_{d(\alpha)}A(\ast)$, and in general there is no homomorphism $RO(C_p) \rightarrow \mathbb{k}$. As a result, we must choose a noncanonical set map $RO(C_p) \rightarrow \mathbb{k}$, and $\mu_\alpha\mu_\beta$ will wind up being $\mu_{\alpha+\beta}$ plus an error term. This complication will be compounded when we start thinking about algebras over $\widetilde{H}_{C_p}^*(S^0)$, e.g. $\widetilde{H}_{C_p}^*(B_{C_p}SO(2)) \cong \widetilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_\mathbb{C}))$, the complex projective space on a complete universe $\mathcal{U}_\mathbb{C}$ ¹³.

Definition 2.74. Extend the d of Definition 2.71 to a set map $d: RO(C_p) \rightarrow \mathbb{Z}$ by reinterpreting the definition

$$d(\alpha) = 1^{a_1}2^{a_2} \dots \left(\frac{p-1}{2}\right)^{a_{(p-1)/2}}$$

as follows. If $a_j \geq 0$, then j^{a_j} is as expected. Otherwise, let $\text{inv}(j)$ be the smallest positive integer such that $j \cdot \text{inv}(j) \equiv 1 \pmod{p}$, and interpret j^{a_j} to mean $(\text{inv}(j))^{-a_j} \in \mathbb{k}$.

We can similarly define a map of sets $d: RO(C_p) \rightarrow \mathbb{k}$ by using $(j^{-1})^{-a_j}$ when j is invertible in \mathbb{k} , and $(\text{inv}(j))^{-a_j}$ when it is not.

We will usually rely on context to distinguish whether $d(\alpha)$ refers to an element of \mathbb{F}_p , an integer, or an element of \mathbb{k} .

We are now in a position to define explicit generators of $\widetilde{H}_{C_p}^\alpha(S^0)$ for relevant α . The notation is intended to be parallel to the notation given for $p = 2$. Note that the trivial representation has dimension $(1, 1)$, and each λ_j has dimension $(0, 2)$. No virtual representation has dimension $(1, 0)$, but each $2 - \lambda_j$ has dimension $(2, 0)$.

Definition 2.75. For a generic $\alpha \in RO(C_p)$, write $\alpha = a_0 + \sum_{j=1}^{(p-1)/2} a_j \lambda_j$. Define generators $1, \mu_\alpha, \nu_\alpha, \epsilon_\alpha$, and ξ_α in some of the Mackey functors $\widetilde{H}_{C_p}^\alpha(S^0)$ as follows.

1. Let $1 \in \widetilde{H}_{C_p}^0(S^0)(\bullet)$ be the image of $1 \in A(\bullet)$ under the unit map $A \rightarrow \widetilde{H}_{C_p}^*(S^0)$. This is also the image of the identity map $S^0 \rightarrow S^0$ under the Hurewicz map. For consistency with item 2, we write $\mu_0 = 1$.
2. If $\alpha \neq 0$ has dimension $(0, 0)$, let

$$W = \sum_{j \text{ s.t. } a_j < 0} (-a_j) \lambda_j.$$

Then $\alpha + W$ and W are honest representations with $|\alpha + W| = |W| = 2m$ for some $m > 0$, so each is the sum of m nontrivial irreducible representations. Using our ordering of the λ_j , we can then write $\alpha = \sum_{j=1}^m \alpha_j$, where each α_j is the difference

¹³The $B_{C_p}SO(2)$ indicates that this space classifies real oriented two-plane bundles; see Section 4.4.

of two distinct irreducible representations; that is, $\alpha_j = \lambda_{k_j} - \lambda_{\ell_j}$ for distinct integers $1 \leq k_j, \ell_j \leq \frac{p-1}{2}$. Then our explicit identification of the λ_j implies that the map $\mathbb{C}^m \rightarrow \mathbb{C}^m$ sending

$$(z_1, \dots, z_m) \mapsto (z_1^{d(\alpha_1)}, \dots, z_m^{d(\alpha_m)})$$

extends to one-point compactifications to give an equivariant map $S^W \rightarrow S^{\alpha+W}$. The image of this under the Hurewicz map is an element of

$$\underline{\tilde{H}}_{-\alpha}^{C_p}(S^0)(\bullet) \cong \underline{\tilde{H}}_{C_p}^{\alpha}(S^0)(\bullet),$$

which we call μ_{α} .

3. If α has dimension $(2n, 0)$ for some integer n , then $|\alpha| = 0$ and we can define W as in item 2. The only difference is that α may now have a nonzero number of trivial representations, so $d(\alpha)$ no longer provides a way of constructing an equivariant map $S^W \rightarrow S^{\alpha+W}$. However, we can use our ordering of the irreducible representations to produce a canonical nonequivariant identification $S^W \rightarrow S^{\alpha+W}$. This induces an isomorphism at the \ast level

$$\underline{\tilde{H}}_{C_p}^0(S^0)(\ast) \cong \underline{\tilde{H}}_{C_p}^{\alpha}(S^{\alpha})(\ast) \xrightarrow{\cong} \underline{\tilde{H}}_{C_p}^{\alpha}(S^0).$$

Let ι_{α} be the image of $r_{\rho}(1)$ under this isomorphism.

4. If α has dimension $(0, 2m)$ for some $m > 0$, then we can write α non-uniquely as $\alpha_0 + V$ for some honest representation V of dimension $(0, 2m)$ and $\alpha_0 \in RO(C_p)$ of dimension $(0, 0)$. The inclusion $S^0 \hookrightarrow S^V$ then induces a map on cohomology $\underline{\tilde{H}}_{C_p}^{\alpha_0}(S^0) \rightarrow \underline{\tilde{H}}_{C_p}^{\alpha}(S^0)$. We define ϵ_{α} to be the image of μ_{α_0} under this map. It is proved in [Lew88, Lemma A.11] that ϵ_{α} is independent of the choice of α_0 and V .
5. If α has dimension $(-2n, 0)$ for $n > 0$, then $\underline{\tilde{H}}_{C_p}^{\alpha}(S^0) \cong R$, for which r_{ρ} is an isomorphism. Let ξ_{α} be $r_{\rho}^{-1}(\iota_{\alpha})$, so $r_{\rho}(\xi_{\alpha}) = \iota_{\alpha}$.
6. Select any generator of $\underline{\tilde{H}}_{C_p}^{3-2\lambda_1}(S^0)(\bullet)$ and call it $\nu_{3-2\lambda_1}$.

For $p = 2$, each copy of $\langle \mathbb{k}/2 \rangle$ in the fourth quadrant had a unique generator at the \bullet level. Further, for each such generator, multiplying by a power of ϵ gave a generator of a copy of $L_{-}(\bullet)$, which in turn was the image of a power of ι under t_{ρ} . Sadly, neither of these applies when $p > 2$, since there are no virtual representations of dimension $(2n + 1, 0)$. It is therefore necessary to make a noncanonical choice of generator in at least one of the fourth-quadrant torsion groups; we have chosen $\nu_{3-2\lambda_1}$. Fortunately, this one noncanonical choice is sufficient.

Proposition 2.76. *Suppose $x \in \underline{\tilde{H}}_{C_p}^{\alpha}(S^0)(\bullet)$ for some α in the fourth quadrant, so $|\alpha| < 0$ and $|\alpha^G| > 0$. Let $n, m \geq 0$. For every $\beta \in RO(C_p)$ of dimension $(0, 0)$, γ of dimension $(0, 2m)$, and δ of dimension $(-2n, 0)$, x is uniquely divisible by μ_{β} , ϵ_{γ} , and ξ_{δ} . That is, there are elements $\mu_{\beta}^{-1}x$, $\epsilon_{\gamma}^{-1}x$, and $\xi_{\delta}^{-1}x$ with the expected properties.*

Putting all of these definitions and propositions together, we can get a picture of the multiplicative structure of $\tilde{H}_{C_p}^*(S^0)$. For the most part, the structure is analogous to the simpler structure of Theorem 2.68: there are now multiple generators ι_α , but $\iota_\alpha \iota_\beta = \iota_{\alpha+\beta}$. Similarly, $\epsilon_\alpha \epsilon_\beta = \epsilon_{\alpha+\beta}$, $\mu_\alpha \epsilon_\beta = \epsilon_{\alpha+\beta}$, and $\xi_\alpha \xi_\beta = \xi_{\alpha+\beta}$. However, $\mu_\alpha \xi_\beta = d(\alpha) \xi_{\alpha+\beta}$; and the noncanonical choices of the μ_α for α of dimension $(0, 0)$ lead to some complications. For example, $r_\rho(\mu_\alpha \mu_\beta) = d(\alpha) d(\beta) \iota_{\alpha+\beta}$, but $r_\rho(\mu_{\alpha+\beta}) = d(\alpha + \beta) \iota_{\alpha+\beta}$, and in general $d(\alpha + \beta) \neq d(\alpha) d(\beta)$.

Theorem 2.80 (Lewis). *As an algebra over $A(\star) \cong \mathbb{k}$,*

$$\tilde{H}_{C_p}^*(S^0)(\star) \cong \mathbb{k}[\iota_{\pm(\lambda_2 - \lambda_1)}, \dots, \iota_{\pm(\lambda_{(p-1)/2} - \lambda_1)}, \iota_{\pm(2 - \lambda_1)}] / \sim$$

where the relation \sim is given by $\iota_{-\alpha} \iota_\alpha = r_\rho(1)$ for each ι_α appearing in the list of generators.

As an algebra over $A(\bullet) = A(C_p)$, the classical Burnside ring, $\tilde{H}_{C_p}^*(S^0)(\bullet)$ is graded commutative, as discussed in Proposition 2.58 and Subsection 2.5.2. It is generated by the elements

$$\begin{aligned} \mu_\alpha &\in \tilde{H}_{C_p}^\alpha(S^0)(\bullet) \cong_{d(\alpha)} A(\bullet) && \text{for } \alpha = \pm(\lambda_j - \lambda_1), \quad 2 \leq j \leq \frac{p-1}{2} \\ \epsilon_{\lambda_1} &\in \tilde{H}_{C_p}^{\lambda_1}(S^0)(\bullet) \cong \langle \mathbb{k} \rangle(\bullet) \\ \xi_{\lambda_1-2} &\in \tilde{H}_{C_p}^{\lambda_1-2}(S^0)(\bullet) \cong R(\bullet) \\ \epsilon_{\lambda_1}^{-m} \kappa_0 &\in \tilde{H}_{C_p}^{-m\lambda_1}(S^0)(\bullet) \cong \langle \mathbb{k} \rangle(\bullet) && \text{for each } m \geq 1 \\ \epsilon_{\lambda_1}^{-m} \xi_{\lambda_1-2}^{-n} \nu_{3-2\lambda_1} &\in \tilde{H}_{C_p}^\alpha(S^0)(\bullet) \cong \langle \mathbb{k}/p \rangle(\bullet) && \text{for each } m, n \geq 1 \text{ and} \\ &&& \alpha = (3 - 2\lambda_1) - m\lambda_1 - n(\lambda_1 - 2) \end{aligned}$$

subject to the following relations. First, the underlying graded $A(\bullet)$ -module of $\tilde{H}_{C_p}^*(S^0)(\bullet)$ is as described in Theorem 2.72; this forces the vanishing of some products. Further, the Frobenius relation $t_\rho(x) \cdot y = t_\rho(x \cdot r_\rho(y))$ dictates relations involving products with $t_\rho(r_\rho(1))$ and the generators $t_\rho(\iota_\alpha^m)$. Finally, we have the following relations; the restrictions on the dimensions of the representations involved can be determined from context and so are left

implicit.

$$\begin{aligned}
\epsilon_\alpha \epsilon_\beta &= \epsilon_{\alpha+\beta} \\
\mu_\alpha \epsilon_\beta &= \epsilon_{\alpha+\beta} \\
\xi_\alpha \xi_\beta &= \xi_{\alpha+\beta} \\
\mu_\alpha \xi_\beta &= d(\alpha) \xi_{\alpha+\beta} \\
\epsilon_\alpha \xi_\beta &= d(\delta - \beta) \epsilon_\gamma \xi_\delta && \text{whenever } \alpha + \beta = \gamma + \delta \\
\epsilon_\beta^{-1} \kappa_\alpha &= \epsilon_\delta^{-1} \kappa_\gamma && \text{whenever } \alpha + \delta = \beta + \gamma \\
\epsilon_\beta (\epsilon_\beta^{-1} \kappa_\alpha) &= \kappa_\alpha \\
\mu_\alpha \mu_\beta &= \mu_{\alpha+\beta} + \left(\frac{d(\alpha)d(\beta) - d(\alpha + \beta)}{p} \right) t_\rho(\iota_{\alpha+\beta})
\end{aligned}$$

The generators of $\widetilde{H}_{C_p}^*(S^0)(\bullet)$ and $\widetilde{H}_{C_p}^*(S^0)(\star)$ are related by $r_\rho \mu_\alpha = d(\alpha) \iota_\alpha$, for each α of dimension $(0, 0)$, and $r_\rho \xi_\alpha = \iota_\alpha$.

Remark 2.81. It follows from Theorem 2.80 and the Frobenius relations that $\kappa_\alpha \kappa_\beta = \kappa_{\alpha+\beta}$ and $\mu_\alpha \kappa_\beta = \kappa_{\alpha+\beta}$. Also, if $x \in \widetilde{H}_{C_p}^\beta(S^0)(\bullet)$ for some β in the fourth quadrant, then $\kappa_\alpha x = 0$ because $px = 0$ and $t_\rho(\iota_\alpha)x = t_\rho(\iota_\alpha r_\rho(x)) = 0$.

Note that, in choosing algebra generators at the \bullet level, it is sufficient to choose μ_α for α of the form $\lambda_j - \lambda_1$, since any other representation of dimension $(0, 0)$ is a sum of these. Then for any (m, n) , as long as we have a generator of $\widetilde{H}_{C_p}^\alpha(S^0)(\bullet)$ for some α of dimension (m, n) , multiplication by the μ_α will give generators of $\widetilde{H}_{C_p}^{\alpha'}(S^0)(\bullet)$ for every other α' of dimension (m, n) . This explains the somewhat arbitrary appearance of λ_1 in the set of generators in Theorem 2.80.

CHAPTER 3

COHOMOLOGY OF C_p -CW-COMPLEXES WITH CELLS IN EVEN DIMENSIONS

3.1 A freeness theorem

In this section, we will temporarily ignore the algebra structure of $\widetilde{H}_{C_p}^*(-)$, and view it as a module over $\widetilde{H}_{C_p}^*(S^0)$.

When dealing with the nonequivariant singular cohomology of a CW-complex, it is immediate from the definitions that a space with cells only in even dimensions must have cohomology which is free as a module over the cohomology of a point, because all boundary maps in the singular chain complex are zero. In this section, we will describe a similar result for $\widetilde{H}_{C_p}^*(-)$, and give some applications.

In Definition 2.4, we defined a G -CW-complex with trivial cells. If we are interested in putting CW structures on spaces found in nature, this is a very restrictive definition, and so we make the following definition.

Definition 3.1. A G -CW-complex X is the colimit of a sequence of subspaces $X_n \subset X$ defined as follows. X_0 is a finite G -set. At each stage, X_{n+1} is formed from X_n by attaching cells of the form $G \times_K D(V)$ along the boundary $G \times_K S(V)$. Here $K < G$ is a subgroup, V is a K -representation, and $D(V)$ and $S(V)$ are the unit disk and unit sphere, respectively, of V . We do not place restrictions on the dimension of V . Since the n in X_n has no geometric meaning, we refer to X_n as the n^{th} **filtration** of X . A G -CW-complex is **finite** if it is built from finitely many cells.

For example, a representation sphere S^V has a G -CW structure with one zero cell G/G , i.e. $G \times_G D(0)$, and one cell $G \times_G D(V)$ attached to the zero cell by collapsing the boundary $G \times_G S(V)$ to a point.

This new definition includes the trivial G -CW-complexes of Definition 2.4, since for a trivial representation \mathbb{R}^n , $G/K \times D^n = G \times_K D(\mathbb{R}^n)$.

In general, it is not clear what the appropriate definition of an “even-dimensional” cell is in this context. We can make the following definition, however, which agrees with intuition for $G = C_p$.

Definition 3.2. A cell $G \times_K DV$ is **even-dimensional** if $|V^L|$ is even for every subgroup L of K .

Return now to the group $G = C_p$. The only subgroups of C_p are C_p itself and the trivial subgroup. Thus the cells are either of the form $C_p \times D^n$ or $C_p \times_{C_p} D(V) = D(V)$ for a (possibly trivial) C_p -representation V . Using the definition above, $C_p \times D^n$ is even-dimensional if n is even; $D(V)$ is even-dimensional if $|V|$ and $|V^{C_p}|$ are both even.¹

Consider for a moment a C_p -CW-complex X which is built by attaching one cell $C_p \times_{K_n} D(V_n)$ at each stage n . We then have a cofiber sequence

$$X_{n-1} \longrightarrow X_n \longrightarrow C_{p+} \wedge_{K_n} S^{V_n},$$

inducing a long exact sequence in cohomology,

$$\xrightarrow{\partial} \underline{\tilde{H}}_{C_p}^*(C_{p+} \wedge_{K_n} S^{V_n}) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_n) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_{n-1}) \xrightarrow{\partial} \underline{\tilde{H}}_{C_p}^{*+1}(C_{p+} \wedge_{K_n} S^{V_n}) \longrightarrow$$

From the previous sections, we know that

$$\underline{\tilde{H}}_{C_p}^*(C_{p+} \wedge_{\{e\}} S^n) \cong \underline{\tilde{H}}_{C_p}^*(S^n) \square A_{\star} \cong \underline{\tilde{H}}_{C_p}^{*-n}(S^0) \square A_{\star}$$

and

$$\underline{\tilde{H}}_{C_p}^*(S^V) \cong \underline{\tilde{H}}_{C_p}^{*-V}(S^0),$$

both free modules over $\underline{\tilde{H}}_{C_p}^*(S^0)$. If we knew from an inductive hypothesis that $\underline{\tilde{H}}_{C_p}^*(X_{n-1})$ was free as a module over the graded Green functor $\underline{\tilde{H}}_{C_p}^*(S^0)$, and if we could additionally show that the boundary map ∂ was zero², then the long exact sequence would break up into short exact sequences. Using the freeness of $\underline{\tilde{H}}_{C_p}^*(X_{n-1})$, we could then conclude that

$$\underline{\tilde{H}}_{C_p}^*(X_n) \cong \underline{\tilde{H}}_{C_p}^*(X_{n-1}) \oplus \underline{\tilde{H}}_{C_p}^*(C_{p+} \wedge_{K_n} S^{V_n}),$$

making $\underline{\tilde{H}}_{C_p}^*(X_n)$ free as well. The same reasoning would hold if we allowed finitely many cells to be attached to X at each stage, by the additivity axiom for cohomology. Thus, in such a situation, each X_n would have cohomology which was free over the cohomology of a point. Further, since each $\underline{\tilde{H}}_{C_p}^*(X_n) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_{n-1})$ would then be a projection, the appropriate \lim^1 term would vanish. This would allow us to conclude that $\underline{\tilde{H}}_{C_p}^*(X)$ was the product of copies of $\underline{\tilde{H}}_{C_p}^*(S^0)$ and $\underline{\tilde{H}}_{C_p}^*(C_{p+})$ in the appropriate dimensions.

In the examples we will discuss in this thesis, this product agrees with the corresponding coproduct; so the above sketch outlines a method for showing that the cohomology of such spaces X is free with generators corresponding to the cells attached. The theorem below, a slightly corrected version of Theorem 2.6 in [Lew88], gives several examples of spaces X

¹Recall that, when $p > 2$, $|V|$ and $|V^{C_p}|$ always have the same parity, so either both or neither are even. They may have different parities for $p = 2$, but we still use the same definition of even-dimensional: both must be even.

²Something not necessarily forced by dimensional considerations!

for which the sketch above can be fleshed out.³ These examples are not exhaustive, but they are sufficient for the applications we have in mind.

Recall the notation for shifted Mackey functors from Definition 2.27 on page 16, interpreted for $G = C_p$ on page 26.

Theorem 3.3 (Lewis). *Suppose that X is a C_p -CW-complex meeting the following conditions.*

- *Each X_n is a finite C_p -CW-complex.*
- *All cells of X have even dimension.*
- *Suppose V and W are C_p -representations such that DW is in the n^{th} filtration X_n for some n and DV is one of the cells attached to X_n to create X_{n+1} . If $|V| > |W|$, then $|V^{C_p}| \geq |W^{C_p}|$.*
- *For each positive integer N , only finitely many cells of the form $D(V)$ have $|V| \leq N$, only finitely many have $|V^{C_p}| \leq N$, and only finitely many cells $C_p \times D^n$ have $n \leq N$.*

Then $\tilde{H}_{C_p}^*(X_+)$ is free as a module over the graded Green functor $\tilde{H}_{C_p}^*(S^0)$. Further, $\tilde{H}_{C_p}^*(X_+)$ decomposes as a direct sum. Its summands consist of one copy of $\tilde{H}_{C_p}^*(X_{0+})$, one copy of $\Sigma^V \tilde{H}_{C_p}^*(S^0)$ for each cell of the form $D(V)$ with $|V| > 0$, and one copy of $\Sigma^{2n} \tilde{H}_{C_p}^*(C_{p+}) \cong \Sigma^{2n} \tilde{H}_{C_p}^*(S^0) \square A_{\star}$ for each cell of the form $C_p \times D^{2n}$, $n > 0$.

Proof. As outlined above, it suffices to show two things: that the boundary maps are all zero in the long exact sequences associated to the cofiber sequences

$$X_{n-1} \longrightarrow X_n \longrightarrow X_n/X_{n-1},$$

and that the product $\tilde{H}_{C_p}^*(X_+)$ agrees with the corresponding coproduct. For ease of notation, suppose that each X_n is formed by attaching a single cell to X_{n-1} ; the proof for the general situation is exactly the same, but the notation is more cumbersome.

We will first address the question of the boundary maps ∂ . There are two cases to consider, corresponding to the two types of cell in a C_p -CW-complex. First, suppose that X_n is formed by attaching a cell $C_p \times D^{2m}$ to X_{n-1} , giving the long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{H}_{C_p}^*(C_{p+} \wedge S^{2m}) \longrightarrow \tilde{H}_{C_p}^*(X_{n+}) \longrightarrow \tilde{H}_{C_p}^*(X_{n-1+}) \xrightarrow{\partial} \tilde{H}_{C_p}^{*+1}(C_{p+} \wedge S^{2m}) \longrightarrow \cdots$$

We know that $\tilde{H}_{C_p}^*(C_{p+} \wedge S^{2m}) \cong \tilde{H}_{C_p}^{*-2m}(S^0) \square A_{\star}$, so

$$\tilde{H}_{C_p}^{\alpha+1}(C_{p+} \wedge S^{2m}) \cong \begin{cases} 0 & |\alpha| \neq 2m - 1 \\ A_{\star} & |\alpha| = 2m - 1. \end{cases}$$

³[Lew88, Theorem 2.6] says that X_0 should consist of a single orbit, but this is not a necessary restriction. [Lew88] also does not put restrictions on the dimensions of the cells, which means that its product may not be the same as the desired coproduct.

By hypothesis, $\tilde{H}_{C_p}^*(X_{n-1+})$ is a direct sum of free $\tilde{H}_{C_p}^*(S^0)$ -modules lying in even dimensions. The forgetful functor from $\tilde{H}_{C_p}^*(S^0)$ -modules to graded Mackey functors is right adjoint to the free functor $M \mapsto \tilde{H}_{C_p}^*(S^0) \square M$. It follows that a map of $\tilde{H}_{C_p}^*(S^0)$ -modules out of $\tilde{H}_{C_p}^*(X_{n-1+})$ is the same as a map of Mackey functors out of an appropriate sum of shifted copies of A and A_{\star} . However, these shifted copies of A and A_{\star} all lie in even dimensions, and $\tilde{H}_{C_p}^{*+1}(C_{p+} \wedge S^{2m})$ is zero in all even dimensions. The vanishing of the target thus forces $\partial = 0$.

Next, suppose that X_n is formed by attaching a cell $D(V)$ to X_{n-1} , so that we have the long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{H}_{C_p}^*(S^V) \longrightarrow \tilde{H}_{C_p}^*(X_{n+}) \longrightarrow \tilde{H}_{C_p}^*(X_{n-1+}) \xrightarrow{\partial} \tilde{H}_{C_p}^{*+1}(S^V) \longrightarrow \cdots$$

As before, ∂ is given by a map of Mackey functors out of an appropriate sum of copies of A and A_{\star} . $\tilde{H}_{C_p}^{*+1}(S^V) \cong \tilde{H}_{C_p}^{*+1-V}(S^0)$, where V has even dimension, so we may use our knowledge of $\tilde{H}_{C_p}^*(S^0)$ to examine the structure of the target. Whether p is odd or even, the only even degrees in which $\tilde{H}_{C_p}^{*+1-V}(S^0)$ is nonzero correspond to the copies of $\langle \mathbb{k}/p \rangle$ in the fourth quadrant. Suppose that a copy of A in dimension W has a nonzero target in $\tilde{H}_{C_p}^{*+1-V}(S^0)$. It follows that $(|W^{C_p}|, |W|) = (|V^{C_p}| - 1 + 2k + 1, |V| - 1 - 2\ell)$ for some positive integers k and ℓ . Hence $|V| > |W|$ but $|V^{C_p}| < |W^{C_p}|$, which contradicts the assumption in the theorem about the dimensions of the cells of X . Thus ∂ is zero on the components of $\tilde{H}_{C_p}^*(X_{n-1+})$ corresponding to cells $D(W)$. Similarly, a map out of A_{\star} in the source is determined by selecting an element of $\langle \mathbb{k}/p \rangle(\star)$ in the corresponding dimension in the target. Since $\langle \mathbb{k}/p \rangle(\star) = 0$, ∂ must also be zero on these summands.

It follows that, for complexes X meeting the hypotheses of the theorem, the boundary maps ∂ are all zero. Hence, by induction, $\tilde{H}_{C_p}^*(X_{n+})$ is free on generators in dimensions corresponding to the cells of X_n for each n .

Since $\tilde{H}_{C_p}^*(X_{n+})$ is the direct sum of $\tilde{H}_{C_p}^*(X_{n-1+})$ and a free summand, and $\tilde{H}_{C_p}^*(X_{n+}) \longrightarrow \tilde{H}_{C_p}^*(X_{n-1+})$ is the projection, the appropriate \lim^1 term vanishes and we see that

$$\tilde{H}_{C_p}^*(X_+) = \prod_V \tilde{H}_{C_p}^{*-V}(S^0) \times \prod_m \tilde{H}_{C_p}^{*-2m}(S^0)_{\star}$$

where the products run over the dimensions of the cells in X . The final hypothesis of the theorem guarantees that only finitely many cells of X contribute to the cohomology in any given dimension, because the cohomology of a point vanishes throughout the “third quadrant,” i.e. when $|\alpha^{C_p}| < 0$ and $|\alpha| < 0$. Hence the infinite product above agrees with the corresponding coproduct, and $\tilde{H}_{C_p}^*(X_+)$ is free on generators in dimensions corresponding to the cells of X . \square

Remark 3.4. An analogous but slightly stronger result holds for homology, by a similar argument. Since homology commutes with colimits, we may remove the final hypothesis of the theorem.

Remark 3.5. Ferland and Lewis have shown in [FL04] that, for homology $\widetilde{H}_*^{C_p}(-)$, a similar result with a twist holds if we weaken the third hypothesis of the theorem to say that there are only finitely many cells $D(V)$ attached after $D(W)$ such that $|V| > |W|$ but $|V^{C_p}| < |W^{C_p}|$. In this situation, it is possible to have nonzero boundary maps ∂ . However, Ferland and Lewis show that the resulting homology $\widetilde{H}_*^{C_p}(X_+)$ is still free, but on generators whose dimensions they are unable to determine! In fact, this brings up a related issue: even if $\mathbb{k} = \mathbb{Z}$, it is possible to have two isomorphic free $\widetilde{H}_*^{C_p}(S^0)$ modules whose generators are in different dimensions. The situation becomes either worse or better, depending on point of view, when $\mathbb{k} = \mathbb{Q}$; in that situation, all the possible dimensions for the generators of $\widetilde{H}_*^{C_p}(X_+)$ yield isomorphic free $\widetilde{H}_*^{C_p}(S^0)$ modules.

The main application we have in mind for Theorem 3.3 is $X = \mathbb{C}P(\mathcal{U}_{\mathbb{C}})$, the space of complex lines in a complete complex universe $\mathcal{U}_{\mathbb{C}}$. By a complete universe, we mean a complex C_p -representation which contains countably infinitely many copies of each finite-dimensional representation. For concreteness, we will view $\mathcal{U}_{\mathbb{C}}$ as follows. Let ϕ be the complex one-dimensional representation of C_p whose underlying real representation is the λ_1 of Subsection 2.5.5, and write ϕ^n to mean $\phi^{\otimes n} = \phi \otimes \cdots \otimes \phi$. Then

$$\{\phi, \phi^2, \dots, \phi^{p-1}, \phi^p = 1\}$$

gives a complete set of the irreducible complex representations of C_p . We may then view $\mathcal{U}_{\mathbb{C}}$ as the direct sum

$$\begin{aligned} \mathcal{U}_{\mathbb{C}} &= 1 \oplus \phi \oplus \phi^2 \oplus \cdots \oplus \phi^{p-1} \oplus \phi^p \oplus \phi^{p+1} \oplus \cdots \\ &\cong 1 \oplus \phi \oplus \phi^2 \oplus \cdots \oplus \phi^{p-1} \oplus 1 \oplus \phi \oplus \cdots \end{aligned}$$

built up from the flag of subrepresentations $\mathcal{U}_{\mathbb{C}}^0 \subset \mathcal{U}_{\mathbb{C}}^1 \subset \mathcal{U}_{\mathbb{C}}^2 \subset \cdots$, where we define $\mathcal{U}_{\mathbb{C}}^n := \bigoplus_{k=0}^n \phi^k$.

The usual nonequivariant Schubert cell decomposition generalizes to give a C_p -CW structure on $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$. Explicitly, suppose that we are given a complex line in $\mathcal{U}_{\mathbb{C}}$. We can use the decomposition $\mathcal{U}_{\mathbb{C}} = 1 \oplus \phi \oplus \phi^2 \oplus \cdots$ to choose a basis for $\mathcal{U}_{\mathbb{C}}$; in the coordinates for this basis, our line can be viewed as the set of \mathbb{C} -scalar multiples of a point with coordinates $(x_0, x_1, \dots, x_{N-1}, 1, 0, 0, \dots)$. That is, we choose any point in the line and then normalize so that the last nonzero coordinate is 1. Then, for a fixed N , the set of all points of the form $(x_0, x_1, \dots, x_{N-1}, 1, 0, 0, \dots)$ is the interior of a disk of real dimension $2N$. When we consider the coordinate-wise action of C_p on this disk, we see that it is the interior of the C_p -representation $D(\phi^{-N}\mathcal{U}_{\mathbb{C}}^{N-1})$, with boundary in $D(\phi^{-(N-1)}\mathcal{U}_{\mathbb{C}}^{N-2})$.

Thus, if we define $\omega_N := \phi^{-N}\mathcal{U}_{\mathbb{C}}^{N-1} = \phi^{-N}(1 \oplus \phi \oplus \cdots \oplus \phi^{N-1})$, we see that $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ has a C_p -CW structure with one cell $D(\omega_N)$ of real dimension $(|\omega_N^{C_p}|, |\omega_N|) = (2\lfloor \frac{N}{p} \rfloor, 2N)$ for each integer $N \geq 0$. These dimensions are plotted for $p = 3$ in Figure 3.6.

Corollary 3.7. *Let $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ be the space of complex lines in a complete complex universe $\mathcal{U}_{\mathbb{C}}$. Then the C_p -CW structure on $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ described above meets the requirements of*

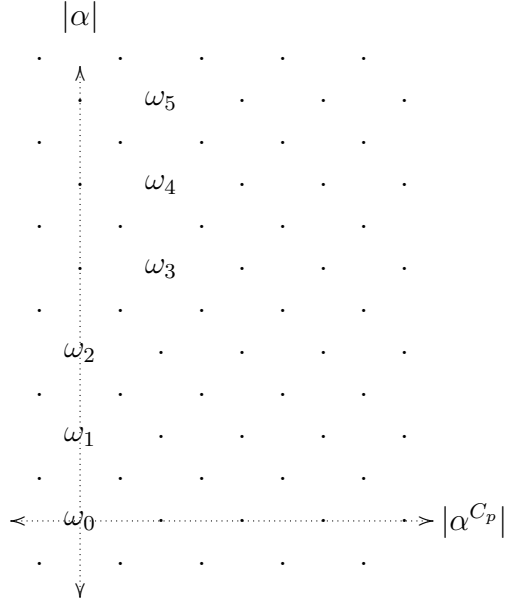


Figure 3.6: The equivariant dimensions of the first few Schubert cell representations ω_N for $p = 3$. The dimensions of virtual representations are marked with dots; dimensions not corresponding to virtual representations are left blank.

Theorem 3.3, so as a module over $\tilde{H}_{C_p}^*(S^0; A)$,

$$\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+; A) \cong \bigoplus_{N \geq 0} \Sigma^{\omega_N} \tilde{H}_{C_p}^*(S^0; A)$$

where Σ^{ω_N} denotes the suspension by the underlying real representation of the complex representation ω_N . \square

Remark 3.8. Notice that choosing a different flag of subrepresentations of $\mathcal{U}_{\mathbb{C}}$ could result in a different C_p -CW structure on $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$, with cells $D(V)$ in *different* dimensions! The resulting cohomologies must be isomorphic as $\tilde{H}_{C_p}^*(S^0)$ -modules, of course, but in general the isomorphism is not induced by a space level map, because there is no equivariant cellular approximation theorem—a G -map between G -CW-complexes need not be G -homotopic to a cellular map.

We also have the following easy consequence of Theorem 3.3.

Corollary 3.9. *Let Y be a CW-complex with cells only in even dimensions, viewed as a C_p -space with the trivial action. Then there is an isomorphism of $\tilde{H}_{C_p}^*(S^0)$ -modules*

$$\tilde{H}_{C_p}^*(Y_+) \cong \tilde{H}_{C_p}^*(S^0) \otimes \tilde{H}^*(Y_+; \mathbb{k}).$$

Proof. The notation simply means that $\widetilde{H}_{C_p}^*(Y_+)$ is the free $\widetilde{H}_{C_p}^*(S^0)$ -module on generators in dimensions of the cells of Y , which all have the trivial C_p action. \square

3.2 Tools for Computing Multiplicative Structure

So far we have not said anything about the multiplicative structure of $\widetilde{H}_{C_p}^*(X_+)$ for G -CW-complexes X satisfying the hypotheses of Theorem 3.3. In this section we will give a result which partially addresses the issue of multiplicative structure.

To begin with, for CW-complexes with a trivial C_p action and cells only in even dimensions, the multiplicative structure of $\widetilde{H}_{C_p}^*(-)$ is determined by the usual nonequivariant cup product structure on $\widetilde{H}^*(-)$.

Proposition 3.10. *Suppose Y is a based CW-complex with cells only in even dimensions such that $\widetilde{H}^*(Y) \cong \operatorname{colim} \widetilde{H}^*(Y_m)$, the colimit of the cohomologies of the skeleta, and similarly for $\widetilde{H}_{C_p}^*(Y)$. Then the isomorphism of Corollary 3.9 is an isomorphism of graded Green functors.*

Proof. We will use the representing spectra of the cohomology theories in question. Let Y be a (nonequivariant) CW-complex with cells only in even dimensions; we will also write Y when viewing it as a C_p -CW-complex with trivial C_p -action. Let $H\mathbb{k}$ be the representing spectrum for nonequivariant cohomology $\widetilde{H}^*(-; \mathbb{k})$, and HA the representing spectrum for $\widetilde{H}_{C_p}^*(-; A)$.

The category of Mackey functors is tensored over \mathbb{k} -modules, and genuine C_p -spectra are tensored over nonequivariant spectra. Thus, in the usual way, the isomorphism $\mathbb{k} \otimes A \rightarrow A$ induces a map $H\mathbb{k} \wedge HA \rightarrow HA$. Combining this with the usual abstract nonsense provides a map of function spectra

$$F(Y, H\mathbb{k}) \wedge F(S^0, HA) \rightarrow F(Y \wedge S^0, H\mathbb{k} \wedge HA) \rightarrow F(Y, HA).$$

Taking homotopy groups or Mackey functors, as appropriate, then produces a map of graded Green functors

$$\varphi: \widetilde{H}^*(Y; \mathbb{k}) \otimes \widetilde{H}_{C_p}^*(S^0) \rightarrow \widetilde{H}_{C_p}^*(Y),$$

natural in Y . We claim that this map is an isomorphism. This can be proved using induction, the long exact sequence in cohomology, and the fact that our Y has cells only in even dimensions. For convenience in what follows, given a ring B , we will write $\mathcal{G}B$ to mean the Green functor $B \otimes \widetilde{H}_{C_p}^*(S^0)$. Thus φ above is a map $\mathcal{G}\widetilde{H}^*(Y; \mathbb{k}) \rightarrow \widetilde{H}_{C_p}^*(Y)$.

Write Y_m for the m -skeleton of Y as a nonequivariant CW-complex. Then we have cofiber sequences $Y_{2m} \rightarrow Y_{2m+2} \rightarrow Y_{2m+2}/Y_{2m}$ for each m , and Y_{2m+2}/Y_{2m} is a wedge of copies of S^{2m+2} . The boundary maps in the associated long exact sequence are zero, so we have a short exact sequence

$$0 \rightarrow \widetilde{H}^n(Y_{2m+2}/Y_{2m}) \rightarrow \widetilde{H}^n(Y_{2m+2}) \rightarrow \widetilde{H}^n(Y_{2m}) \rightarrow 0$$

for each n ; \mathbb{k} coefficients are implicit in the notation. Since the arrows are all induced by maps of spaces, we in fact have a short exact sequence of \mathbb{k} -algebras

$$0 \longrightarrow \tilde{H}^*(Y_{2m+2}/Y_{2m}) \longrightarrow \tilde{H}^*(Y_{2m+2}) \longrightarrow \tilde{H}^*(Y_{2m}) \longrightarrow 0.$$

Since Y has cells only in even dimensions, $\tilde{H}^*(Y_{2m})$ is a free \mathbb{k} -module for each m , and thus tensoring the above short exact sequence with the Mackey functor $\underline{\tilde{H}}_{C_p}^*(S^0)$ gives another short exact sequence

$$0 \longrightarrow \mathcal{G}\tilde{H}^*(Y_{2m+2}/Y_{2m}) \longrightarrow \mathcal{G}\tilde{H}^*(Y_{2m+2}) \longrightarrow \mathcal{G}\tilde{H}^*(Y_{2m}) \longrightarrow 0.$$

Since our map φ above is natural, we get a map of short exact sequences of graded Green functors

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G}\tilde{H}^*(Y_{2m+2}/Y_{2m}) & \longrightarrow & \mathcal{G}\tilde{H}^*(Y_{2m+2}) & \longrightarrow & \mathcal{G}\tilde{H}^*(Y_{2m}) \longrightarrow 0 \\ & & \varphi \downarrow & & \varphi \downarrow & & \varphi \downarrow \\ 0 & \longrightarrow & \underline{\tilde{H}}_{C_p}^*(Y_{2m+2}/Y_{2m}) & \longrightarrow & \underline{\tilde{H}}_{C_p}^*(Y_{2m+2}) & \longrightarrow & \underline{\tilde{H}}_{C_p}^*(Y_{2m}) \longrightarrow 0 \end{array}$$

It is clear from the suspension axiom that

$$\varphi: \mathcal{G}\tilde{H}^*(Y_{2m+2}/Y_{2m}) \longrightarrow \underline{\tilde{H}}_{C_p}^*(Y_{2m+2}/Y_{2m})$$

is an isomorphism. We may then do an inductive argument; the right-hand vertical arrow is an isomorphism by the inductive hypothesis, and it follows from the Five Lemma that the middle arrow is also an isomorphism. It follows by induction that φ is an isomorphism on the $2m$ -skeleton of Y for every m . Since by assumption the cohomology of Y is the colimit of the cohomologies of the skeleta, it follows that

$$\varphi: \tilde{H}^*(Y) \otimes \underline{\tilde{H}}_{C_p}^*(S^0) \longrightarrow \underline{\tilde{H}}_{C_p}^*(Y)$$

is an isomorphism. □

For a C_p -CW-complex X satisfying the hypotheses of Theorem 3.3, this result allows us to compare the unknown ring $\underline{\tilde{H}}_{C_p}^*(X_+)(\bullet)$ with more familiar objects, as follows. The map $C_{p+} \wedge X_+ \longrightarrow X_+$ coming from the projection $\rho: \star \longrightarrow \bullet$ induces a homomorphism of Green functors

$$\rho^*: \underline{\tilde{H}}_{C_p}^*(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^*(C_{p+} \wedge X_+).$$

We have previously seen that $\underline{\tilde{H}}_{C_p}^*(C_{p+} \wedge X_+) \cong \underline{\tilde{H}}_{C_p}^*(X_+)_{\star} = \underline{\tilde{H}}_{C_p}^*(X_+) \square A_{\star\star}$, using the notation for shifted Mackey functors from page 16 and page 26. The map ρ^* can also be viewed as the map $\underline{\tilde{H}}_{C_p}^*(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_+)_{\star}$ induced by the projection $\star \longrightarrow \bullet$, and we will switch back and forth between these views in what follows. Since $\underline{\tilde{H}}_{C_p}^*(X_+)_{\star} \cong \tilde{H}^{|\star|}(X_+)$, the target of ρ^* is something we can already compute.

It also follows from the definition of a C_p -CW-complex and from our definition of “even dimensional” that, if X satisfies the hypotheses of Theorem 3.3, then X^{C_p} is a CW-complex with trivial C_p action, having cells only in even dimensions. Thus Proposition 3.10 applies, so we know $\underline{\tilde{H}}_{C_p}^*(X_+^{C_p})$ as a Green functor. The inclusion $i: X^{C_p} \hookrightarrow X$ induces $\underline{\tilde{H}}_{C_p}^*(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_+^{C_p})$. Putting these together, we see that the map

$$\rho^* \oplus i^*: \underline{\tilde{H}}_{C_p}^*(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^*(X_+)_{\star} \oplus \underline{\tilde{H}}_{C_p}^*(X_+^{C_p})$$

is a homomorphism of Green functors.

We have the following surprising result.⁴

Theorem 3.11. *Let X be a C_p -CW-complex satisfying the hypotheses of Theorem 3.3, so that $\underline{\tilde{H}}_{C_p}^*(X_+)$ is a free $\underline{\tilde{H}}_{C_p}^*(S^0)$ -module. Further suppose that the ground ring \mathbb{k} has no elements of order p . Then for any $\alpha \in RO(C_p)$ of even dimension, the map*

$$\rho^* \oplus i^*: \underline{\tilde{H}}_{C_p}^\alpha(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(X_+)_{\star} \oplus \underline{\tilde{H}}_{C_p}^\alpha(X_+^{C_p})$$

is a monomorphism. Under the same hypotheses, the map

$$\rho^* \oplus \hat{i}^*: \underline{\tilde{H}}_{C_p}^\alpha(X_+) \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(X_+)_{\star} \oplus \left(\langle \mathbb{k} \rangle \square \underline{\tilde{H}}_{C_p}^\alpha(X_+^{C_p}) \right)$$

becomes a monomorphism after tensoring with $\mathbb{k}[1/p]$, for any α . Here \hat{i}^* is the composite

$$\underline{\tilde{H}}_{C_p}^\alpha(X_+) \xrightarrow{i^*} \underline{\tilde{H}}_{C_p}^\alpha(X_+^{C_p}) = A \square \underline{\tilde{H}}_{C_p}^\alpha(X_+^{C_p}) \xrightarrow{1 \square \text{id}} \langle \mathbb{k} \rangle \square \underline{\tilde{H}}_{C_p}^\alpha(X_+^{C_p});$$

the final arrow comes from the map $A \longrightarrow \langle \mathbb{k} \rangle$ taking $\text{id} \mapsto 1 \in \mathbb{k}$.

The proof depends on the following crucial lemma.

Theorem 3.12. *The statement of Theorem 3.11 holds when X_+ is replaced by a sphere of the form $C_{p+} \wedge S^m$ or S^V for a C_p -representation V .*

Note that this lemma does not immediately follow from the suspension isomorphism and the statement for S^0 , because suspension by representation spheres does not preserve fixed points. The proof appears below, after the proof of Theorem 3.11.

By the additivity axiom, it follows from Theorem 3.12 that the statement of the theorem also holds for wedge sums of spheres $C_{p+} \wedge S^{2m}$ and S^V . Given this, the proof of Theorem 3.11 is a straightforward induction argument.

Proof of Theorem 3.11. We proceed by induction. By Theorem 3.12, both maps of Theorem 3.11 are monomorphisms for finite C_p -sets.

⁴Flawed proof given by Lewis in [Lew88]; it has been corrected by the addition of Theorem 3.12.

Now suppose that the result is known for X_{n-1} , and consider X_n . Since X satisfies the hypotheses of Theorem 3.3, we know that the boundary maps in the long exact sequence associated to the cofibration

$$X_{n-1} \longrightarrow X_n \longrightarrow X_n/X_{n-1}$$

are all zero. Since long exact sequences associated to cofibrations are natural, the map $\rho^* \oplus i^*$ induces the commuting diagram of Mackey functors below; the columns are exact.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\tilde{H}_{C_p}^\alpha(X_n/X_{n-1}) & \xrightarrow{\rho^* \oplus i^*} & \tilde{H}_{C_p}^\alpha(X_n/X_{n-1})_{\star} \oplus \tilde{H}_{C_p}^\alpha((X_n/X_{n-1})^{C_p}) \\
\downarrow & & \downarrow \\
\tilde{H}_{C_p}^\alpha(X_{n+}) & \xrightarrow{\rho^* \oplus i^*} & \tilde{H}_{C_p}^\alpha(X_{n+})_{\star} \oplus \tilde{H}_{C_p}^\alpha(X_{n+}^{C_p}) \\
\downarrow & & \downarrow \\
\tilde{H}_{C_p}^\alpha(X_{n-1+}) & \xrightarrow{\rho^* \oplus i^*} & \tilde{H}_{C_p}^\alpha(X_{n-1+})_{\star} \oplus \tilde{H}_{C_p}^\alpha(X_{n-1+}^{C_p}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

If α has even dimension, the bottom horizontal arrow is a monomorphism by the induction hypothesis, and the top one by Theorem 3.12. Hence, by the Five Lemma, the middle arrow is a monomorphism as well. The same holds for the corresponding diagram with horizontal arrows given by $\rho^* \oplus \hat{i}^*$, for any α .

Thus, by induction, the theorem holds for $\tilde{H}_{C_p}^*(X_n)$, for each n . Since X satisfies the hypotheses of Theorem 3.3, $\tilde{H}_{C_p}^*(X) \cong \text{colim } \tilde{H}_{C_p}^*(X_n)$, so the theorem must hold for $\tilde{H}_{C_p}^*(X)$ as well. \square

We now turn to Theorem 3.12. Since there are multiple parts to the argument, we will break the proof into several smaller lemmas. These are then used to prove Theorem 3.12 on page 62.

Lemma 3.13. *The assertions of Theorem 3.11 hold at the \star level, for any C_p -space X .*

Proof. At the \star level,

$$\rho^* : \tilde{H}_{C_p}^*(X_+)_{\star} \longrightarrow \tilde{H}_{C_p}^*(X_+)_{\star \times \star} \cong \bigoplus_{g \in C_p} \tilde{H}_{C_p}^*(X_+)_{\star}$$

is the diagonal map, a monomorphism. Thus $\rho^* \oplus i^*$ and $\rho^* \oplus \hat{i}^*$ are monomorphisms as well, and the latter remains a monomorphism after tensoring with $\mathbb{k}[1/p]$. \square

It remains only to prove the result at the \bullet level. For the lemmas below, recall from Example 2.51 that

$$\langle \mathbb{k} \rangle \square M = \langle M(\bullet) / (\text{im } t_\rho) \rangle,$$

and that $\tilde{H}_{C_p}^*(X)(\bullet) = \tilde{H}_{C_p}^*(X)$ and $\tilde{H}_{C_p}^*(X)(\ast) = \tilde{H}_{C_p}^*(C_{p+} \wedge X)$ by definition.

Lemma 3.14. *If X is a finite C_p -set, then the map*

$$\rho^* \oplus i^* : \tilde{H}_{C_p}^\alpha(X_+) \longrightarrow \tilde{H}_{C_p}^\alpha(C_{p+} \wedge X_+) \oplus \tilde{H}_{C_p}^\alpha(X_+^{C_p})$$

is a monomorphism for α of even dimension, and

$$\rho^* \oplus \hat{i}^* : \tilde{H}_{C_p}^\alpha(X_+) \longrightarrow \tilde{H}_{C_p}^\alpha(C_{p+} \wedge X_+) \oplus \left(\tilde{H}_{C_p}^\alpha(X_+^{C_p}) / (\text{im } t_\rho) \right)$$

becomes a monomorphism for every α after tensoring with $\mathbb{k}[1/p]$.

Proof. The first assertion holds by inspection, since i^* is an isomorphism when the set is $X_+ = S^0$, and ρ^* is the diagonal map, a monomorphism, when $X_+ = C_{p+}$. For the same reason, the second assertion holds when $X_+ = C_{p+}$.

It thus suffices to consider the second assertion when $X_+ = S^0$. There are three cases to consider, based on the dimension of α .

- If $|\alpha| = 0$ and $|\alpha^{C_p}| \neq 0$, then $\tilde{H}_{C_p}^\alpha(S^0)$ is one of R , L , R_- , or L_- . For R , L , and R_- , ρ^* is a monomorphism and remains one after inverting p . For L_- , ρ^* becomes a monomorphism after inverting $p = 2$. Thus $\rho^* \oplus \hat{i}^*$ is a monomorphism as well, in each case.
- If α has dimension $(0, 0)$, we know that $\tilde{H}_{C_p}^\alpha(S^0) = {}_d A$ for some d prime to p , and that $\tilde{H}_{C_p}^\alpha(S^0)_{\ast} = A_{\ast}$. We can also identify $\langle {}_d A(\bullet) / (\text{im } t_\rho) \rangle \cong \mathbb{k}$. It follows that $(\rho^* \oplus \hat{i}^*)(\bullet)$ is the map $\mathbb{k} \oplus \mathbb{k} \longrightarrow \mathbb{k} \oplus \mathbb{k}$ given by $\begin{pmatrix} d & 1 \\ p & 0 \end{pmatrix}$. This is a monomorphism and remains so after tensoring with $\mathbb{k}[1/p]$.
- If $|\alpha| \neq 0$, then $\tilde{H}_{C_p}^\alpha(S^0)$ is $\langle \mathbb{k} \rangle$ or $\langle \mathbb{k}/p \rangle$, so $\text{im } t_\rho = 0$ in $\tilde{H}_{C_p}^\alpha(S^0)(\bullet)$ and \hat{i}^* is an isomorphism. Thus $\rho^* \oplus \hat{i}^*$ is a monomorphism.

Putting these results together gives the statement of the lemma. □

Lemma 3.15. *If λ is the underlying real two dimensional representation of a nontrivial irreducible complex representation of C_p , then the map*

$$\rho^* \oplus i^* : \tilde{H}_{C_p}^\alpha(S^\lambda) \longrightarrow \tilde{H}_{C_p}^\alpha(C_{p+} \wedge S^\lambda) \oplus \tilde{H}_{C_p}^\alpha\left((S^\lambda)_+^{C_p}\right)$$

is a monomorphism for α of even dimension, and

$$\rho^* \oplus \hat{i}^* : \tilde{H}_{C_p}^\alpha(S^\lambda) \longrightarrow \tilde{H}_{C_p}^\alpha(C_{p+} \wedge S^\lambda) \oplus \left(\tilde{H}_{C_p}^\alpha\left((S^\lambda)_+^{C_p}\right) / (\text{im } t_\rho) \right)$$

becomes a monomorphism for every α after tensoring with $\mathbb{k}[1/p]$.

Proof. Write $X = S^\lambda$, for brevity. Then $X^{C_p} = S^0$ and $S^0 \hookrightarrow S^\lambda$ is the inclusion of the points at 0 and ∞ . We can give X the structure of a C_p -CW-complex with trivial cells, as follows. We let $X_0 = S^0 = X^{C_p}$; X_1 is given by attaching a cell $C_p \times D^1$ to X_0 with endpoints the two distinct points of X_0 ; and $X_2 = X$ is given by attaching a cell $C_p \times D^2$ by gluing in copies of D^2 between consecutive copies of D^1 in X_1 . Although this C_p -CW-structure does not have cells only in even dimensions, we know the cohomology of the cofibers $X_1/X_0 = S^1 \wedge C_{p+}$ and $X_2/X_1 = S^2 \wedge C_{p+} = S^\lambda \wedge C_{p+}$, and we can use this to learn about the map induced by $S^0 \hookrightarrow S^\lambda$ on cohomology.

From here, we need to consider the cases $p = 2$ and $p > 2$ separately; the arguments are similar but slightly more complicated for $p = 2$.

We will begin by assuming p is an odd prime. First, consider the cofibration sequence

$$S^0 \longrightarrow X_1 \longrightarrow X_1/S^0 = S^1 \wedge C_{p+} \longrightarrow S^1 \wedge S^0 \longrightarrow \dots$$

The map $S^1 \wedge C_{p+} \longrightarrow S^1 \wedge S^0$ is induced by the projection $\rho: \ast \longrightarrow \bullet$, hence agrees with the restriction r_ρ on cohomology. It follows that, for any α , we have an exact sequence

$$\underline{\tilde{H}}_{C_p}^{\alpha-1}(S^0) \xrightarrow{\rho^*} \underline{\tilde{H}}_{C_p}^{\alpha-1}(C_{p+}) \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(X_1) \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(S^0) \xrightarrow{\rho^*} \underline{\tilde{H}}_{C_p}^\alpha(C_{p+}).$$

This breaks up into a short exact sequence

$$0 \longrightarrow (\text{coker } \rho^*)^{\alpha-1} \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(X_1) \longrightarrow (\ker \rho^*)^\alpha \longrightarrow 0. \quad (3.16)$$

At the \bullet level, ρ^* agrees with the map

$$r_\rho: \underline{\tilde{H}}_{C_p}^*(S^0)(\bullet) \longrightarrow \underline{\tilde{H}}_{C_p}^*(S^0)(\ast) \cong \underline{\tilde{H}}_{C_p}^*(C_{p+})(\bullet),$$

so we can use our knowledge of $\underline{\tilde{H}}_{C_p}^*(S^0)$ to describe the kernel and cokernel of ρ^* . Of the Mackey functors appearing in $\underline{\tilde{H}}_{C_p}^*(S^0)$ for $p > 2$, r_ρ fails to be surjective only for L ; it fails to be injective for $\langle \mathbb{k} \rangle$, $\langle \mathbb{k}/p \rangle$, and ${}_d A$. A pictorial representation of $(\ker \rho^*)(\bullet)$ and $(\text{coker } \rho^*)(\bullet)$ for $p > 2$ is shown in Figure 3.17. For every α , at least one of $(\text{coker } \rho^*)^{\alpha-1}$ and $(\ker \rho^*)^\alpha$ vanishes, so the short exact sequence of (3.16) splits; $\underline{\tilde{H}}_{C_p}^\alpha(X_1) \cong (\text{coker } \rho^*)^{\alpha-1} \oplus (\ker \rho^*)^\alpha$, and the map $\underline{\tilde{H}}_{C_p}^\alpha(X_1) \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(S^0)$ induced by $S^0 \hookrightarrow X_1$ is the projection onto $\ker \rho^*$.

Similarly, in the cofibration sequence

$$X_1 \longrightarrow S^\lambda \longrightarrow S^\lambda \wedge C_{p+} \longrightarrow S^1 \wedge X_1 \longrightarrow \dots$$

the map $S^\lambda \longrightarrow S^\lambda \wedge C_{p+}$ is exactly the transfer associated to the projection $\rho: \ast \longrightarrow \bullet$; see, e.g. [May96, page 83]. Thus the induced map on cohomology is given by $\underline{\tilde{H}}_{C_p}^*(C_{p+})(\bullet) \cong \underline{\tilde{H}}_{C_p}^*(S^0)(\ast) \xrightarrow{t_\rho} \underline{\tilde{H}}_{C_p}^*(S^0)(\bullet)$. It follows as before that the cohomology long exact sequence arising from the cofibration sequence above breaks up into short exact sequences

$$0 \longrightarrow (\text{coker } t_\rho)^{\alpha-\lambda} \longrightarrow \underline{\tilde{H}}_{C_p}^\alpha(X_1) \longrightarrow (\ker t_\rho)^{\alpha-\lambda+1} \longrightarrow 0 \quad (3.18)$$

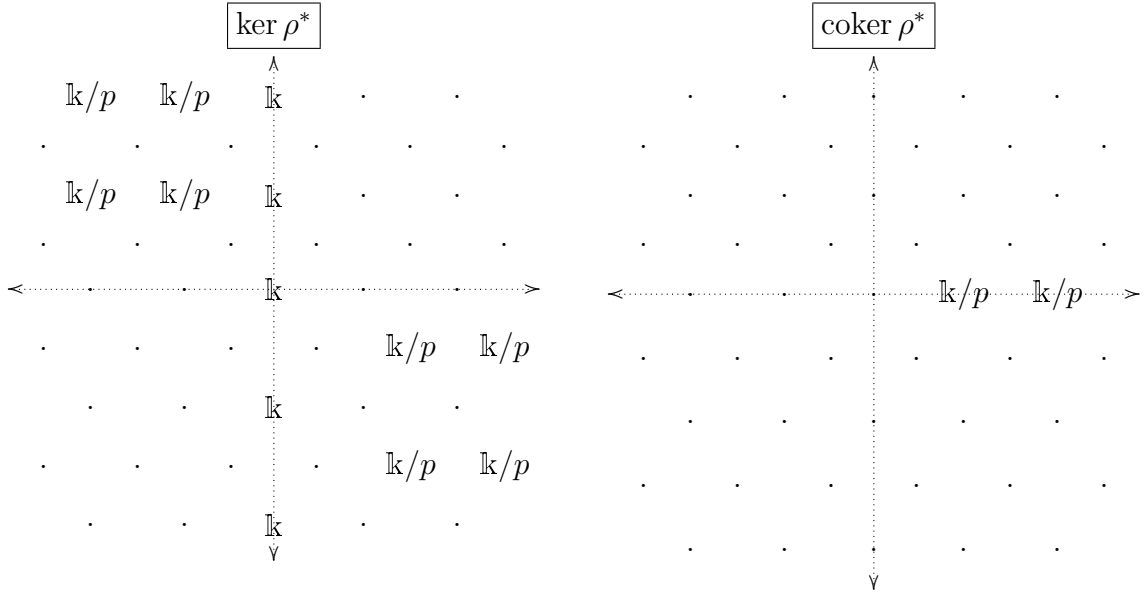


Figure 3.17: A plot showing the kernel and cokernel of ρ at the \bullet level; p is odd in this picture, and \mathbb{k} has no torsion of order p . Compare to Figure 2.73. As always, the axes represent $|\alpha^{C_p}|$ and $|\alpha|$.

where t_ρ denotes the transfer map on $\tilde{H}_{C_p}^*(S^0)$. By assumption \mathbb{k} has no torsion of order p , so $\ker t_\rho$ vanishes for every Mackey functor appearing in $\tilde{H}_{C_p}^*(S^0)$ when $p > 2$. So $\tilde{H}_{C_p}^\alpha(X_1) \cong (\text{coker } t_\rho)^{\alpha-\lambda}$, and $\tilde{H}_{C_p}^*(S^\lambda) \rightarrow \tilde{H}_{C_p}^*(X_1)$ is the projection.⁵ A pictorial representation of $(\text{coker } t_\rho)(\bullet)$ is shown in Figure 3.19.

Putting all of this together, we see that the inclusion of fixed points $S^0 \hookrightarrow S^\lambda$ induces the composite of two projections followed by an inclusion

$$\begin{aligned} \tilde{H}_{C_p}^\alpha(S^\lambda) &\cong \tilde{H}_{C_p}^{\alpha-\lambda}(S^0) \rightarrow (\text{coker } t_\rho)^{\alpha-\lambda} \cong \\ &\cong (\text{coker } \rho^*)^{\alpha-1} \oplus (\ker \rho^*)^\alpha \rightarrow (\ker \rho^*)^\alpha \hookrightarrow \tilde{H}_{C_p}^\alpha(S^0) \end{aligned}$$

at the \bullet level of the cohomology Mackey functors. If we restrict to representations α of even dimension, or if we tensor everything in sight with $\mathbb{k}[1/p]$, then $(\text{coker } \rho^*)^{\alpha-1}$ vanishes, and the kernel of $i^*: \tilde{H}_{C_p}^*(S^\lambda)(\bullet) \rightarrow \tilde{H}_{C_p}^*(S^0)(\bullet)$ is exactly the image of t_ρ in $\tilde{H}_{C_p}^*(S^\lambda)$. The kernel of ρ^* is, of course, the kernel of r_ρ , and so we are interested in the intersection of the kernel of r_ρ and the image of t_ρ .

However, recall that in any Mackey functor M , the composite $r_\rho t_\rho$ is the trace map of the C_p -action on $M(\ast)$. Since $p > 2$ and \mathbb{k} has no p -torsion, this trace map is a monomorphism for every Mackey functor appearing in $\tilde{H}_{C_p}^*(S^\lambda) \cong \tilde{H}_{C_p}^{\ast-\lambda}(S^0)$. Thus $(\ker r_\rho)$

⁵Note that this is compatible with $\tilde{H}_{C_p}^\alpha(X_1) \cong (\text{coker } \rho^*)^{\alpha-1} \oplus (\ker \rho^*)^\alpha$.

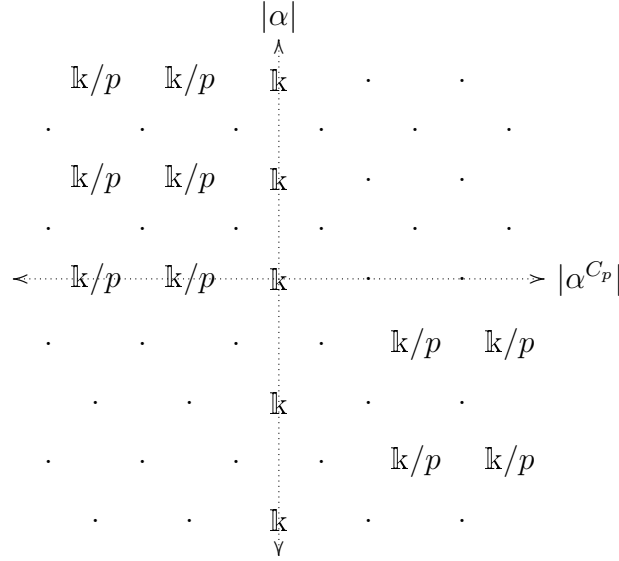


Figure 3.19: A plot showing the cokernel of t_ρ at the \bullet level, again for p odd. Compare to $\tilde{H}_{C_p}^*(S^0)(\bullet)$ in Figure 2.73.

and $(\text{im } t_\rho)$ have trivial intersection, and the map $\rho^* \oplus i^*$ is a monomorphism for every α of even dimension.

Similarly, the image of i^* in $\tilde{H}_{C_p}^\alpha(S^0)$ is the kernel of r_ρ , and thus has empty intersection with the image of t_ρ . It follows that $\rho^* \oplus \hat{i}^*$ becomes a monomorphism after inverting p , for every α . This completes the proof for odd primes.

We next turn to the proof for $p = 2$, where the existence of a nontrivial real one-dimensional representation creates additional dimensions α where $\tilde{H}_{C_2}^\alpha(S^0) \neq 0$. In particular, there are α for which $(\text{coker } \rho^*)^{\alpha-1}$ and $(\ker \rho^*)^\alpha$ are both nonzero, and the short exact sequence of (3.16) does not split. Additionally, $\ker t_\rho \neq 0$ for the Mackey functors L_- and R_- , and all three terms in the short exact sequence of (3.18) are nonzero for $\alpha = \zeta$.

Fortunately, we also have some more explicit knowledge of the spaces involved. We have $\lambda = \zeta \oplus \zeta$, i.e. two copies of the one-dimensional sign representation; $X_1 = S^\zeta$, whose cohomology is known; and the inclusion $X_1 = S^\zeta \hookrightarrow S^{2\zeta} = S^\lambda$ is the suspension of $S^0 \hookrightarrow S^\zeta$.

The arguments for $p > 2$ apply equally well for $p = 2$ in dimensions $\alpha = a_0 + a_1\zeta$ where a_1 is even, and in fact for dimensions where one of the terms in (3.16) and also $(\ker t_\rho)^{\alpha-\lambda+1}$ in (3.18) vanish. Thus we can concentrate on the exceptions. If we generate a picture from Figure 2.63 in the same way that Figure 3.17 was generated from Figure 2.73, we see that these dimensions are exactly those α of the form $a + (1 - a)\zeta$ for some integer a . These all have $|\alpha| = 1$, hence are not of even dimension; so we need only consider α for which $\tilde{H}_{C_2}^\alpha(S^\lambda)(\bullet) \otimes \mathbb{k}[1/2] \neq 0$. The only such α is $\alpha = \zeta$; using the suspension isomorphism,

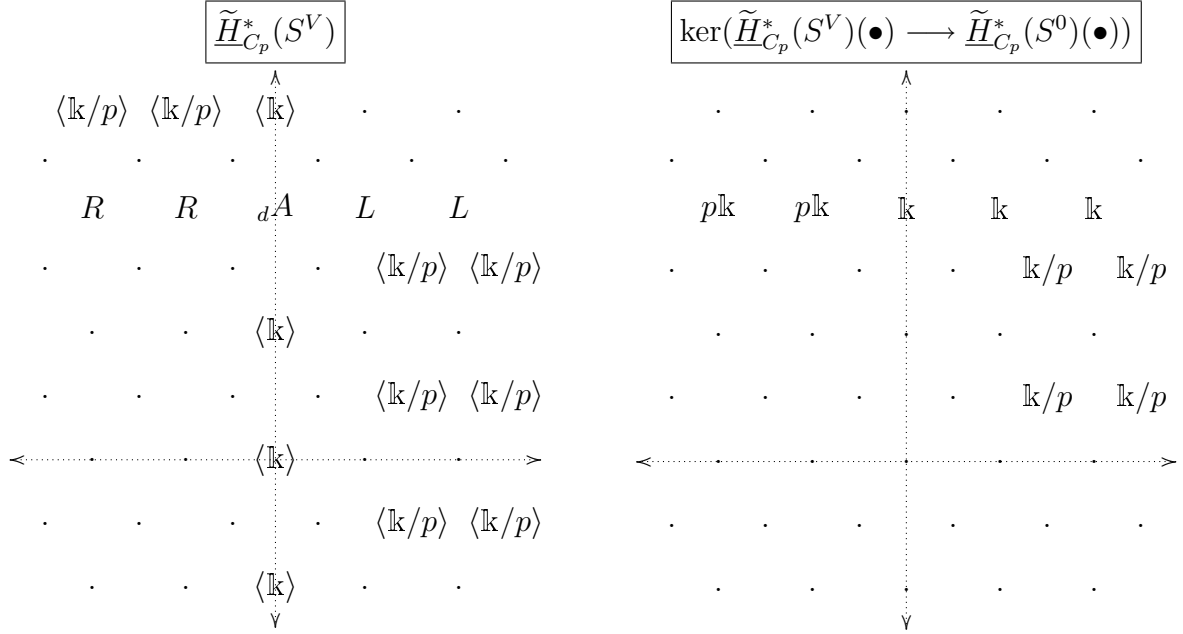


Figure 3.21: $\tilde{H}_{C_p}^*(S^V)$ and the kernel of $\tilde{H}_{C_p}^*(S^V)(\bullet) \rightarrow \tilde{H}_{C_p}^*(S^0)(\bullet)$, shown for $|V| = 4$ and $p > 2$.

$i^*: \tilde{H}_{C_2}^\zeta(S^{2\zeta}) \rightarrow \tilde{H}_{C_2}^\zeta(S^0)$ factors as

$$\tilde{H}_{C_2}^{-\zeta}(S^0) \rightarrow \tilde{H}_{C_2}^0(S^0) \rightarrow \tilde{H}_{C_2}^\zeta(S^0).$$

Using (3.16), we see the left-hand map is a monomorphism with image $\ker r_\rho \subset \tilde{H}_{C_2}^0(S^0)$. From (3.18), we see that the kernel of the right-hand map is $\text{im } t_\rho$. Since $\ker r_\rho \cap \text{im } t_\rho = 0$ in $\tilde{H}_{C_2}^0(S^0)(\bullet) = A(\bullet)$, it follows that $i^*: \mathbb{k} \rightarrow \mathbb{k}$ is a monomorphism and remains so after inverting $p = 2$. Since $\text{im } t_\rho = 0$ in $\tilde{H}_{C_2}^\zeta(S^0)$, this completes the proof of the lemma. \square

Lemma 3.20. *Let V be a C_p -representation with $V^{C_p} = 0$. Then the kernel of the map of \mathbb{k} -modules $i^*: \tilde{H}_{C_p}^*(S^V) \rightarrow \tilde{H}_{C_p}^*(S^0)$ induced by $S^0 \hookrightarrow S^V$ is given by*

$$(\ker i^*)^\alpha = \begin{cases} (\text{im } t_\rho)^\alpha & |\alpha| = |V| \\ \tilde{H}_{C_p}^\alpha(S^V) & 0 < |\alpha| < |V| \text{ and } |\alpha^{C_p}| > 0 \\ 0 = (\text{im } t_\rho)^\alpha & \text{otherwise.} \end{cases}$$

The kernel of i^* is shown pictorially for odd primes p in Figure 3.21. Note that, for $|\alpha| \neq |V|$, the kernel is nonzero only in odd dimensions.

Proof. We know from the proofs of Lemma 3.15 that, for a nontrivial irreducible representation λ and $\alpha \in RO(C_p)$, $S^0 \hookrightarrow S^\lambda$ induces the map

$$\begin{aligned} \tilde{H}_{C_p}^{\alpha+\lambda}(S^\lambda) &\cong \tilde{H}_{C_p}^\alpha(S^0) \twoheadrightarrow (\text{coker } t_\rho)^\alpha \cong \\ &\cong (\text{coker } \rho^*)^{\alpha+\lambda-1} \oplus (\ker \rho^*)^{\alpha+\lambda} \twoheadrightarrow (\ker \rho^*)^{\alpha+\lambda} \hookrightarrow \tilde{H}_{C_p}^{\alpha+\lambda}(S^0), \end{aligned}$$

whose kernel is as described in the statement of the lemma. In particular, if $|\alpha|$ is even, the kernel of the map $\tilde{H}_{C_p}^\alpha(S^0) \longrightarrow \tilde{H}_{C_p}^{\alpha+\lambda}(S^0)$ is $(\text{im } t_\rho)^\alpha$, and the image is $(\ker r_\rho)^{\alpha+\lambda}$. If $|\alpha|$ is odd, then the map is a surjection onto the target.

Now consider any representation V with $V^{C_p} = 0$, so we can write

$$V = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_\ell,$$

where each λ_i is a nontrivial irreducible representation of C_p . Then the sequence of inclusions $S^0 \hookrightarrow S^{\lambda_1} \hookrightarrow \cdots \hookrightarrow S^{\lambda_1 \oplus \cdots \oplus \lambda_\ell}$ induces maps

$$\tilde{H}_{C_p}^\alpha(S^0) \longrightarrow \tilde{H}_{C_p}^{\alpha+\lambda_\ell}(S^0) \longrightarrow \cdots \longrightarrow \tilde{H}_{C_p}^{\alpha+\lambda_\ell+\cdots+\lambda_1}(S^0)$$

where each arrow is a map of the form described above. The description of the kernel then follows from the description above and the fact that $(\text{im } t_\rho) \cap (\ker r_\rho) = 0$ in $\tilde{H}_{C_p}^*(S^0)$. \square

We are finally in a position to prove the more general statement of Theorem 3.12. As a reminder, the claim is that, when $X = C_{p+} \wedge S^m$ or $X = S^V$ for a C_p -representation V ,

$$\rho^* \oplus i^* : \tilde{H}_{C_p}^\alpha(X) \longrightarrow \tilde{H}_{C_p}^\alpha(X)_{\heartsuit} \oplus \tilde{H}_{C_p}^\alpha(X^{C_p})$$

is a monomorphism, and

$$\rho^* \oplus \hat{i}^* : \tilde{H}_{C_p}^\alpha(X) \longrightarrow \tilde{H}_{C_p}^\alpha(X)_{\heartsuit} \oplus \left(\langle \mathbb{k} \rangle \square \tilde{H}_{C_p}^\alpha(X^{C_p}) \right)$$

becomes one after inverting p in \mathbb{k} .

Proof of Theorem 3.12. For spheres $C_{p+} \wedge S^m$, ρ^* is already a monomorphism, and remains one after tensoring with $\mathbb{k}[1/p]$. The result follows.

Next consider spheres S^V . Lemma 3.13 shows that $\rho^* \oplus i^*$ and $\rho^* \oplus \hat{i}^*$ are monomorphisms at the \heartsuit level, so we restrict attention to the \bullet level. Suspension by a trivial sphere preserves fixed points, and so we may assume without loss of generality that $V^{C_p} = 0$ and so $(S^V)^{C_p} = S^0$.

We know that, for any α , the kernel of $\tilde{H}_{C_p}^\alpha(S^V) \longrightarrow \tilde{H}_{C_p}^\alpha(C_{p+} \wedge S^V)$ is exactly the kernel of $r_\rho : \tilde{H}_{C_p}^\alpha(S^V)(\bullet) \longrightarrow \tilde{H}_{C_p}^\alpha(S^V)(\heartsuit)$; for $|\alpha| \neq |V|$, r_ρ is the zero map on $\tilde{H}_{C_p}^\alpha(S^V)$. From Lemma 3.20, we know that the kernel of i^* is the image of t_ρ in these dimensions. As in Lemma 3.15, it follows that $(\ker \rho^* \oplus i^*)^\alpha = 0$ when $|\alpha| = |V|$, since $(\text{im } t_\rho) \cap (\ker r_\rho) = 0$ in $\tilde{H}_{C_p}^*(S^V)$. In these dimensions, $\text{im } t_\rho = 0$ in the image, and so $(\ker \rho^* \oplus \hat{i}^*)^\alpha = 0$ as well.

For $|\alpha| \neq |V|$ of even dimension, i^* is a monomorphism, and so $\rho^* \oplus i^*$ is as well. The kernel of i^* disappears for all $|\alpha| \neq |V|$ after inverting p , and the image of i^* is contained in the kernel of r_ρ , which does not intersect t_ρ . It follows that $r_\rho \oplus \hat{i}^*$ becomes a monomorphism in all dimensions after tensoring with $\mathbb{k}[1/p]$. \square

3.3 Cohomology of Complex Projective Spaces

We have already seen in Corollary 3.7 that, as a module over $\tilde{H}_{C_p}^*(S^0)$, the cohomology $\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+)$ of the complex projective space on a complete universe is free on generators corresponding to the cells $D(\omega_N)$ in the equivariant Schubert cell decomposition; ω_N is the complex representation

$$\omega_N = \phi^{-N}(1 \oplus \phi \oplus \cdots \oplus \phi^{N-1}),$$

where ϕ has underlying real representation λ_1 . For ease of notation, we will also use ω_N to denote the underlying real representation of ω_N , relying on context to determine whether a real or complex representation is needed.

In the remainder of this section we will sketch how to use Theorem 3.11 to find the multiplicative structure of $\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+)$, since it will be relevant to the computation of $\tilde{H}_{C_p}^*(B_{C_p}O(2)_+)$ in Chapter 5. We will use the Dress pairing of Lemma 2.17 and Proposition 2.47 and focus on $\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+(\bullet)) = \tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+)$, since the nonequivariant cohomology $\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+(\ast))$ is known. Further, since we are working with unreduced cohomology in this section, we will write $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ rather than $\tilde{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})_+)$ and $H_{C_p}^*(pt)$ rather than $\tilde{H}_{C_p}^*(S^0)$. Full details of all results presented below, for $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ and for the more general computation of $H_{C_p}^*(\mathbb{C}P(V))$ for any C_p -representation V , can be found in [Lew88].

We would first like to describe the maps $\rho^* \oplus i^*$ and $\rho^* \oplus \hat{i}^*$ of Theorem 3.11 explicitly; we will start by examining $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p}$. A point of $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ is a line in $\mathcal{U}_{\mathbb{C}}$ through the origin and a point $x = (x_0, x_1, \dots, x_{N-1}, 1, 0, \dots)$. This line will lie in $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p}$ if gx is a \mathbb{C} -scalar multiple of x . It is clear that this happens only when all nonzero coordinates x_j come from isomorphic representations ϕ^j . Since there are p irreducible complex representations of C_p , it follows that

$$\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p} \simeq \prod_{N=0}^{p-1} \mathbb{C}P((\phi^N)^{\oplus \infty}) = \prod_{N=0}^{p-1} \mathbb{C}P^\infty.$$

Thus $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p}) \cong \bigoplus_p H_{C_p}^*(\mathbb{C}P^\infty)$. Since finite products and coproducts of \mathbb{k} -modules agree⁶, the map $i^*: H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})) \rightarrow H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p})$ is determined by p maps $i_N^*: H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})) \rightarrow H_{C_p}^*(\mathbb{C}P((\phi^N)^{\oplus \infty}))$; and so $\rho^* \oplus i^*$ is determined by ρ^* together with the maps i^* . In this context, Theorem 3.11 tells us that, for even-dimensional α , two elements $x, y \in H_{C_p}^\alpha(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ are the same if and only if $\rho^*(x) = \rho^*(y)$ and $i_N^*(x) = i_N^*(y)$

⁶The same is true of Mackey functors.

for each $0 \leq N \leq p-1$. An analogous statement holds for every α and \hat{i}^* after tensoring with $\mathbb{k}[1/p]$.

Since we know the multiplicative structures of the targets of ρ^* , i^* , and \hat{i}^* , we will be able to identify the products of generators in $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ once we have computed the values of ρ^* and i_N^* or \hat{i}_N^* on these generators.

Proposition 3.22 (Lewis). *Let i_N^* for $0 \leq N \leq p-1$ be the components of $i^*: H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})) \rightarrow H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}})^{C_p})$, as described above. Then there are elements $C \in H_{C_p}^{\omega_p}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ and, for each $1 \leq j \leq p-1$, $D_j \in H_{C_p}^{\omega_j}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$, satisfying the following conditions.*

1. *If we write $D_0 = 1$, then $\{D_j C^n\}_{0 \leq j \leq p-1, n \geq 0}$ gives a complete set of generators of $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ as a module over $H_{C_p}^*(pt)$.*

2. *Given a chosen generator z of $\underline{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))(\star) \cong H^{|\star|}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$,*

$$\rho^*(D_j) = z^j \text{ and } \rho^*(C) = z^p.$$

3. *Let $H_{C_p}^*(\mathbb{C}P((\phi^N)^{\oplus \infty})) \cong \mathbb{k}[z_N] \otimes H_{C_p}^*(pt)$. Using brackets $[-]$ to denote the image of an element of $H_{C_p}^*(-)$ in $H_{C_p}^*(-)/(\text{im } t_\rho)$, we have the following. For each $0 \leq N \leq p-1$,*

$$\hat{i}_N^*(D_j) = [d_{N,j} \epsilon_{\omega_j}] \text{ and } \hat{i}_N^*(C) = [\epsilon_{\omega_{p-1}} z_N]$$

where $d_{N,j}$ is given by

$$d_{N,j} = \begin{cases} 0 & N < j \\ 1 & N = j \\ \prod_{i=0}^{j-1} d(\lambda_{|i-N|} - \lambda_{|i-j|}) & N > j \end{cases}$$

Proof. See [Lew88, Construction 6.1]. The idea is to identify appropriate generators in the cohomology of certain small finite projective spaces, and then pass to the projective spaces of interest by attaching cells and looking at the corresponding long exact sequences in cohomology. \square

Proposition 3.22 puts us in a position to completely describe the multiplicative structure of $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$. Since the $\{D_j C^n\}$ give a complete set of additive generators, dimensional considerations show that $(D_j C^n)(D_k C^m)$ must be a $H_{C_p}^*(pt)$ -linear combination of the elements $D_0 C^{n+m}$ through $D_{p-1} C^{n+m}$ and, if $j+k \geq p$, of $D_0 C^{n+m+1}$ through $D_{j+k-p} C^{n+m+1}$. After quotienting by $\text{im } t_\rho$ and inverting p , the contributions from the $D_j C^{n+m+1}$ vanish. Using Proposition 3.22 and the fact that each \hat{i}_N^* is a map of algebras, this leaves us with p equations in p unknowns, which can be used to determine the coefficients of $D_0 C^{n+m}$ through $D_{p-1} C^{n+m}$. We can then go through a similar process using i_N^* to determine the coefficients of $D_0 C^{n+m+1}$ through $D_{j+k-p} C^{n+m+1}$.

Unfortunately, for odd primes, there does not appear to be a good closed form for these coefficients, since their values depend on our noncanonical choices for the $d(\alpha)$, which in turn depend on the “mod p inverses” $\text{inv}(j)$. Even with $\mathbb{k} = \mathbb{Q}$, there is some difficulty, since \mathbb{F}_p is not a subfield of \mathbb{Q} . Thus we will simply say the following.

Theorem 3.23 (Lewis). *As a graded commutative algebra over $H_{C_p}^*(pt)$, $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_C))$ is generated by elements C in dimension ω_p and D_j in dimension ω_j for each $1 \leq j \leq p-1$. C generates a polynomial subalgebra of $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_C))$, and a complete set of additive generators of $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_C))$ is given by the elements $\{D_j C^n\}_{0 \leq j \leq p-1, n \geq 0}$. If $j \leq k \leq p-1$, then each product $D_j D_k = D_k D_j$ is given by a linear combination over $H_{C_p}^*(pt)$ of the elements D_j through D_{j+k} , if $j+k \leq p-1$, and of D_j through D_{p-1} and $D_0 C = C$ through $D_{j+k-p} C$ if $j+k \geq p$.*

However, when $p = 2$, the issues with $d(\alpha)$ vanish, and we have the following description of the algebra structure.

Theorem 3.24 (Lewis). *As a graded commutative algebra over $H_{C_2}^*(pt)$, $H_{C_2}^*(\mathbb{C}P(\mathcal{U}_C))$ is generated by elements $D_1 \in H_{C_2}^{2\zeta}(\mathbb{C}P(\mathcal{U}_C))$ and $C \in H_{C_2}^{2(1+\zeta)}(\mathbb{C}P(\mathcal{U}_C))$, satisfying the single relation*

$$D_1^2 = \epsilon^2 D_1 + \xi C.$$

CHAPTER 4

SPECTRAL SEQUENCES FOR LOCAL COEFFICIENTS

4.1 The nonequivariant situation

Let X be a path-connected based space with universal cover \tilde{X} . Let $\pi = \pi_1(X)$ and let π act on the right of \tilde{X} by deck transformations. As always, fix a ground ring \mathbb{k} . Let M be a left and N be a right module over the group ring $\mathbb{k}[\pi]$. Let C_* be the normalized singular chain complex functor with coefficients in \mathbb{k} .

Definition 4.1. Define the homology of X with coefficients in M by

$$H_*(X; M) = H_*(C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} M).$$

Define the cohomology of X with coefficients in N by

$$H^*(X; N) = H^*(\text{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), N)).$$

Functoriality in M and N for fixed X is clear. For a based map $f: X \rightarrow Y$, where $\pi_1(Y) = \rho$, and for a left $\mathbb{k}[\rho]$ -module P , we may regard P as a $\mathbb{k}[\pi]$ -module by pullback along $\pi_1(f)$, and then, using the standard functorial construction of the universal cover, we obtain

$$f_*: H_*(X; f^*P) \rightarrow H_*(Y; P).$$

Cohomological functoriality is similar. The definition goes back to Eilenberg [Eil47], and has the homology of spaces and the homology of groups as special cases, as discussed below. It deserves more emphasis than it is usually given because it implies spectral sequences for the calculation of homology and cohomology with local coefficients, as we shall recall.

Example 4.2. If π acts trivially on M and N , then $H_*(X; M)$ and $H^*(X; N)$ are the usual homology and cohomology groups of X with coefficients in M and N . We can identify $C_*(X)$ with $C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} \mathbb{k}$, where $\mathbb{k}[\pi]$ acts trivially on \mathbb{k} . This implies the identifications

$$C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} M \cong C_*(X; M) \quad \text{and} \quad \text{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), N) \cong C^*(X; N).$$

Example 4.3. If $X = K(\pi, 1)$, then $H_*(X; M)$ and $H^*(X; N)$ are the usual homology and cohomology groups of π with coefficients in M and N since $C_*(\tilde{X})$ is a $\mathbb{k}[\pi]$ -free resolution of \mathbb{k} . That is,

$$H_*(K(\pi, 1); M) = \text{Tor}_*^{\mathbb{k}[\pi]}(\mathbb{k}, M) \quad \text{and} \quad H^*(K(\pi, 1); M) = \text{Ext}_{\mathbb{k}[\pi]}^*(\mathbb{k}, N).$$

Example 4.4. If $M = \mathbb{k}[\pi] \otimes_{\mathbb{k}} C$ and $N = \text{Hom}_{\mathbb{k}}(\mathbb{k}[\pi], C)$ for a \mathbb{k} -module C , then

$$H_*(X; M) \cong H_*(\tilde{X}, C) \quad \text{and} \quad H^*(X; N) \cong H^*(\tilde{X}; C).$$

Remark 4.5. If we replace N by M (viewed as a right $\mathbb{k}[\pi]$ -module) in the cohomology case of the previous example, then we are forced to impose finiteness restrictions and consider cohomology with compact supports; compare [Hat02, 3H.5].

We have spectral sequences that generalize the last two examples. When π acts trivially on M and N , they can be thought of as versions of the Serre spectral sequence of the evident fibration $\tilde{X} \rightarrow X \rightarrow K(\pi, 1)$.

Theorem 4.6 (Eilenberg Spectral Sequence). *There are spectral sequences*

$$E_{p,q}^2 = \text{Tor}_{p,q}^{\mathbb{k}[\pi]}(H_*(\tilde{X}), M) \implies H_{p+q}(X; M)$$

and

$$E_2^{p,q} = \text{Ext}_{\mathbb{k}[\pi]}^{p,q}(H_*(\tilde{X}), N) \implies H^{p+q}(X; N).$$

Up to notation, these are the spectral sequences given by Cartan and Eilenberg in [CE56, p. 355].

Proof. In the E^2 and E_2 terms, p is the homological degree and q is the internal grading on $H_*(\tilde{X})$. Let $\varepsilon: P_* \rightarrow M$ be a $\mathbb{k}[\pi]$ -projective resolution of M and form the bicomplex

$$C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} P_*;$$

the theorem comes from looking at the two spectral sequences associated to this bicomplex and converging to a common target.

If we filter $C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} P_*$ by the degrees of $C_*(\tilde{X})$, we get a spectral sequence whose E^0 -term has differential $\text{id} \otimes d$. Since $C_*(\tilde{X})$ is a projective $\mathbb{k}[\pi]$ module, the resulting E^1 -term is $C_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} M$, the resulting E^2 -term is $H_*(X; M)$, and $E^2 = E^\infty$. Since E^∞ is concentrated in degree $q = 0$, there is no extension problem; we have identified the target as claimed in the theorem.

Filtering the other way, by the degrees of P_* , we obtain a spectral sequence whose E^0 -term has differential $d \otimes \text{id}$. The resulting E^1 -term is $H_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} P_*$ and the resulting E^2 -term is $\text{Tor}_{*,*}^{\mathbb{k}[\pi]}(H_*(\tilde{X}), M)$. This gives the first statement of the theorem.

The argument in cohomology is similar, starting from the bicomplex

$$\text{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), I^*)$$

for an injective resolution $\eta: N \rightarrow I^*$ of N . □

We record an immediate corollary.

Corollary 4.7. *Let π be a finite group of order n and \mathbb{k} be a field of characteristic prime to n . Then*

$$H_*(X; M) \cong H_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} M \quad \text{and} \quad H^*(X; N) \cong \text{Hom}_{\mathbb{k}[\pi]}(H_*(\tilde{X}), N).$$

Proof. Since $\mathbb{k}[\pi]$ is semi-simple, $E_{p,q}^2 = 0$ and $E_2^{p,q} = 0$ for $p > 0$. Therefore the spectral sequences collapse to the claimed isomorphisms. \square

If π acts trivially on $H_*(\tilde{X})$, so that $H_*(\tilde{X}) \cong H_*(\tilde{X}) \otimes_{\mathbb{k}[\pi]} \mathbb{k}$, then the situation simplifies even further. For ease of notation, let M_π denote the coinvariants M/IM , where $I \subset \mathbb{k}[\pi]$ is the augmentation ideal, and let N^π denote the fixed points of N .

Corollary 4.8. *Suppose that we are in the situation of Corollary 4.7 and that π acts trivially on $H_*(\tilde{X})$. Then*

$$H_*(X; M) \cong H_*(\tilde{X}; M_\pi) \quad \text{and} \quad H^*(X; N) \cong H^*(\tilde{X}; N^\pi).$$
 \square

It remains to identify the homology and cohomology groups of Definition 4.1 with classical (co)homology with local coefficients. To do this, we first need to reconcile our coefficient $\mathbb{k}[\pi]$ -modules with the classical definition of a local coefficient system.

In Definition 4.1, we took M and N to be left and right modules over the group ring $\mathbb{k}[\pi]$ and took $C_*(\tilde{X})$ to be the normalized singular chains of \tilde{X} . A (left or right) $\mathbb{k}[\pi]$ -module M is the same as a (covariant or contravariant) functor from π , viewed as a category with a single object, to the category of \mathbb{k} -modules.

As usual, given a space X , let ΠX be the fundamental groupoid of X ; this is a category whose objects are the points of X and whose morphism sets are homotopy classes of paths between fixed endpoints. By definition, a **local coefficient system** \mathcal{M} on X is a functor (covariant or contravariant depending on context, corresponding to our left and right $\mathbb{k}[\pi]$ -module distinction above) from the fundamental groupoid ΠX to the category of \mathbb{k} -modules. When X is path-connected with basepoint x_0 , the category $\pi = \pi_1(X)$ with single object x_0 is a skeleton of ΠX . Therefore a coefficient system \mathcal{M} is determined by its restriction M to π .

Whitehead [Whi78, VI.3.4 and 3.4*] (see also Hatcher [Hat02, 3H.4]) proves the following result and ascribes it to Eilenberg [Eil47].

Theorem 4.9 (Eilenberg). *For path-connected spaces X and covariant and contravariant local coefficient systems \mathcal{M} and \mathcal{N} on X , the classical homology and cohomology with local coefficients $H_*(X; \mathcal{M})$ and $H^*(X; \mathcal{N})$ are naturally isomorphic to the homology and cohomology groups $H_*(X; M)$ and $H^*(X; N)$, where M and N are the restrictions of \mathcal{M} and \mathcal{N} to π .*

Therefore Theorem 4.6 gives a way to compute the additive structure of homology and cohomology with local coefficients. In particular, if $f: E \rightarrow X$ is a fibration with fiber F and path-connected base space X , it gives a means to compute the homology and cohomology with local coefficients that appear in

$$E_{*,*}^2 = H_*(X; \mathcal{H}_*(F; \mathbb{k})) \quad \text{and} \quad E_2^{*,*} = H^*(X; \mathcal{H}^*(F; \mathbb{k}))$$

of the Serre spectral sequences for the computation of $H_*(E; \mathbb{k})$ and $H^*(E; \mathbb{k})$.

Even the case when π is finite of order n and \mathbb{k} is a field of characteristic prime to n often occurs in practice. More generally, the spectral sequences of Theorem 4.6 help make the Serre spectral sequence amenable to explicit calculation in the presence of non-trivial local coefficient systems.

Note that we have not yet addressed the multiplicative structure of the cohomological Eilenberg spectral sequence; this will be discussed in Section 6.1. However, even without the multiplicative structure, we can already do one example. It will be useful to have the following simple consequence of Definition 4.1.

Proposition 4.10. *Let X be a space and π its fundamental group. For any $\mathbb{k}[\pi]$ -module M , $H^0(X; M) \cong M^\pi$, the π -fixed points of M . If M is a $\mathbb{k}[\pi]$ -algebra, the isomorphism is as algebras.*

Proof. We would like to identify the kernel of

$$\mathrm{Hom}_{\mathbb{k}[\pi]}(C_0(\tilde{X}), M) \longrightarrow \mathrm{Hom}_{\mathbb{k}[\pi]}(C_1(\tilde{X}), M).$$

Since \tilde{X} is connected, $H_0(\tilde{X}) \cong \mathbb{k}$ with the trivial π action. Since $\mathrm{Hom}_{\mathbb{k}[\pi]}$ is left exact, it follows that $H^0(X; M) \cong \mathrm{Hom}_{\mathbb{k}[\pi]}(\mathbb{k}, M) \cong M^\pi$, as claimed. \square

In particular, in the Serre spectral sequence, $E_2^{0,t} \cong \mathcal{H}^t(F; \mathbb{k})^\pi$. If we are in a situation where $E_2^{s,t}$ vanishes for $s > 0$, then Proposition 4.10 completely describes the multiplicative structure on the E_2 page.

Example 4.11. Let $\mathbb{Z}/2$ be the cyclic group of order two, identified with $\{\pm 1\}$ when convenient; we write $\mathbb{Z}/2$ instead of C_2 because we will later want to distinguish the role of structural groups and groups of equivariance. Consider the fibration

$$B\mathrm{det}: BO(2) \longrightarrow B\mathbb{Z}/2$$

with fiber $BSO(2)$. Take coefficients in a finite field $\mathbb{k} = \mathbb{F}_q$ with q an odd prime, so that $\mathbb{k}[\pi]$ is semisimple. The action of the base on the fiber is nontrivial; the Serre spectral sequence for this fibration has

$$E_2^{s,t} = H^s(B\mathbb{Z}/2; \mathcal{H}^t(BSO(2); \mathbb{F}_q)) \implies H^{s+t}(BO(2); \mathbb{F}_q)$$

We know that $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$ with universal cover $S^\infty \simeq pt$, so Corollary 4.8 applies, and $BSO(2) \simeq \mathbb{C}P^\infty$. We also know $H^*(\mathbb{C}P^\infty; \mathbb{F}_q) \cong \mathbb{F}_q[x]$, a polynomial algebra on one generator x in degree two, and that the fundamental group $\pi_1(B\mathbb{Z}/2) \cong \mathbb{Z}/2$ acts on $H^*(\mathbb{C}P^\infty)$ by $x \mapsto -x$.

By Corollary 4.8, we thus have

$$E_2^{s,t} \cong H^s(S^\infty; \mathcal{H}^t(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2})$$

in the Serre spectral sequence; $\mathcal{H}^t(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2}$ is either 0 or \mathbb{F}_q , depending on t . $E_2^{s,t}$ thus vanishes for $s > 0$, so the Serre spectral sequence collapses with no extension problems. Using the observation after Proposition 4.10, we see

$$H^*(BO(2); \mathbb{F}_q) \cong \mathcal{H}^*(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2} \cong \mathbb{F}_q[x^2].$$

The isomorphism is of \mathbb{k} -algebras. We have thus shown the well-known fact that $H^*(BO(2); \mathbb{F}_q)$ is polynomial on one generator (the Pontrjagin class) in degree four. We will later examine an equivariant version of this example.

4.2 Equivariant generalizations

Heading towards an equivariant generalization of Theorem 4.6, we first rephrase the definition of (co)homology with local coefficients. In Section 4.1, we effectively defined homology and cohomology with local coefficients by restricting a local coefficient system $\mathcal{M}: \Pi X \rightarrow \mathbb{k}\text{-mod}$ to a $\mathbb{k}[\pi]$ -module $M: \pi \rightarrow \mathbb{k}\text{-mod}$.

Rather than restricting \mathcal{M} to π , we could instead redefine \tilde{X} to be the universal cover functor $\Pi X^{\text{op}} \rightarrow \text{Top}$ that sends a point $x \in X$ to the space $\tilde{X}(x)$ of equivalence classes of paths starting at x and sends a path γ from x to y to the map $\tilde{X}(y) \rightarrow \tilde{X}(x)$ given by precomposition with γ . Since π is a skeleton of ΠX , the following definition is equivalent to Definition 4.1 when X is connected. By Theorem 4.9, there is no conflict with the classical notation for homology with local coefficients. Let $\text{Ch}_{\mathbb{k}}$ denote the category of chain complexes of \mathbb{k} -modules.

Definition 4.12 (Reformulation of Definition 4.1). Let $\mathcal{M}: \Pi X \rightarrow \mathbb{k}\text{-mod}$ and $\mathcal{N}: \Pi X^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ be functors and let $C_*(\tilde{X}): \Pi X^{\text{op}} \rightarrow \text{Top} \rightarrow \text{Ch}_{\mathbb{k}}$ be the composite of the universal cover functor with the functor C_* . Define the homology of X with coefficients in \mathcal{M} to be

$$H_*(X; \mathcal{M}) = H_*(C_*(\tilde{X}) \otimes_{\Pi X} \mathcal{M})$$

where $\otimes_{\Pi X}$ is the tensor product of functors (which is given by an evident coequalizer diagram). Similarly, define

$$H^*(X; \mathcal{N}) = H^*\left(\text{Hom}_{\Pi X}(C_*(\tilde{X}), \mathcal{N})\right)$$

where $\text{Hom}_{\Pi X}$ is the hom of functors (also known as natural transformations; alternatively, given by an evident equalizer diagram).

Note that our distinctions between left and right and between covariant and contravariant are purely semantic above, since we are dealing with groups and groupoids. However, we are about to consider (Bredon) equivariant homology and cohomology. Here the fundamental “groupoid” is only an EI-category (endomorphisms are isomorphisms) and the distinction is essential. There is an equivariant Serre spectral sequence, due to Moerdijk and Svensson [MS93], but it has not yet had significant calculational applications. The essential

reason is the lack of a way to compute its E^2 -terms. However, the results of Section 4.1 generalize nicely to compute Bredon homology and cohomology with local coefficients.

Definition 4.12 generalizes directly to the equivariant case. Recall our discussion of Bredon cohomology in Section 2.1; for this subsection, we will think about integer-graded Bredon cohomology, which has coefficient systems rather than Mackey functors as coefficients. From now on, let X be a G -space; as in the rest of this thesis, we take G to be a finite group.¹ Following tom Dieck [tD87], we can define the **fundamental EI-category** $\Pi_G X$ to be the category whose objects are pairs (K, x) , where $x \in X^K$; a morphism from (K, x) to (L, y) consists of a G -map $\alpha: G/K \rightarrow G/L$, determined by $\alpha(eH) = gK$, together with a homotopy class $\text{rel endpoints } [\gamma]$ of paths from x to $\alpha^*(y) = gy$. Here $\alpha^*: X^L \rightarrow X^K$ is the map given by $\alpha^*(z) = gz$, which makes sense because $g^{-1}Hg \subset L$.

Likewise, we follow tom Dieck in defining the equivariant universal cover \tilde{X} to be the functor $\tilde{X}: (\Pi_G X)^{\text{op}} \rightarrow \text{Top}$ which sends (K, x) to $\tilde{X}^K(x)$, the space of equivalence classes of paths in X^K starting at x . For a morphism $(\alpha, [\gamma]): (K, x) \rightarrow (L, y)$, $\tilde{X}(\alpha, [\gamma]): \tilde{X}(L, y) \rightarrow \tilde{X}(K, x)$ takes a class of paths $[\beta]$ starting at $y \in X^L$ to the class of the composite $(\alpha^*\beta)\gamma$.

We can now define equivariant (co)homology with local coefficients. In fact, Definition 4.12 applies almost verbatim: we need only add G to the notations. We repeat the definition for emphasis.

Definition 4.13 (Equivariant generalization of Definition 4.12). Let X be a G -space and write $\Pi = \Pi_G X$. Let $\mathcal{M}: \Pi \rightarrow \mathbb{k}\text{-mod}$ and $\mathcal{N}: \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ be functors and let $C_*^G(\tilde{X}): \Pi^{\text{op}} \rightarrow \text{Top} \rightarrow \text{Ch}_{\mathbb{k}}$ be the composite of the equivariant universal cover functor with the functor C_* . Define the homology of X with coefficients in \mathcal{M} to be

$$H_*^G(X; \mathcal{M}) = H_*(C_*^G(\tilde{X}) \otimes_{\Pi} \mathcal{M})$$

and the cohomology of X with coefficients in \mathcal{N} by

$$H_G^*(X; \mathcal{N}) = H^*\left(\text{Hom}_{\Pi}(C_*^G(\tilde{X}), \mathcal{N})\right).$$

Note that we could also take Π to be a skeleton $\text{skel}(\Pi_G X)$.

Inserting G into the notations, the proofs in Whitehead or Hatcher [Whi78, Hat02] apply to show that this definition of Bredon (co)homology with local coefficients is naturally isomorphic to the Bredon (co)homology with local coefficients, as defined in Mukherjee and Pandey [MP01], which they in turn show is naturally isomorphic to the (co)homology with local coefficients, as defined and used by Moerdijk and Svensson in [MS93] to construct the equivariant Serre spectral sequence of a G -fibration $f: E \rightarrow B$.

We quickly review the homological algebra needed for the equivariant generalization of Theorem 4.6. Since $\mathbb{k}\text{-mod}$ is an abelian category, the categories $[\Pi, \mathbb{k}\text{-mod}]$ and $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$ of functors from Π to $\mathbb{k}\text{-mod}$ are also abelian, with kernels and cokernels

¹The results in this subsection apply equally well to discrete groups, and with a little more detail, we could generalize to topological groups.

defined levelwise. These categories have enough projectives and injectives, which by the Yoneda lemma are related to the represented functors.

Specifically, let \mathbb{k} - denote the free \mathbb{k} -module functor $\mathbf{Set} \rightarrow \mathbb{k}\text{-mod}$. Given an object $(K, x) \in \Pi$, let $P_{K,x}$ be the covariant represented functor $\Pi \rightarrow \mathbb{k}\text{-mod}$ given on objects by

$$P_{K,x}(L, y) = \mathbb{k}\Pi((K, x), (L, y)).$$

By the Yoneda lemma, each $P_{K,x}$ is projective. Therefore, given a functor \mathcal{M} , we can construct an epimorphism $\mathcal{P} \rightarrow \mathcal{M}$ with \mathcal{P} projective by taking \mathcal{P} to be a direct sum of representables

$$\mathcal{P} = \bigoplus_{(K,x)} \bigoplus_{\mathcal{M}(K,x)} P_{K,x},$$

one for each element of each \mathbb{k} -module $\mathcal{M}(K, x)$. Similarly, there are contravariant represented functors $P^{K,x}: \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ given by

$$P^{K,x}(L, y) = \mathbb{k}\Pi((L, y), (K, x)).$$

The same argument shows that these are projective and that $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$ has enough projectives.

The construction of the injective objects is dual but perhaps less familiar. Given a \mathbb{k} -module C and $(K, x) \in \Pi$, we define a functor $I_{K,x,C}: \Pi \rightarrow \mathbb{k}\text{-mod}$ by

$$I_{K,x,C}(L, y) = \text{Hom}_{\mathbb{k}}(P^{K,x}(L, y), C).$$

Whenever C is an injective \mathbb{k} -module, $I_{K,x,C}$ is an injective object in $[\Pi, \mathbb{k}\text{-mod}]$. This comes from a more general fact. For any coefficient system $\mathcal{A}: \Pi \rightarrow \mathbb{k}\text{-mod}$, there is a tensor-hom adjunction

$$[\Pi, \mathbb{k}\text{-mod}](\mathcal{A}, I_{K,x,C}) \cong \mathbb{k}\text{-mod}(\mathcal{A} \otimes_{\Pi} P^{K,x}, C)$$

where again \otimes_{Π} is the tensor product of functors. The tensor product of any functor with a representable functor $P^{K,x}$ is given by evaluation at (K, x) . Putting these two facts together, we have that a natural transformation from \mathcal{A} to $I_{K,x,C}$ is given by the same data as a homomorphism of \mathbb{k} -modules from $\mathcal{A}(K, x)$ to C . It is then clear that, if C is an injective \mathbb{k} -module, $I_{K,x,C}$ must be an injective object of $[\Pi, \mathbb{k}\text{-mod}]$, as desired. Given any $\mathcal{N}: \Pi \rightarrow \mathbb{k}\text{-mod}$, we can construct an injective coefficient system \mathcal{I} and a monomorphism $\mathcal{N} \rightarrow \mathcal{I}$ as follows. Choose monomorphisms $\mathcal{N}(K, x) \hookrightarrow C_{K,x}$ for each (K, x) with $C_{K,x}$ injective, and define \mathcal{I} to be the product of injective functors

$$\mathcal{I} = \prod_{(K,x)} I_{K,x,C_{K,x}}.$$

It can be checked that the evident map $\mathcal{N} \rightarrow \mathcal{I}$ is a monomorphism. Thus $[\Pi, \mathbb{k}\text{-mod}]$ has enough injectives. The functors $I^{K,x,C} = \text{Hom}_{\mathbb{k}}(P_{K,x}(-), C)$ show that $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$ has

enough injectives as well.

Finally, we define $\mathrm{Tor}^\Pi(\mathcal{N}, \mathcal{M})$ in the obvious way. It is the homology of the complex of \mathbb{k} -modules that is obtained by taking the tensor product of functors of \mathcal{N} with a projective resolution of the functor \mathcal{M} . We define $\mathrm{Ext}_\Pi(\mathcal{N}_1, \mathcal{N}_2)$ similarly, taking the hom of functors of \mathcal{N}_1 with an injective resolution of \mathcal{N}_2 .²

The following equivariant analogue of the nonequivariant statement that $C_*(\tilde{X})$ is a free $\mathbb{k}[\pi]$ -module should be a standard first observation in equivariant homology theory, but the author has not seen it in the literature. The nonequivariant assertion, while obvious, is the crux of the proof of Theorem 4.6. Let $P^{K,x} = \mathbb{k}\Pi(-, (K, x))$, as above.

Lemma 4.14. *With $\Pi = \Pi_G X$, each functor $C_n^G(\tilde{X}): \Pi^{\mathrm{op}} \rightarrow \mathbb{k}\text{-mod}$ is a direct sum of representable functors $\bigoplus_{(K_i, x_i)} P^{K_i, x_i}$.*

Granting this result for the moment, we can prove the equivariant generalization of Theorem 4.6.

Theorem 4.15 (Equivariant Eilenberg Spectral Sequence). *With $\Pi = \Pi_G X$, there are spectral sequences*

$$E_{p,q}^2 = \mathrm{Tor}_{p,q}^\Pi(\mathcal{H}_*(\tilde{X}), \mathcal{M}) \implies H_{p+q}^G(X; \mathcal{M})$$

and

$$E_2^{p,q} = \mathrm{Ext}_\Pi^{p,q}(\mathcal{H}_*(\tilde{X}), \mathcal{N}) \implies H_G^{p+q}(X; \mathcal{N}).$$

Here the functor $\mathcal{H}_*(\tilde{X}): \Pi^{\mathrm{op}} \rightarrow \mathbb{k}\text{-mod}$ is the homology of the chain complex functor $C_*^G(\tilde{X})$; that is, $\mathcal{H}_*(\tilde{X})(K, x)$ is the homology of the chain complex $C_*(\tilde{X})(K, x)$.

Proof. Let $\varepsilon: \mathcal{P}_* \rightarrow \mathcal{M}$ be a projective resolution of \mathcal{M} . As in the nonequivariant theorem, form the bicomplex of \mathbb{k} -modules $C_*(\tilde{X}) \otimes_\Pi \mathcal{P}_*$. Since the tensor product of a functor with a representable functor is given by evaluation,

$$P^{K,x} \otimes_\Pi \mathcal{M} \cong \mathcal{M}(K, x),$$

tensoring with such projective modules is exact.

In particular, if we filter our bicomplex by degrees of $C_*(\tilde{X})$, then $d^0 = \mathrm{id} \otimes d$. By Lemma 4.14, each $C_n^G(\tilde{X})$ is projective, and so we get a spectral sequence with E^1 -term $C_*(\tilde{X}) \otimes_\Pi \mathcal{M}$. Thus the resulting $E^2 = E^\infty$ term is $H_*^G(X, \mathcal{M})$, exactly as in the nonequivariant case.

If we instead filter by degrees of \mathcal{P}_* , so $d^0 = d \otimes \mathrm{id}$, then the E^1 term is $\mathcal{H}_*(\tilde{X}) \otimes_\Pi \mathcal{P}_*$ and the E^2 term is $\mathrm{Tor}_{*,*}^\Pi(\mathcal{H}_*(\tilde{X}), \mathcal{M})$, as desired.

The construction of the second spectral sequence is similar, starting from an injective resolution $\eta: \mathcal{N} \rightarrow \mathcal{I}^*$. \square

²Alternatively, we could define $\mathrm{Ext}_\Pi(\mathcal{N}_1, \mathcal{N}_2)$ by taking a projective resolution of \mathcal{N}_1 ; however, it is the definition above which gives rise to the desired spectral sequence.

Proof of Lemma 4.14. The proof is analogous to that of the nonequivariant result, but more involved. We may identify $C_n^G(\tilde{X})(K, x)$ with the free \mathbb{k} -module on generators given by the nondegenerate singular n -simplices $\sigma: \Delta^n \rightarrow \tilde{X}(K, x)$. We must show that these free \mathbb{k} -modules piece together appropriately into a free functor. More specifically, by the Yoneda lemma, each $\sigma: \Delta^n \rightarrow \tilde{X}(L, y)$ determines a natural transformation

$$\iota_\sigma: P^{L,y} \rightarrow C_n^G(\tilde{X})$$

that takes $\text{id} \in \Pi((L, y), (L, y))$ to σ . We thus obtain a natural transformation

$$\bigoplus_{\{\tau\}} P^{L_\tau, y_\tau} \rightarrow C_n^G(\tilde{X})$$

from any set of nondegenerate n -simplices $\{\tau: \Delta^n \rightarrow \tilde{X}(L_\tau, y_\tau)\}$. We must show that there is a set $\{\tau\}$ such that the resulting natural transformation is a natural isomorphism, that is, a levelwise isomorphism. This amounts to showing that the following statements hold for our choice of generators τ and each object (K, x) .

1. (Injectivity) For any arrows $(\alpha_1, [\gamma_1])$ and $(\alpha_2, [\gamma_2])$ in Π with source (K, x) and any generators τ_1 and τ_2 , $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$ must imply that both $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$ and $\tau_1 = \tau_2$.
2. (Surjectivity) For every $\sigma: \Delta^n \rightarrow \tilde{X}(K, x)$, there must be a generator τ and an arrow $(\alpha, [\gamma])$ such that $\sigma = (\alpha, [\gamma])^* \tau$.

Fixing n , define the generating set as follows. Regard the initial vertex v of Δ^n as a basepoint. Recall that $\tilde{X}(L, y)$ is the universal cover of X^L defined with respect to the basepoint $y \in X^L$, so that the equivalence class of the constant path $c_{L,y}$ at y is the basepoint of \tilde{X}^L . In choosing our generating set, we restrict attention to based maps $\sigma: \Delta^n \rightarrow \tilde{X}(L_\sigma, y_\sigma)$ that are non-degenerate n -simplices of \tilde{X}^{L_σ} . Such maps σ are in bijective correspondence with based nondegenerate n -simplices $\sigma_0: \Delta^n \rightarrow X^{L_\sigma}$. The correspondence sends σ to its composite with the end-point evaluation map $p: \tilde{X}(L_\sigma, y_\sigma) \rightarrow X^{L_\sigma}$ and sends σ_0 to the map $\sigma: \Delta^n \rightarrow \tilde{X}(L_\sigma, y_\sigma)$ that sends a point $a \in \Delta^n$ to the image under σ_0 of the straight-line path from v to a . Restrict further to those σ that cannot be written as a composite

$$\Delta^n \xrightarrow{\rho} \tilde{X}(L', y') \xrightarrow{(\alpha, \gamma)^*} \tilde{X}(L_\sigma, y_\sigma)$$

for any non-isomorphism $(\alpha, \gamma): (L', y') \rightarrow (L_\sigma, y_\sigma)$ in Π . Note that, for each such σ , we can obtain another such σ by composing with the isomorphism $(\xi, \delta)^*$ induced by an isomorphism (ξ, δ) in Π . We say that the resulting maps σ are equivalent, and we choose one τ in each equivalence class of such based singular n -simplices σ .

It remains to verify that the natural transformation defined by this set $\{\tau\}$ is an isomorphism. This is straightforward but somewhat tedious and technical.

For the injectivity, suppose that $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$, where τ_1, τ_2 are in our generating set and

$$\begin{aligned}\tau_1: \Delta^n &\longrightarrow \tilde{X}(L_1, y_1), & (\alpha_1, [\gamma_1]) &\in \Pi((K, x), (L_1, y_1)) \\ \tau_2: \Delta^n &\longrightarrow \tilde{X}(L_2, y_2), & (\alpha_2, [\gamma_2]) &\in \Pi((K, x), (L_2, y_2)).\end{aligned}$$

Since $\tau_i(v) = c_{(L_i, y_i)}$ for $i = 1, 2$, we see that $(\alpha_i, [\gamma_i])^* \tau_i$ must take v to $[\gamma_i]$. Since $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$, this means that $[\gamma_1] = [\gamma_2]$; call this path class $[\gamma]$. In turn, this implies that $\alpha_1^* y_1 = \alpha_2^* y_2$; call this point $z \in X^K$, so that $[\gamma]$ is a path from x to z . Since $(\alpha_i, [\gamma_i]) = (\alpha_i, [c_z]) \circ (\text{id}, [\gamma])$ and $(\text{id}, [\gamma])$ is an isomorphism in Π , we must have

$$(\alpha_1, [c_z])^* \tau_1 = (\alpha_2, [c_z])^* \tau_2.$$

In particular, if we compose each side of this equation with p , we obtain

$$p \circ (\alpha_1, [c_z])^* \tau_1 = p \circ (\alpha_2, [c_z])^* \tau_2$$

as maps $\Delta^n \longrightarrow X^K$. Since we have commutative diagrams

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\tau_i} & \tilde{X}(L_i, y_i) \xrightarrow{(\alpha_i, [c_z])^*} \tilde{X}(K, z) \\ & & \downarrow p \qquad \qquad \downarrow p \\ & & X^{L_i} \xrightarrow{\alpha_i^*} X^K \end{array}$$

for each i , this implies that we have a commutative square

$$\begin{array}{ccc} \Delta^n & \xrightarrow{p \circ \tau_1} & X^{L_1} \\ p \circ \tau_2 \downarrow & & \downarrow \alpha_1^* \\ X^{L_2} & \xrightarrow{\alpha_2^*} & X^K \end{array}$$

If the maps $\alpha_i: G/K \longrightarrow G/L_i$ are defined by elements $g_i \in G$, this implies that the common composite $\Delta^n \longrightarrow X^K$ factors through the fixed-point sets $X^{g_i L_i g_i^{-1}}$ for each i , and hence through X^L , where L is the smallest subgroup containing $g_1 L_1 g_1^{-1}$ and $g_2 L_2 g_2^{-1}$. Since $L_i \subset g_i^{-1} L g_i$, the maps $\alpha_i: G/K \longrightarrow G/L_i$ factor through the maps $\beta_i: G/L_i \longrightarrow G/L$ specified by $\beta_i(eK_i) = g_i L$ and there result factorizations of the τ_i as

$$\Delta^n \longrightarrow \tilde{X}(g_i^{-1} L g_i, y_i) \xrightarrow{(q, [c_z])^*} \tilde{X}(L, y_i),$$

where q denotes either quotient map $G/L_i \longrightarrow G/g_i^{-1} L g_i$. By our choice of the generators τ , this can only happen if $g_i^{-1} L g_i = L_i$, giving $g_1 L_1 g_1^{-1} = g_2 L_2 g_2^{-1}$. In terms of g_1 and g_2 , we see that our equation $(\alpha_1, [c_z])^* \tau_1 = (\alpha_2, [c_z])^* \tau_2$ says that $g_1 \tau_1 = g_2 \tau_2$, that is, $\tau_2 = g_2^{-1} g_1 \tau_1$. Since $g_2^{-1} g_1$ defines an isomorphism $G/L_2 \longrightarrow G/L_1$, we again see by our choice of the generators τ that $\tau_1 = \tau_2$ and that $g_2^{-1} g_1 \in L_1 = L_2$. This in turn implies

that the maps $\alpha_i: G/K \rightarrow G/L_i$ defined by the g_i are identical. The conclusion is that $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$ implies $\tau_1 = \tau_2$ and $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$, as desired.

It only remains to show that we have accounted for all elements of $C_n(\tilde{X})(K, x)$. For any map $\sigma: \Delta^n \rightarrow \tilde{X}(K, x)$, $\sigma(v)$ is a homotopy class of paths from (K, x) to (K, x') in X^K . Call this class $[\gamma]$. Then $(\text{id}, [\gamma])$ is an isomorphism with inverse $(\text{id}, [\gamma^{-1}])$, and $\sigma' = (\text{id}, [\gamma^{-1}])^* \sigma$ takes v to the homotopy class of the constant path at (K, x') ; it follows that $\sigma = (\text{id}, [\gamma])^* \sigma'$. Similarly, if σ' factors through $\tilde{X}(L, y)$ for some L properly containing a conjugate of K , then by definition $\sigma = (\alpha, [\gamma])^* \tau$ for some $\tau: \Delta^n \rightarrow \tilde{X}(L, y)$. We can choose a τ that does not itself factor and is in our chosen set of generators. \square

4.3 Some remarks on the Serre Spectral Sequence

We say that a map $f: E \rightarrow X$ is a **G -fibration** if $f^K: E^K \rightarrow X^K$ is a fibration for every subgroup K of G . Observe that, if $x \in X^K$, then its preimage $f^{-1}(x) \subset E$ is necessarily a K -space. As previously mentioned, Moerdijk and Svensson in [MS93] develop an equivariant Serre spectral sequence for G -fibrations of G -spaces, which we will now describe. They use integer-graded Bredon cohomology.

Given a coefficient system $M: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ and a subgroup $K < G$, there is a restricted coefficient system $M|_K: \mathcal{O}_K^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$ given by $M|_K(K/L) := M(G/L)$. We may thus define a local coefficient system

$$h_G^q(f; M): \Pi_G X^{\text{op}} \rightarrow \mathbf{Ab}$$

to be the functor which acts on objects by

$$(K, x) \mapsto H_G^q(G \times_K f^{-1}(x); M) \cong H_K^q(f^{-1}(x); M|_K).$$

It is defined on morphisms via lifting of paths.

The main result of [MS93] is the following spectral sequence.

Theorem 4.16 (Moerdijk and Svensson). *For any G -fibration $f: E \rightarrow X$ and any coefficient system $M: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$, there is a natural spectral sequence*

$$E_2^{s,t}(M) = H_G^s(X; h_G^t(f; M)) \implies H_G^{s+t}(E; M).$$

Further, this spectral sequence carries a product structure, in the sense that there is a natural pairing of spectral sequences

$$E_r^{s,t}(M) \otimes E_r^{s',t'}(N) \longrightarrow E_r^{s+s',t+t'}(M \otimes N)$$

converging to the standard pairing

$$H_G^*(E; M) \otimes H_G^*(E; N) \xrightarrow{\sim} H_G^*(E; M \otimes N).$$

On the E_2 page, this pairing agrees with the standard pairing

$$H_G^s(X; h_G^t(f, M)) \otimes H_G^{s'}(X; h_G^{t'}(f, N)) \longrightarrow H_G^{s+s'}(X; h_G^{t+t'}(f, M \otimes N)).^3$$

The tensor product $M \otimes N$ above is a levelwise tensor product, $(M \otimes N)(G/K) = M(G/K) \otimes N(G/K)$, which is distinct from the box product \square discussed in Section 2.2.

Although we have been using the integer-graded equivariant cohomology originally defined by Bredon in this chapter, equivariant cohomology is more naturally graded on $RO(G)$, as we discussed in Chapter 2. As we mentioned there, for any coefficient system M which can be extended to a Mackey functor, the Bredon cohomology theory $H_G^*(-; M)$ can be extended to an $RO(G)$ -graded theory. That is, for every virtual representation ω , we have a functor $H_G^\omega(-; M)$. We pictured these functors as lying in a two-dimensional plane in Chapter 2, but another way of visualizing this extra data is to say that we have one integer-graded theory $\{H_G^{V+n}\}_{n \in \mathbb{Z}}$ for each representation V containing no trivial subrepresentations. Each of these theories $H_G^{V+*}(-; M)$ can be used to define local coefficient systems $h_G^{V+t}(f, M)$. Kronholm shows the following in his thesis [Kro09].

Theorem 4.17 (Kronholm). *For each real representation V , there is a natural spectral sequence*

$$E_2^{s,t}(M, V) = H_G^s(X; h_G^{V+t}(f, M)) \implies H_G^{V+s+t}(E; M).$$

Further, for each $V, V' \in RO(G)$, there is a pairing

$$E_r^{s,t}(M, V) \otimes E_r^{s',t'}(N, V') \longrightarrow E_r^{s+s',t+t'}(M \otimes N, V + V')$$

converging to the standard pairing on E_∞ and agreeing with the standard pairing on E_2 .

We have an analogue of Proposition 4.10, as well.

Proposition 4.18. $H_G^0(X; \mathcal{M}) \cong \text{Hom}_\Pi(\underline{\mathbb{k}}, \mathcal{M})$, where $\underline{\mathbb{k}}$ is the constant functor.

Proof. As in Proposition 4.10, this comes from identifying $\mathcal{H}_0(\tilde{X}) \cong \underline{\mathbb{k}}$ and from the left exactness of Hom_Π . \square

Proposition 4.19. *Suppose that X is G -connected, in the sense that each X^K is nonempty and connected, and let $x \in X^G$. Then $\text{Hom}_\Pi(\underline{\mathbb{k}}, \mathcal{M})$ is isomorphic to a sub- \mathbb{k} -module of $\mathcal{M}(G, x)$.*

Proof. Since X is G -connected, (G, x) is a weakly terminal object in $\Pi_G X$, i.e. for every (K, y) there is a map $(K, y) \longrightarrow (G, x)$. It follows that an element of $\text{Hom}_\Pi(\underline{\mathbb{k}}, \mathcal{M})$ is determined by the map of \mathbb{k} -modules $\mathbb{k} \longrightarrow \mathcal{M}(G, x)$. \square

³As usual in the Serre spectral sequence, the standard pairing on the E_2 page incorporates a sign $(-1)^{s't}$ relative to the cup product pairing.

4.4 Equivariant classifying spaces

For the remainder of this thesis, with the exception of classical structure groups, groups named with Greek letters will be viewed as structure groups and those named with Latin letters will be viewed as ambient groups of equivariance. Suppose that we are given a structure group Γ and a group of equivariance G . Then, as discussed in [May96] and many other places, there is a notion of a principal (G, Γ) -bundle, namely a projection to Γ -orbits $E \rightarrow B = E/\Gamma$ of a Γ -free $(G \times \Gamma)$ -space E . Such equivariant bundles are classified by universal principal bundles $E_G\Gamma \rightarrow B_G\Gamma$, where $E_G\Gamma$ is a space whose fixed point sets $(E_G\Gamma)^\Lambda$ are empty when $\Lambda \subset G \times \Gamma$ intersects $\Gamma = \{e\} \times \Gamma$ nontrivially and contractible when $\Lambda \cap \Gamma = \{e\}$.

As should be expected, for a fixed group G , the equivariant classifying space construction can be made functorial; that is, there are functors

$$\begin{aligned} E_G: \mathbf{Grp} &\longrightarrow G\text{-Top} \\ B_G: \mathbf{Grp} &\longrightarrow G\text{-Top}. \end{aligned}$$

It will be helpful to pick particular functors E_G and B_G , using the categorical two-sided bar construction. Given any groups G and Γ , we may take

$$\begin{aligned} E_G\Gamma &:= B(T_\Gamma, \mathcal{O}_{G \times \Gamma}, O_{G \times \Gamma}) \\ B_G\Gamma &:= (E_G\Gamma)/\Gamma \end{aligned}$$

where $\mathcal{O}_{G \times \Gamma}$ is the orbit category, $O_{G \times \Gamma}: \mathcal{O}_{G \times \Gamma} \rightarrow \text{Top}$ is given by viewing an orbit $(G \times \Gamma)/\Lambda$ as a topological space, and $T_\Gamma: \mathcal{O}_{G \times \Gamma}^{\text{op}} \rightarrow \text{Top}$ is the functor which takes

$$(G \times \Gamma)/\Lambda \longmapsto \begin{cases} pt & \text{if } \Lambda \cap \Gamma = \{e\} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since the functor $O_{G \times \Gamma}$ lands in $(G \times \Gamma)\text{-Top}$, $E_G\Gamma$ is a $(G \times \Gamma)$ -space, and it is easy to check that it has the correct fixed points.

The bar construction $B(-, -, -)$ is a functor from the category of triples (T, \mathcal{C}, S) to Top . Here \mathcal{C} is a category and S, T are respectively a covariant and a contravariant functor $\mathcal{C} \rightarrow \text{Top}$. A morphism $(T_1, \mathcal{C}_1, S_1) \rightarrow (T_2, \mathcal{C}_2, S_2)$ in the category of triples consists of a functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ together with natural transformations $S_1 \rightarrow S_2 \circ F$ and $T_1 \rightarrow T_2 \circ F^{\text{op}}$. It follows that, for fixed G , we can make $E_G(-)$ into a functor $\mathbf{Grp} \rightarrow \text{Top}$ as follows. Given a homomorphism $\varphi: \Gamma_1 \rightarrow \Gamma_2$, we apply $B(-, -, -)$ to the morphism of triples given by the functor

$$F: \mathcal{O}_{G \times \Gamma_1} \rightarrow \mathcal{O}_{G \times \Gamma_2}: (G \times \Gamma_1)/\Lambda \longmapsto (G \times \Gamma_2)/((\text{id} \times \varphi)(\Lambda)),$$

with the obvious natural transformations $O_{G \times \Gamma_1} \rightarrow O_{G \times \Gamma_2} \circ F$ and $T_{\Gamma_1} \rightarrow T_{\Gamma_2} \circ F^{\text{op}}$ (for the latter, note that if $\Lambda \cap \Gamma_1 = \{e\}$, then also $(\text{id} \times \varphi)(\Lambda) \cap \Gamma_2 = \{e\}$). Since the

morphism $E_G\Gamma_1 \rightarrow E_G\Gamma_2$ induced by $\Gamma_1 \rightarrow \Gamma_2$ is Γ_1 -equivariant, there is an induced map $B_G\Gamma_1 \rightarrow B_G\Gamma_2$, making $B_G(-)$ a functor.

The following result will be useful later.

Proposition 4.20. *Fix a group G . Corresponding to any short exact sequence of structure groups*

$$1 \rightarrow \Gamma \xrightarrow{\varphi} \Upsilon \xrightarrow{\psi} \Sigma \rightarrow 1$$

there is a pullback square in the category of G -spaces

$$\begin{array}{ccc} B_G\Gamma & \longrightarrow & E_G\Sigma \\ \downarrow / \Sigma & & \downarrow / \Sigma \\ B_G\Upsilon & \xrightarrow{B_G\psi} & B_G\Sigma \end{array}$$

Proof. By functoriality of $E_G(-)$ and the definition of $B_G(-)$, the map ψ induces a commutative diagram

$$\begin{array}{ccc} E_G\Upsilon & \xrightarrow{E_G\psi} & E_G\Sigma \\ \downarrow / \Upsilon & & \downarrow / \Upsilon = / \Sigma \\ B_G\Upsilon & \xrightarrow{B_G\psi} & B_G\Sigma \end{array}$$

Further, the projection $E_G\Upsilon \rightarrow B_G\Upsilon = (E_G\Upsilon)/\Upsilon$ factors as

$$E_G\Upsilon \rightarrow (E_G\Upsilon)/\Gamma \xrightarrow{/\Sigma} (E_G\Upsilon)/\Upsilon$$

and since the action of Γ on $E_G\Sigma$ is trivial, $E_G\Upsilon \rightarrow E_G\Sigma$ also factors through $(E_G\Upsilon)/\Gamma \cong B_G\Gamma$. We thus get the commutative square described in the proposition. Viewing $E_G\Sigma$ as a G -space via the inclusion $G \hookrightarrow G \times \Sigma$, this is a diagram in the category of G -spaces. The square induces a homeomorphism on the fibers of the vertical maps, and hence is a pullback in the category of G -spaces, as desired. \square

Example 4.21. We have already studied one example of a classifying space: the complex projective space $\mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ on a complete complex universe $\mathcal{U}_{\mathbb{C}}$ is a model for $B_GSO(2)$, the equivariant classifying space of the circle group. Corollary 3.7, Proposition 3.22, and Theorem 3.23 established the structure of $\underline{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ as an algebra over $\underline{H}_{C_p}^*(pt)$ for $G = C_p$, provided the ground ring \mathbb{k} has no torsion of order p . In the next chapter, we will use this result to compute the cohomology of $B_{C_p}O(2)$ for $\mathbb{k} = \mathbb{F}_q$.

CHAPTER 5

EXAMPLE: THE COHOMOLOGY OF $B_{C_p}O(2)$

As before, let C_p be the cyclic group of order p . In this chapter, we will calculate the $RO(C_p)$ -graded Bredon cohomology

$$H_{C_p}^*(B_{C_p}O(2); A)$$

of the equivariant classifying space $B_{C_p}O(2)$ for ground ring $\mathbb{k} = \mathbb{F}_q$ and for an odd prime $p \neq q$. By way of motivation, we know from Section 3.3 what $H_{C_p}^*(B_{C_p}S^1)$ is, and historically $O(2)$ is often the first test case to try after S^1 . Let D_1 through D_p and C be the elements described in Proposition 3.22 and Theorem 3.23.

Theorem 5.1. *Let p, q be distinct odd primes. Then, as an algebra over $H_{C_p}^*$, $H_{C_p}^*(B_{C_p}O(2); A)$ is isomorphic to the subalgebra of $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A)$ generated by the elements $D_2, D_4, \dots, D_{p-1}, D_1C, \dots, D_{p-2}C, C^2$.*

By analogy with Example 4.11, we will approach Theorem 5.1 via the short exact sequence of structure groups

$$1 \longrightarrow SO(2) \longrightarrow O(2) \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

and the induced fibration $f: B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$, where again $\mathbb{Z}/2$ is the cyclic group of order 2. We will first identify the C_p -action on the fibers of f , then explicitly describe the coefficient systems $h^{V+t}(f, A)$ and $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$, and finally prove the theorem.

5.1 Identifying the fibers of $f: B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$

We will begin by identifying models for the equivariant classifying spaces under consideration. Recall that nonequivariantly, the universal bundle $E\mathbb{Z}/2 \longrightarrow B\mathbb{Z}/2$ has as a model $S^\infty \longrightarrow \mathbb{R}P^\infty$, where $S^\infty = S(\mathbb{R}^\infty)$ is the unit sphere in \mathbb{R}^∞ and $\mathbb{R}P^\infty = \mathbb{R}P(\mathbb{R}^\infty)$ is the infinite-dimensional real projective space. In general, for any group G , let $\mathcal{U}_{\mathbb{R}}$ be a direct sum containing countably infinitely many copies of each real representation of G . Then $S(\mathcal{U}_{\mathbb{R}}) \longrightarrow \mathbb{R}P(\mathcal{U}_{\mathbb{R}})$ is a model for $E_G\mathbb{Z}/2 \longrightarrow B_G\mathbb{Z}/2$. The $G \times \mathbb{Z}/2$ action on $S(\mathcal{U}_{\mathbb{R}})$ comes from the G action on $\mathcal{U}_{\mathbb{R}}$ and the $\mathbb{Z}/2$ action by multiplication by -1 . When $G = C_p$ is cyclic of prime order, however, we can choose a simpler model.

Lemma 5.2. *If p is an odd prime, then $E_{C_p}\mathbb{Z}/2 \longrightarrow B_{C_p}\mathbb{Z}/2$ has as a model $S^\infty \longrightarrow \mathbb{R}P^\infty$ with the trivial C_p -action on both spaces.*

Proof. To verify this claim, it suffices to check the fixed-point sets of S^∞ . Since $\mathbb{Z}/2$ acts freely on S^∞ , the fixed points $(S^\infty)^\Lambda$ are certainly empty when $\Lambda \cap \mathbb{Z}/2$ is nontrivial. So we need only check that the fixed point set is contractible whenever $\Lambda \cap \mathbb{Z}/2$ is trivial.

Note that the subgroups $\Lambda \subset C_p \times \mathbb{Z}/2$ which intersect $\mathbb{Z}/2$ trivially are the “twisted diagonal subgroups” $\Lambda = \Delta_{\rho, K} = \{(h, \rho(h)) \mid h \in K\}$, for K a subgroup of C_p and $\rho: K \rightarrow \mathbb{Z}/2$ a homomorphism. However, since p is an odd prime, the only homomorphism $K \rightarrow \mathbb{Z}/2$ is the trivial homomorphism, and so $\Delta_{\rho, K} = K \times \{e\}$. This acts trivially on S^∞ , so $(S^\infty)^\Lambda = S^\infty \simeq pt$, as desired. \square

Similarly, $SO(2)$ is the circle \mathbb{T} . Letting $\mathcal{U}_\mathbb{C}$ again be the direct sum of countably infinitely many copies of each irreducible complex representation of C_p , an analysis of the fixed-point sets of $S(\mathcal{U}_\mathbb{C})$ gives the following well-known result.

Lemma 5.3. *For any prime p , $E_{C_p}SO(2) \rightarrow B_{C_p}SO(2)$ has as a model $S(\mathcal{U}_\mathbb{C}) \rightarrow CP(\mathcal{U}_\mathbb{C})$. The $C_p \times SO(2)$ action on $S(\mathcal{U}_\mathbb{C})$ comes from the C_p action on $\mathcal{U}_\mathbb{C}$ and the usual circle action on the complex plane.* \square

We are now in a position to use Proposition 4.20 for the short exact sequence

$$1 \rightarrow SO(2) \rightarrow O(2) \xrightarrow{\det} \mathbb{Z}/2 \rightarrow 1.$$

We have a pullback square in the category of C_p -spaces

$$\begin{array}{ccc} B_{C_p}SO(2) & \longrightarrow & E_{C_p}\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ B_{C_p}O(2) & \longrightarrow & B_{C_p}\mathbb{Z}/2 \end{array}$$

By Lemma 5.2, the C_p -actions on $E_{C_p}\mathbb{Z}/2$ and $B_{C_p}\mathbb{Z}/2$ are trivial. It follows that $E_{C_p}\mathbb{Z}/2$ is C_p -contractible, and so we have proved the following about our map $f: B_{C_p}O(2) \rightarrow B_{C_p}\mathbb{Z}/2$.

Lemma 5.4. *For each point $x \in B_{C_p}\mathbb{Z}/2$, the fiber $f^{-1}(x)$ is C_p -homotopy equivalent to $B_{C_p}SO(2) = CP(\mathcal{U}_\mathbb{C})$.* \square

5.2 The local coefficient system $h_{C_p}^{V+t}(f, A)$

Recall from Theorem 4.17 that the equivariant Serre spectral sequence for a C_p -fibration $f: E \rightarrow X$ has

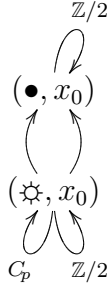
$$E_2^{s,t}(M, V) = H_{C_p}^s(X; h_{C_p}^{V+t}(f, M)) \implies H_{C_p}^{V+s+t}(E; M).$$

Choose the coefficient ring $\mathbb{k} = \mathbb{F}_q$, the finite field with q elements, for an odd prime $q \neq p$. As in Theorem 5.1, let $M = A$. We must first analyze the local coefficient systems

$$h_{C_p}^{V+t}(f, A): (K, x) \mapsto H_{C_p}^{V+t}(C_p \times_K f^{-1}(x)).$$

We may start by taking a skeleton Π of the category $\Pi_{C_p} B_{C_p} \mathbb{Z}/2$. Again using Lemma 5.2, we see that each fixed-point set $(B_{C_p} \mathbb{Z}/2)^K$ is nonempty and connected with fundamental group $\mathbb{Z}/2$, so Π has two objects (C_p, x_0) and $(\{e\}, x_0)$. Recall from Section 2.4 that we denoted the orbit $C_p/\{e\}$ by \star and C_p/C_p by \bullet . We will continue that convention in this section. For convenience, and to bring out the parallels, we will also write (\bullet, x_0) and (\star, x_0) in place of (C_p, x_0) and $(\{e\}, x_0)$, respectively.

Recalling that a map $(K, x) \rightarrow (L, y)$ consists of a C_p -map $\alpha: C_p/K \rightarrow C_p/L$ and a homotopy class of paths $[\gamma]$ from x to α^*y , we see that there are two endomorphisms of (\bullet, x_0) . One of these is the identity, and the other, κ , squares to the identity. Similarly, there are two morphisms $(\star, x_0) \rightarrow (\bullet, x_0)$, and composition with κ exchanges them. Finally, the endomorphisms of (\star, x_0) are in bijection with $C_p \times \mathbb{Z}/2$; when precomposing with the two morphisms $(\star, x_0) \rightarrow (\bullet, x_0)$, only the $\mathbb{Z}/2$ factor has an effect. We may visualize Π as follows:



We can then explicitly describe the coefficient system $h_{C_p}^{V+T}(f, A)$. Let ρ be the projection $\star \rightarrow \bullet$.

Proposition 5.5. *The functor $h_{C_p}^{V+T}(f, A): \Pi \rightarrow \mathbb{k}\text{-mod}$ takes*

$$\begin{aligned} (\bullet, x_0) &\longmapsto H_{C_p}^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A) = \underline{H}_{C_p}^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))(\star) \\ (\star, x_0) &\longmapsto H^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A(\star)) = \underline{H}_{C_p}^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))(\bullet) \end{aligned}$$

The functor is determined on morphisms by the following:

1. The image of $(\rho, [c_{(\star, x_0)}])$ is the image of ρ in the underlying contravariant coefficient system of the Mackey functor $\underline{H}_{C_p}^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$;
2. For each map $g: \star \rightarrow \star$, the image of $(g, [c_{(\star, x_0)}])$ is the image of g in the underlying contravariant coefficient system of $\underline{H}_{C_p}^{V+T}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$;
3. Let D_1 through D_{p-1} and C be the algebra generators described in Theorem 3.23. The nontrivial automorphism of (\bullet, x_0) acts by the identity on the D_{2k} and by multiplication by -1 on the D_{2k-1} and C .

This may be visualized by the diagram below.

$$\begin{array}{c}
\mathbb{Z}/2 \\
\downarrow \\
H_{C_p}^{V+t}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A) \\
\downarrow \\
H^{|V+t|}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A(\star)) \\
\downarrow \\
\begin{array}{cc}
C_p & \mathbb{Z}/2
\end{array}
\end{array}$$

Note that the second downward arrow is given by composing the maps of (1) and (3).

Proof. Lemma 5.4 identifies the value of $h_{C_p}^{V+t}(f; A)$ on objects.

For item (1), the downward arrow induced by the morphism $(\rho, [c_{(\star, x_0)}])$ is simply the map on cohomology induced by the space-level map

$$C_p \times f^{-1}(x) \longrightarrow f^{-1}(x).$$

This is the same as the map $\underline{H}_{C_p}^{V+t}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))(\bullet) \longrightarrow \underline{H}_{C_p}^{V+t}(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))(\star)$ induced by the span

$$\begin{array}{ccc}
& \star & \\
\swarrow & & \searrow \\
\star & & \bullet
\end{array}$$

A similar argument identifies the map in item (2).

It remains to identify the $\mathbb{Z}/2$ action. Recall that any $B_G\Gamma$ is of the nonequivariant homotopy type of the classifying space $B\Gamma$. In particular, our fibration $B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$ corresponds to the nonequivariant fiber sequence

$$BSO(2) \longrightarrow BO(2) \xrightarrow{B\det} B\mathbb{Z}/2$$

We know that $H^*(BSO(2); \mathbb{F}_q) \cong \mathbb{F}_q[x]$, a polynomial algebra on a generator x in degree 2, and that $\pi_1 B\mathbb{Z}/2$ acts by -1 on x . This determines the $\mathbb{Z}/2$ action at the \star level in the Mackey functor $\underline{H}_{C_p}^{V+t}$, and thus at the \star level in $h_{C_p}^{V+t}(f; A)$. The algebra generators of $\underline{H}_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}))$ at the \bullet level are D_1 through D_{p-1} and C , where D_j restricts to x^j at the \star level and C restricts to x^p . It follows that the action of $\mathbb{Z}/2$ at the \bullet level must be by -1 on the elements D_{2k-1} and C , and by the identity on the D_{2k} . \square

5.3 The local coefficient system $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$

We will continue to write Π for $\Pi_{C_p} B_{C_p}\mathbb{Z}/2$. Recall that $\mathcal{H}_*(\widetilde{X})$ is the coefficient system $\Pi_G X^{\text{op}} \longrightarrow \mathbb{k}\text{-mod}$ which takes $(K, x) \mapsto H_*(\widetilde{X^K}(x))$. In our case, since $B_{C_p}\mathbb{Z}/2 \cong \mathbb{R}P^\infty$ with trivial C_p -action, it follows that $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$ is the constant functor at $H_*(S^\infty) \cong$

$H_*(pt)$. We will continue to take our coefficient ring $\mathbb{k} = \mathbb{F}_q$ for $q \neq p$, so $H_*(pt) \cong \mathbb{F}_q$ concentrated in dimension 0.

One reason that $\mathbb{k} = \mathbb{F}_q$ is such a convenient choice in this example is that the constant functor at $H_*(pt)$ is projective.

Proposition 5.6. *The constant functor $\underline{\mathbb{F}}_q: \Pi^{op} \rightarrow \mathbb{F}_q\text{-mod}$ is a direct summand of a representable functor and hence a projective object in the category $[\Pi^{op}, \mathbb{F}_q\text{-mod}]$.*

Proof. Consider the represented functor $\mathbb{F}_q\Pi(-, (\bullet, x_0))$. We see by inspection that $\mathbb{F}_q\Pi((K, x_0), (\bullet, x_0)) \cong \mathbb{F}_q[\mathbb{Z}/2]$ for both possible values of K . For an appropriate choice of basis, we can display this as

$$\begin{array}{c}
 \begin{array}{c} \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \\ \curvearrowright \\ \mathbb{F}_q \oplus \mathbb{F}_q \end{array} \\
 \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \\
 \begin{array}{c} \mathbb{F}_q \oplus \mathbb{F}_q \\ \curvearrowleft \\ \text{triv} \quad \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \end{array}
 \end{array}$$

That is, the nontrivial element of $\mathbb{Z}/2$ acts by interchanging the basis elements, on both the top and the bottom. The action of C_p on the bottom is trivial, and the downward maps behave as shown. If we take the new basis given by the change-of-coordinates matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (using the fact that $q \neq 2$), we see that $\mathbb{F}_q\Pi(-, (\bullet, x_0))$ breaks up as the direct sum of two functors, one of which is our constant functor $\underline{\mathbb{F}}_q$.

$$\mathbb{F}_q\Pi(-, (\bullet, x_0)) \cong \begin{array}{c} \text{id} \\ \curvearrowright \\ \mathbb{F}_q \\ \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) \\ \text{id} \quad \text{id} \\ \curvearrowleft \\ \mathbb{F}_q \\ \text{triv} \quad \text{id} \end{array} \oplus \begin{array}{c} -1 \\ \curvearrowright \\ \mathbb{F}_q \\ \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) \\ \text{id} \quad -1 \\ \curvearrowleft \\ \mathbb{F}_q \\ \text{triv} \quad -1 \end{array}$$

□

5.4 The calculation of $H_{C_p}^*(B_{C_p}O(2); A)$

We are now prepared to prove Theorem 5.1. Fix odd primes $p \neq q$, and continue to take $\mathbb{k} = \mathbb{F}_q$. We will also continue to make heavy use of the identification of $B_{C_p}\mathbb{Z}/2$ in Lemma 5.2.

Proof of Theorem 5.1. We will use the equivariant Eilenberg spectral sequence to identify the E_2 page of the Serre spectral sequence for $f: B_{C_p}O(2) \rightarrow B_{C_p}\mathbb{Z}/2$ and then show that the Serre spectral sequence collapses with no extension problems.

Since $B_{C_p}\mathbb{Z}/2$ is the constant functor at S^∞ , the relevant equivariant Eilenberg spectral sequence in this case is

$$\mathrm{Ext}_\Pi^{u,v} \left(\mathcal{H}_*(S^\infty), h_{C_p}^{V+t}(f; A) \right) \Longrightarrow H_{C_p}^{u+v}(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)).$$

As before, in the E_2 term, u is the homological degree and v is the internal grading on \mathcal{H}_* . Since $\mathcal{H}_v(S^\infty)$ is either 0 or \mathbb{F}_q , both of which are projective, $\mathrm{Hom}_\Pi(\mathcal{H}_v(S^\infty), -)$ is exact, and so all Ext terms with $u > 0$ vanish. It follows that the spectral sequence collapses at E_2 with no extension problems, and so the E_2 terms of the Serre spectral sequence are given by

$$H_{C_p}^s(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)) \cong \mathrm{Hom}_\Pi \left(\mathcal{H}_s(S^\infty), h_{C_p}^{V+t}(f; A) \right).$$

The homology of S^∞ vanishes for $s > 0$, so in fact the Serre spectral sequence also collapses with no extension problems.

$B_{C_p}\mathbb{Z}/2$ has a trivial C_p action and is C_p -connected, meaning that Proposition 4.18 and Proposition 4.19 apply. Thus we may identify

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong H_{C_p}^0(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)) \hookrightarrow H_{C_p}^{V+t}(B_{C_p}SO(2); A)$$

as algebras over the cohomology of a point.

More specifically, we have

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong \mathrm{Hom}_\Pi(\mathbb{F}_q, h_{C_p}^{V+t}(f; A)).$$

For any \mathcal{N} and any element $\eta \in \mathrm{Hom}_\Pi(\mathbb{F}_q, \mathcal{N})$, η factors through the “fixed subfunctor of \mathcal{N} ,” i.e. the subfunctor

$$\begin{array}{c} \mathcal{N}(\bullet, x_0)^{\mathbb{Z}/2} \\ \downarrow \\ \mathcal{N}(\star, x_0)^{C_p \times \mathbb{Z}/2}. \end{array}$$

Note the two downward arrows must give the same map, and so a single arrow has been drawn above. As already observed, our Π has a weakly terminal object, and so a map in $\mathrm{Hom}_\Pi(\mathbb{F}_q, \mathcal{N})$ is in fact determined by choosing an element of $\mathcal{N}(C_p, x_0)^{\mathbb{Z}/2}$. By inspection of the structure of $\Pi = \Pi_{C_p}B_{C_p}\mathbb{Z}/2$, we see that every element of $\mathcal{N}(C_p, x_0)^{\mathbb{Z}/2}$ defines a map in $\mathrm{Hom}_\Pi(\mathbb{F}_q, \mathcal{N})$, as well. In other words, we have demonstrated that

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong H_{C_p}^{V+t}(B_{C_p}SO(2); A)^{\mathbb{Z}/2}$$

for each V and t , and hence

$$H_{C_p}^*(B_{C_p}O(2); A) \cong H_{C_p}^*(B_{C_p}SO(2); A)^{\mathbb{Z}/2}.$$

By Theorem 3.23 and Proposition 5.5, it follows that $H_{C_p}^*(B_{C_p}O(2); A)$ is the subalgebra of $H_{C_p}^*(B_{C_p}SO(2); A)$ generated by the elements D_{2k} , $D_{2k-1}C$, and C^2 , the generators restricting to an even power of the nonequivariant generator x of $H^*(\mathbb{C}P^\infty)$. \square

In fact, closer examination shows that the Green functor $\underline{H}_{C_p}^*(B_{C_p}O(2))$ is a sub-Green functor of $\underline{H}_{C_p}^*(\mathbb{C}P(\mathcal{U})) = \underline{H}_{C_p}^*(B_{C_p}SO(2))$, again on the generators D_{2k} , $D_{2k-1}C$, and C^2 .

CHAPTER 6

MULTIPLICATIVE STRUCTURE OF THE SPECTRAL SEQUENCE

6.1 Multiplicative structure

The nonequivariant Serre spectral sequence for the computation of $H^*(E; \mathbb{k})$ is a spectral sequence of algebras, and the same is true of its equivariant analogue. The Eilenberg spectral sequence would thus be much more powerful if it could be used to find the multiplicative structure of the E_2 page of the Serre spectral sequence. An elaboration of Whitehead's proof of Theorem 4.9 shows that when \mathcal{N} is \mathbb{k} -algebra valued, $H^*(X; \mathcal{N})$ and $H^*(X; N)$ are isomorphic as \mathbb{k} -algebras, so we need only show that the Eilenberg spectral sequence is multiplicative.

6.1.1 The nonequivariant spectral sequence

Consider a fibration $f: E \rightarrow X$ with fiber F and $\pi := \pi_1 X$. Return for the moment to $\mathbb{k}[\pi]$ -module notation in the nonequivariant case. The Serre spectral sequence has

$$E_2^{s,t} = H^s(X; \mathcal{H}^t(F; \mathbb{k}))$$

where $\mathcal{H}^t(F; \mathbb{k})$ is a $\mathbb{k}[\pi]$ -module. This has the expected pairing

$$H^{s_1}(X; \mathcal{H}^{t_1}(F; \mathbb{k})) \otimes H^{s_2}(X; \mathcal{H}^{t_2}(F; \mathbb{k})) \rightarrow H^{s_1+s_2}(X; \mathcal{H}^{t_1+t_2}(F; \mathbb{k})).$$

We know from Theorem 4.6 that, for each fixed t , there is a spectral sequence

$$E_2^{u,v} = \text{Ext}_{\mathbb{k}[\pi]}^{u,v}(H_*(\tilde{X}), \mathcal{H}^t(F; \mathbb{k})) \implies H^{u+v}(X; \mathcal{H}^t(F; \mathbb{k})).$$

It is convenient to view the collection of all of these spectral sequences as a single trigraded spectral sequence.

Lemma 6.1 (Trigraded Eilenberg Spectral Sequence). *Let X be a path-connected based space with universal cover \tilde{X} and $\pi = \pi_1 X$ (as in Section 4.1). Let N^* be a graded $\mathbb{k}[\pi]$ module. Then there is a spectral sequence*

$$E_2^{t,u,v} = \text{Ext}_{\mathbb{k}[\pi]}^{u,v}(H_*(\tilde{X}), N^t) \implies H^{u+v}(X; N^t).$$

The differentials d_r change tridegrees by $(t, u, v) \mapsto (t, u + r, v - r + 1)$.

Proof. For each t , choose an injective resolution $N^t \rightarrow I^{t,*}$. Then we have a trigraded complex $\mathrm{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), I^{*,*})$. For each fixed t , we have a bicomplex as in Theorem 4.6 and thus a spectral sequence. As t varies, these fit together to form the above trigraded spectral sequence. \square

Now suppose that N^* is a $\mathbb{k}[\pi]$ -algebra. A multiplicative structure on

$$\mathrm{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), I^{*,*})$$

which induces the expected multiplicative structure on $H^*(X; N^*)$ is sufficient to make the trigraded Eilenberg spectral sequence into a spectral sequence of algebras.

To find such a structure, it is enough to show the existence of a product map $\varphi: I^{*,*} \otimes I^{*,*} \rightarrow I^{*,*}$ compatible with the multiplication $N^* \otimes N^* \rightarrow N^*$. We then have the composite

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{k}[\pi]}(C_{v_1}(\tilde{X}), I^{t_1, u_1}) \otimes \mathrm{Hom}_{\mathbb{k}[\pi]}(C_{v_2}(\tilde{X}), I^{t_2, u_2}) & & \\ & \searrow & \\ \mathrm{Hom}_{\mathbb{k}[\pi]}(C_{v_1}(\tilde{X}) \otimes C_{v_2}(\tilde{X}), I^{t_1, u_1} \otimes I^{t_2, u_2}) & & \\ & \searrow & \\ \mathrm{Hom}_{\mathbb{k}[\pi]}(C_{v_1+v_2}(\tilde{X}), I^{t_1+t_2, u_1+u_2}) & & \end{array}$$

where the first arrow takes the tensor product of a pair of maps, and the second is induced by φ and the usual Alexander-Whitney map $C_*(\tilde{X}) \rightarrow C_*(\tilde{X}) \otimes C_*(\tilde{X})$.

In general, it is difficult to come up with a suitable product on $I^{*,*}$. However, there are some situations in which this product exists, and then the cohomological Eilenberg spectral sequence becomes a spectral sequence of algebras.

Theorem 6.2. *Suppose that π is a finite group of order n , \mathbb{k} is a field of characteristic prime to n , and N^* is a $\mathbb{k}[\pi]$ -algebra. Then the spectral sequence of Lemma 6.1 is a spectral sequence of algebras, where the multiplication on the E_2 page comes from the diagonal map on $H_*(\tilde{X})$ and the multiplication of N^* .*

Proof. Since $\mathbb{k}[\pi]$ is semisimple, every $\mathbb{k}[\pi]$ -module is injective and projective. Hence we can choose each $I^{t,*}$ to be the resolution $0 \rightarrow N^t \rightarrow 0 \rightarrow \dots$. Since N^* is an algebra, it is clear that we have $I^{*,*} \otimes I^{*,*} \rightarrow I^{*,*}$ as desired. \square

In fact, it suffices to assume that each N^t has an underlying \mathbb{k} -module which is finitely generated and projective.

Theorem 6.3. *Suppose that $\pi = \pi_1 X$ is a finite group and that N^* is a graded $\mathbb{k}[\pi]$ -module such that the underlying \mathbb{k} -module of each N^t is finitely generated projective. Then the spectral sequence of Lemma 6.1 is a spectral sequence of algebras; the multiplication on E_2 is as in Theorem 6.2.*

To prove Theorem 6.3, we will need to review some terminology and results from [Bro82, Chapter VI]. In the following, let \mathbb{k} be any ring and π be a finite group. (The definitions work for general π , but the propositions require finite groups.) Brown takes $\mathbb{k} = \mathbb{Z}$ throughout, but all proofs work for general \mathbb{k} as well.

Definition 6.4. An injection $M_1 \hookrightarrow M_2$ of $\mathbb{k}[\pi]$ -modules is an **admissible injection** if it is split when regarded as an injection of \mathbb{k} -modules. A long exact sequence of $\mathbb{k}[\pi]$ -modules is **admissible** if it is contractible when viewed as a chain complex of \mathbb{k} -modules.

Definition 6.5. We say that a $\mathbb{k}[\pi]$ -module Q is **relatively injective** if the functor $\mathrm{Hom}_{\mathbb{k}[\pi]}(-, Q)$ takes admissible injections of $\mathbb{k}[\pi]$ -modules to surjections of \mathbb{k} -modules. If M is any $\mathbb{k}[\pi]$ -module, a **relatively injective** resolution of M consists of a complex $0 \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots$ such that each Q^i is relatively injective, together with a weak equivalence $M \rightarrow Q^*$ such that $0 \rightarrow M \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots$ is admissible.

In particular, injective modules are relatively injective and injective resolutions are relative injective resolutions. The following proposition shows that it suffices to consider relative injective resolutions when computing Tor and Ext.

Proposition 6.6. *Let π be a finite group, and let M be a left $\mathbb{k}[\pi]$ -module.*

1. *Any two relative injective resolutions of M are canonically homotopy equivalent.*
2. *If M is projective as a $\mathbb{k}[\pi]$ -module, then M is relatively injective.*
3. *There is a $\mathbb{k}[\pi]$ -module isomorphism $\mathrm{Hom}_{\mathbb{k}}(M, \mathbb{k}) \cong \mathrm{Hom}_{\mathbb{k}[\pi]}(M, \mathbb{k}[\pi])$.*

Proof. These are propositions VI.2.5, VI.2.3, and VI.3.4 in [Bro82]. □

These facts are sufficient to prove the following useful lemma.

Lemma 6.7. *Let M_1, M_2 be $\mathbb{k}[\pi]$ -modules whose underlying \mathbb{k} -modules are finitely generated and projective. Then, for suitable choices of relatively injective resolutions $M_1 \rightarrow I_1^*$ and $M_2 \rightarrow I_2^*$, $I_1^* \otimes I_2^*$ is a relatively injective resolution of $M_1 \otimes M_2$ and*

$$0 \rightarrow M_1 \otimes M_2 \rightarrow (I_1^* \otimes I_2^*)^0 \rightarrow (I_1^* \otimes I_2^*)^1 \rightarrow \dots$$

is contractible when viewed as a complex of \mathbb{k} -modules.

Proof. Since each M_i is finitely generated and projective as a \mathbb{k} -module, the same is true of $\overline{M}_i := \mathrm{Hom}_{\mathbb{k}}(M_i, \mathbb{k}) \cong \mathrm{Hom}_{\mathbb{k}[\pi]}(M_i, \mathbb{k}[\pi])$; additionally, $M_i \cong \overline{\overline{M}}_i = \mathrm{Hom}_{\mathbb{k}}(\overline{M}_i, \mathbb{k}) \cong \mathrm{Hom}_{\mathbb{k}[\pi]}(\overline{M}_i, \mathbb{k}[\pi])$.

Choose $\mathbb{k}[\pi]$ -projective finitely generated resolutions $P_{i,*} \rightarrow \overline{M}_i$. Since \overline{M}_i is projective as a \mathbb{k} -module, $P_{i,*} \rightarrow M_i$ is a homotopy equivalence of complexes of \mathbb{k} -modules. If we then dualize, defining $I_i^n = \overline{P}_{i,n}$, it follows that $M_i \cong \overline{\overline{M}}_i \xrightarrow{\eta_i} I_i^*$ is also a homotopy equivalence of complexes over \mathbb{k} . Since the $P_{i,*}$ are projective $\mathbb{k}[\pi]$ -modules, the same is true of the I_i^* , and so we have relatively injective resolutions of M_1 and M_2 by Proposition 6.6.

Since η_1 and η_2 are homotopy equivalences of chain complexes of \mathbb{k} -modules, it follows that $\eta_1 \otimes \eta_2: M_1 \otimes M_2 \rightarrow I_1^* \otimes I_2^*$ is a homotopy equivalence (over $\mathbb{k} \otimes \mathbb{k} = \mathbb{k}$) as well; see, e.g. [Bro82, Chapter I]. This immediately implies the final conclusion of the theorem. In addition, since

$$0 \rightarrow M_1 \otimes M_2 \rightarrow (I_1^* \otimes I_2^*)^0 \rightarrow (I_1^* \otimes I_2^*)^1 \rightarrow \dots$$

is acyclic as a complex of \mathbb{k} -modules, it must also be acyclic as a complex of $\mathbb{k}[\pi]$ -modules. Using Proposition 6.6 and the fact that the tensor product of projectives is projective, we see that $M_1 \otimes M_2 \rightarrow I_1^* \otimes I_2^*$ is a relatively injective resolution as claimed. \square

Proof of Theorem 6.3. We would like to construct the trigraded Eilenberg spectral sequence in such a way that there is a multiplication on the E_0 page. By Proposition 6.6, we may use the relatively injective resolutions of Lemma 6.7 when defining the trigraded Eilenberg spectral sequence. Thus, for each t , we have a relatively injective resolution $N^t \rightarrow I^{t,*}$, and for each t_1, t_2 , $N^{t_1} \otimes N^{t_2} \rightarrow I^{t_1,*} \otimes I^{t_2,*}$ is also a relatively injective resolution.

Since N^* is an algebra, there are maps $N^{t_1} \otimes N^{t_2} \rightarrow N^{t_1+t_2}$ for each t_1 and t_2 . We would then like to fill in the dotted arrows in the diagram below.

$$\begin{array}{ccccccc} 0 \rightarrow N_{t_1} \otimes N_{t_2} & \rightarrow & (I^{t_1,*} \otimes I^{t_2,*})^0 & \rightarrow & (I^{t_1,*} \otimes I^{t_2,*})^1 & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow N^{t_1+t_2} & \rightarrow & I^{t_1+t_2,0} & \rightarrow & I^{t_1+t_2,1} & \rightarrow & \dots \end{array}$$

By Lemma 6.7, the top row is an admissible exact sequence. The bottom row is a relatively injective resolution of $N^{t_1+t_2}$. We may thus use the universal property of relatively injective modules to successively fill in the dotted arrows, as usual.

These dotted arrows give us maps $I^{t_1,u_1} \otimes I^{t_2,u_2} \rightarrow I^{t_1+t_2,u_1+u_2}$ for each t_1, t_2, u_1, u_2 , and hence $\varphi: I^{*,*} \otimes I^{*,*} \rightarrow I^{*,*}$. This φ , together with the Alexander-Whitney map $C_*(\tilde{X}) \rightarrow C_*(\tilde{X}) \otimes C_*(\tilde{X})$, then gives us a multiplication on

$$\mathrm{Hom}_{\mathbb{k}[\pi]}(C_*(\tilde{X}), I^{*,*}),$$

the E_0 page of our spectral sequence, and hence makes the trigraded Eilenberg spectral sequence into a spectral sequence of algebras. \square

6.1.2 The equivariant spectral sequence

As in the previous section, we may view the equivariant Eilenberg spectral sequence as a tri-graded spectral sequence. Unfortunately, the tricks of Subsection 6.1.1 do not immediately generalize to the equivariant situation. As a result, the following remains a conjecture.

Conjecture 6.8. *In some cases of interest, the trigraded equivariant Eilenberg spectral sequence*

$$E_2^{t,u,v} = \mathrm{Ext}_{\Pi}^{u,v}(\mathcal{H}_*(\tilde{X}), \mathcal{N}^t) \implies \mathcal{H}^{u+v}(X; \mathcal{N}^t)$$

is a spectral sequence of algebras. The multiplication on the E_2 page comes from the diagonal map on $\mathcal{H}_*(\tilde{X})$ and the multiplication of \mathcal{N}^* .

6.2 Sample nonequivariant computations

We are now in a position to use the multiplicative Eilenberg spectral sequence to do some computations with the nonequivariant Serre spectral sequence.

Example 6.9. Again consider the fibration

$$B\det: BO(2) \longrightarrow B\mathbb{Z}/2,$$

but now take $k = \mathbb{Z}$, so that $k[\mathbb{Z}/2]$ is not semisimple. Then the Serre spectral sequence has

$$E_2^{s,t} = H^s(\mathbb{R}P^\infty; \mathcal{H}^t(BSO(2); \mathbb{Z})) \implies H^{s+t}(BO(2); \mathbb{Z})$$

We know that the underlying \mathbb{Z} -algebra (i.e. ring) of $\mathcal{H}^t(BSO(2); \mathbb{Z})$ is the integral polynomial ring $\mathbb{Z}[x]$; since \mathbb{Z} is a projective abelian group, Theorem 6.3 applies. The action of $\mathbb{Z}/2$ on $\mathbb{Z}[x]$ is again given by $x \mapsto -x$. The trigraded Eilenberg spectral sequence (of algebras) then has

$$E_2^{t,u,v} = \text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^u(H_v(S^\infty), \mathbb{Z}[x]^t).$$

Since $S^\infty \simeq pt$, each term with $v \neq 0$ vanishes, and so the spectral sequence collapses with no extension problems. It follows that, in the Serre spectral sequence,

$$E_2^{s,t} = \text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^s(\mathbb{Z}, \mathbb{Z}[x]^t).$$

Since $\mathbb{Z}/2$ acts on x by $x \mapsto -x$, $\mathbb{Z}[x]^t$ is either \mathbb{Z} with the trivial $\mathbb{Z}/2$ action or \mathbb{Z}_- with the action by -1 . We can then find relatively injective resolutions of these groups, coming from projective resolutions of their duals, and use these to calculate the desired Ext groups together with their multiplication, as described by Theorem 6.3.

In fact, in this case, the description of the multiplicative structure is made simpler by the fact that $\mathbb{Z}[x]^{t_1} \otimes \mathbb{Z}[x]^{t_2} \longrightarrow \mathbb{Z}[x]^{t_1+t_2}$ is an isomorphism for every t_1 and t_2 . Thus, if $I^{t_1,*}$ and $I^{t_2,*}$ are the relatively injective resolutions of $\mathbb{Z}[x]^{t_1}$ and $\mathbb{Z}[x]^{t_2}$ given by Theorem 6.3, $I^{t_1,*} \otimes I^{t_2,*}$ is a resolution of $\mathbb{Z}[x]^{t_1+t_2}$, and so it may be used to calculate $\text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^s(\mathbb{Z}, \mathbb{Z}[x]^{t_1+t_2})$. Specifically, if $\alpha_i \in \text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, I^{t_i, s_i})$ for $i = 1, 2$ represent cohomology classes in $\text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^{s_i}(\mathbb{Z}, \mathbb{Z}[x]^{t_i})$, then their product in $\text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^{s_1+s_2}(\mathbb{Z}, \mathbb{Z}[x]^{t_1+t_2})$ is represented by $\alpha_1 \otimes \alpha_2 \in \text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, I^{t_1, s_1} \otimes I^{t_2, s_2})$.

From here, the example can be completed by explicitly choosing relatively injective resolutions I^* of \mathbb{Z} and J^* of \mathbb{Z}_- , as we will outline. Note that, for both \mathbb{Z} and \mathbb{Z}_- , the dual is isomorphic to the original module. For the remainder of this example, write $\mathbb{Z}/2$ multiplicatively with generator σ . As is well known, the following is a resolution for \mathbb{Z} over $\mathbb{Z}[\mathbb{Z}/2]$.

$$\cdots \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

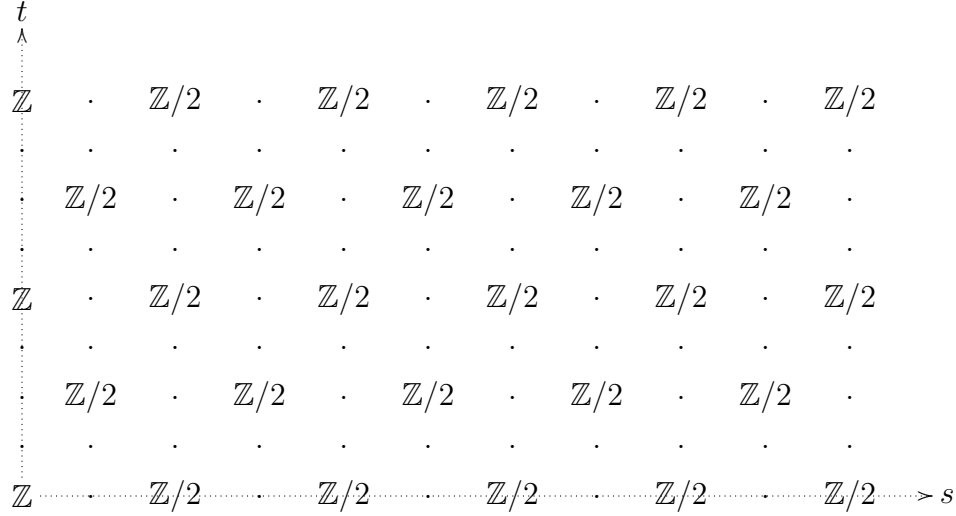


Figure 6.10: $\text{Ext}_{\mathbb{Z}/2}^s(\mathbb{Z}, \mathbb{Z}[x]^t)$. A dot represents the zero group.

Dualizing, we see that we may take I^* to be the relatively injective resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow \dots$$

A similar argument shows that we may take J^* to be the relatively injective resolution

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \longrightarrow \dots$$

Note that $\mathbb{Z}[\mathbb{Z}/2] \otimes \mathbb{Z}[\mathbb{Z}/2] \cong \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$. The tensor products $I^* \otimes I^*$, $I^* \otimes J^*$, $J^* \otimes I^*$, and $J^* \otimes J^*$ all have the form

$$0 \longrightarrow \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2] \longrightarrow \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^{\oplus 2} \longrightarrow \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^{\oplus 3} \longrightarrow \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^{\oplus 4} \longrightarrow \dots$$

where the arrows depend on which of the four tensor products we are considering. It is an instructive exercise at this point to apply the functor $\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}, -)$, take homology, and check that we get the correct Ext groups. We will omit the details, but Figure 6.10 gives a picture of $\text{Ext}_{\mathbb{Z}/2}^s(\mathbb{Z}, \mathbb{Z}[x]^t)$ for varying values of s and t .

We will now describe the multiplicative structure. It is a routine calculation to check, using the above resolutions I^* , J^* , and their tensor products, that the product of any two generators of nonzero groups appearing in the picture above is a generator of the group in the appropriate degree. It follows that we can describe the multiplicative structure on $\text{Ext}_{\mathbb{Z}/2}^*(\mathbb{Z}, \mathbb{Z}[x]^*)$ as follows. Let p_1 be a generator of the copy of \mathbb{Z} at $(s, t) = (0, 4)$, α the generator of the $\mathbb{Z}/2$ at $(2, 0)$, and β the generator of the $\mathbb{Z}/2$ at $(1, 2)$. Then our algebra has the presentation

$$\text{Ext}_{\mathbb{Z}/2}^*(\mathbb{Z}, \mathbb{Z}[x]^*) \cong \mathbb{Z}[p_1, \alpha, \beta] / (2\alpha = 0, 2\beta = 0, \beta^2 = p_1\alpha).$$

Having computed the multiplicative structure of the E_2 page, we are finally ready to consider the Serre spectral sequence for $BO(2) \xrightarrow{B\det} B\mathbb{Z}/2$. For dimensional reasons, all differentials except d_3 must be 0. Note that $d_3\beta$ lands in a copy of $\mathbb{Z}/2$, so $d_3\beta^2 = 0$ by the Leibniz formula. Since $d_3\alpha = 0$, we have $0 = d_3\beta^2 = d_3(p_1\alpha) = (d_3p_1)\alpha$; hence $d_3p_1 = 0$. We are thus left with only $d_3\beta$ to consider.

Lemma 6.11. $H^3(BO(2); \mathbb{Z}) \neq 0$.

Proof. By the Universal Coefficients Theorem, we have (non-canonical) isomorphisms

$$\begin{aligned} H^3(BO(2); \mathbb{Z}) &\cong \text{torsion}(H_2(BO(2); \mathbb{Z})) \oplus \text{free}(H_3(BO(2); \mathbb{Z})) \\ H^2(BO(2); \mathbb{Z}) &\cong \text{torsion}(H_1(BO(2); \mathbb{Z})) \oplus \text{free}(H_2(BO(2); \mathbb{Z})) \\ H_2(BO(2); \mathbb{F}_2) &\cong H_2(BO(2); \mathbb{Z}) \otimes \mathbb{F}_2 \oplus \text{Tor}_1^{\mathbb{Z}}(H_1(BO(2); \mathbb{Z}), \mathbb{F}_2) \end{aligned}$$

Since $\pi_1(BO(2)) \cong \mathbb{Z}/2$, $H_1(BO(2); \mathbb{Z}) \cong \mathbb{Z}/2$ as well. We can see from the E_2 page of the Serre spectral sequence above that $H^2(BO(2); \mathbb{Z}) \cong \mathbb{Z}/2$, so we can conclude from the middle isomorphism that $H_2(BO(2); \mathbb{Z})$ consists entirely of torsion.

The mod 2 homology $H_*(BO(2); \mathbb{F}_2)$ is well known; $H_2(BO(2); \mathbb{F}_2)$ is an \mathbb{F}_2 -vector space of dimension two. Since $H_1(BO(2); \mathbb{Z}) = \mathbb{Z}/2$, we see that

$$H_2(BO(2); \mathbb{Z}) \otimes \mathbb{F}_2$$

is nonzero. Since $H_2(BO(2); \mathbb{Z})$ consists entirely of torsion, we conclude from the first isomorphism that $H^3(BO(2); \mathbb{Z})$ must be nonzero as well. \square

Remark 6.12. The usual method for calculating $H^*(BO(2); \mathbb{Z})$ makes use of the Bockstein spectral sequence and of the well-known fact $H^*(BO(2); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2]$. We could alternatively follow this argument just far enough to see that $H^3(BO(2); \mathbb{Z}) \neq 0$.

Since $H^3(BO(2); \mathbb{Z}) \neq 0$, we see from the E_2 page of the Serre spectral sequence above that $d_3\beta$ must be zero. Hence the spectral sequence collapses at $E_2 = E_\infty$, and so we need only resolve the extension problems.

Abbreviate $H^* := H^*(BO(2); \mathbb{Z})$ for the moment. Since the Serre spectral sequence in question is a spectral sequence of algebras converging to H^* , there is a product-respecting filtration F on H^* such that

$$(F^s H^{t+s}) / (F^{s+1} H^{t+s}) \cong \text{Ext}_{\mathbb{Z}[\mathbb{Z}/2]}^s(\mathbb{Z}, \mathbb{Z}[x]^t) \cong \mathbb{Z}[p_1, \alpha, \beta] / (2\alpha = 0, 2\beta = 0, \beta^2 = p_1\alpha).$$

In particular, we see that $H^2 \cong \mathbb{Z}/2$, generated by α ; and $H^3 \cong \mathbb{Z}/2$, generated by β . For H^4 , we have $H^4 \supset F^1 H^4 = F^2 H^4 = F^3 H^4 = F^4 H^4 = \mathbb{Z}/2 \supset 0$, satisfying the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow H^4 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since \mathbb{Z} is projective, the sequence splits, and $H^4 \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The $\mathbb{Z}/2$ is generated by α^2 , and the \mathbb{Z} by some element which we will call p_1 . A similar argument shows that

$H^6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and either $\alpha p_1 = \beta^2$ or $\alpha p_1 = \beta^2 + \alpha^3$. By replacing p_1 by $p_1 + \alpha^2$, if necessary, we may assume that $\alpha p_1 = \beta^2$. It then follows that, as an algebra,

$$H^* = H^*(BO(2); \mathbb{Z}) \cong \mathbb{Z}[p_1, \alpha, \beta] / (2\alpha = 0, 2\beta = 0, \beta^2 = p_1\alpha).$$

REFERENCES

- [AGH09] Vigleik Angeltveit, Teena Gerhardt, and Lars Hesselholt. On the K -theory of truncated polynomial algebras over the integers. arXiv:0809.3544, April 2009.
- [Bre67] Glen E. Bredon. *Equivariant cohomology theories*. Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin, 1967.
- [Bro82] K. Brown. *Cohomology of Groups*. Springer-Verlag, New York, 1982.
- [CE56] Henri Cartan and Samuel Eilenberg. *Homological Algebra*. Princeton University Press, Princeton, N. J., 1956.
- [Dre73] Andreas W. M. Dress. Contributions to the theory of induced representations. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 183–240. Lecture Notes in Math., Vol. 342. Springer, Berlin, 1973.
- [Eil47] S. Eilenberg. Homology of spaces with operators. I. *Trans. Amer. Math. Soc.*, 61:378–417, 1947. Errata, 62(1947), p. 548.
- [FL04] Kevin K. Ferland and L. Gaunce Lewis, Jr. The $RO(G)$ -graded equivariant ordinary homology of G -cell complexes with even-dimensional cells for $G = \mathbb{Z}/p$. *Mem. Amer. Math. Soc.*, 167(794):viii+129, 2004.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [HHR09] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. On the non-existence of elements of Kervaire invariant one. arXiv:0908.3724, August 2009.
- [Kro09] William C. Kronholm. The $RO(G)$ -graded Serre spectral sequence. arXiv:0908.3827, August 2009.
- [Lew80] L. Gaunce Lewis, Jr. The theory of Green functors. Mimeographed notes, 1980.
- [Lew88] L. Gaunce Lewis, Jr. The $RO(G)$ -graded equivariant ordinary cohomology of complex projective spaces with linear C_p actions. In T. tom Dieck, editor, *Algebraic topology and transformation groups*, volume 1361 of *Lecture Notes in Math.*, pages 53–122, Berlin, 1988. Springer. Proceedings of a conference held in Göttingen, 1987.
- [LM06] L. Gaunce Lewis, Jr. and Michael A. Mandell. Equivariant universal coefficient and Künneth spectral sequences. *Proc. London Math. Soc. (3)*, 92(2):505–544, 2006.

- [LMM81] G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$ -graded cohomology. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):208–212, 1981.
- [LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
- [May96] J.P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 1996. Published for the Conference Board of the Mathematical Sciences, Washington, DC.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [MP01] Goutam Mukherjee and Neeta Pandey. Equivariant cohomology with local coefficients. *Proceedings of the American Mathematical Society*, 130(1):227–232, 2001.
- [MS93] I. Moerdijk and J.A. Svensson. The equivariant Serre spectral sequence. *Proc. Amer. Math. Soc.*, 118:263–278, 1993.
- [MS06] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.
- [tD75] Tammo tom Dieck. *The Burnside ring and equivariant stable homotopy*. Department of Mathematics, University of Chicago, Chicago, Ill., 1975. Lecture notes by Michael C. Bix.
- [tD87] Tammo tom Dieck. *Transformation Groups*. Walter de Gruyter, 1987.
- [Whi78] G.W. Whitehead. *Elements of homotopy theory*. Springer-Verlag, 1978.
- [Wir74] Klaus Wirthmüller. Equivariant homology and duality. *Manuscripta Math.*, 11:373–390, 1974.