$RO(\mathbb{Z}/p)$-graded cohomology of some classifying spaces

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Goal

Learn about equivariant bundle theory by studying the cohomology of classifying spaces.

We would like to find a suitable invariant for studying equivariant characteristic classes. The theory I will choose is $RO(G)$-graded Bredon cohomology, where G is a finite group.

Outline

1. Background
   - Equivariant bundles
   - Bredon cohomology
2. Computational tools
   - The Serre spectral sequence
   - Cohomology with local coefficients
   - The equivariant generalization
3. A Sample Calculation
   - $H^*_{C_p}(B_{C_p}O(2); A_q)$
   - Future directions

Equivariant bundles

Definition

A G-bundle with structure group Γ is a projection $E \xrightarrow{/\Gamma} B$ where B is a G-space and E is a $G \times \Gamma$ space. It is principal if the action of Γ on E is free.

Given G and Γ, there is a universal bundle $E_G\Gamma \longrightarrow B_G\Gamma$ such that principal G-bundles over a space B are in one-to-one correspondence with homotopy classes of maps $B \longrightarrow B_G\Gamma$, via a pullback square

$$\begin{array}{ccc} E & \longrightarrow & E_G\Gamma \\ \big\downarrow{\scriptstyle /\Gamma} & \lrcorner & \big\downarrow{\scriptstyle /\Gamma} \\ B & \longrightarrow & B_G\Gamma \end{array}$$

Bredon cohomology

Equivariant bundles have a rich structure, so we need a cohomology theory whose characteristic classes will tell us a lot about this structure. Borel cohomology $H^*_{\text{Borel}}(X) := H^*(EG \times_G X)$ is not a good choice.

A better candidate is Bredon cohomology, which is built by looking at $H^*(X^K)$ for every subgroup $K < G$.

Definition

The orbit category $\mathscr{O}_G$ has objects the orbits G/K, and morphisms G-set maps. A functor $\mathscr{O}_G^{\text{op}} \longrightarrow R$-mod is called a coefficient system.

Bredon cohomology

Suppose G is a finite group and X is a CW complex with a cellular G-action and n-skeleton X_n. For each n, define a cellular chains coefficient system (with implicit R coefficients)

$$\underline{C}_n(X) : G/K \mapsto H_n(X_n^K, X_{n-1}^K)$$

Definition

Given a coefficient system M, the Bredon cohomology $H^*_G(X; M)$ is the cohomology of the chain complex $\mathrm{Hom}_{\mathscr{O}_G}(\underline{C}_*(X), M)$.

Bredon cohomology

This is still only graded on the integers. For theoretical reasons we would like to have a group $H_G^{V-W}(-; M)$ for every formal difference of G-representations V and W (i.e. the theory should be indexed on $RO(G)$).

Theorem (May)

If M is the underlying coefficient system of a stable coefficient system, or Mackey functor, then $H_G^*(-; M)$ extends to a theory indexed on $RO(G)$.

A stable coefficient system is given by replacing the orbit category $\mathscr{O}_G$ with a stable analogue, the Burnside category $\mathscr{B}_G$, which has the same objects but stable G-maps as morphisms.

Computations

$RO(G)$-graded Bredon cohomology has good theoretical properties and is a good home for characteristic classes.

The only problem with Bredon cohomology: it is very difficult to compute.

Some known computations:

- $H^*_{C_n}(pt; M)$ for any coefficient system M and certain cyclic groups (2, Bob Stong; primes, Gaunce Lewis; 2^k, Hill-Hopkins-Ravenel)
- $H^*_{C_n}(\mathbb{C}P(V); A)$, hence $H^*_{C_n}(B_{C_n}S^1; A) = H^*_{C_n}(B_{C_n}SO(2); A)$ (Gaunce Lewis)
- $H^*_{C_2}(\mathbb{R}P(\mathcal{U}); \underline{C_2})$ (Bill Kronholm)

The Serre spectral sequence

Moerdijk and Svensson have an equivariant Serre spectral sequence which Kronholm has extended to an $RO(G)$-grading.

Theorem (Moerdijk-Svensson, Kronholm)

Let $f : E \longrightarrow X$ be a G-fibration. There is a spectral sequence with

$$E_2^{s,t}(M, V) = H_G^s(X; h_G^{V+t}(f, M)) \Longrightarrow H_G^{V+s+t}(E; M).$$

$h_G^{V+t}(f, M)$ is a local coefficient system corresponding to the cohomology of the fibers of f. We need to learn to compute cohomology with local coefficients.

Cohomology with local coefficients

Definitions

Given a space X:

- ΠX has objects $x \in X$; morphisms x to y are homotopy classes of paths $x \longrightarrow y$.
- A local coefficient system $\mathscr{M}$ is a functor $\Pi X^{\mathrm{op}} \longrightarrow R$-mod.
- $C_n\tilde{X} : \Pi X^{\mathrm{op}} \xrightarrow{\tilde{X}} \mathrm{Top} \xrightarrow{C_n} R$-mod sends $x_0 \mapsto C_n\tilde{X}(x_0)$
  $\tilde{X}$ takes $x_0 \mapsto \tilde{X}(x_0)$, the space of equivalence classes of paths in X starting at x_0.
- $H^*(X; \mathscr{M})$ is the cohomology of $\mathrm{Hom}_{\Pi X}(C_*\tilde{X}, \mathscr{M})$.

Remark: If X is connected, we can replace the groupoid ΠX by a skeleton with one object and $\pi_1 X$ worth of morphisms; then local coefficient systems are the same as $R[\pi_1 X]$-modules and we can write $\mathrm{Hom}_{R[\pi_1 X]}$ instead of $\mathrm{Hom}_{\Pi X}$ above.

The equivariant generalization

Definitions

Given a space X,

- $\Pi_G X$ has objects $(K, x \in X^K)$; a morphism (K, x) to (H, y) consists of $\alpha : G/K \longrightarrow G/H$ together with a homotopy class of paths $x \longrightarrow \alpha^* y$ in X^K.
- A local coefficient system $\mathscr{M}$ is a functor $\Pi_G X^{\mathrm{op}} \longrightarrow R$-mod.
- $C_n\tilde{X} : \Pi_G X^{\mathrm{op}} \longrightarrow \mathrm{Top} \longrightarrow R$-mod sends $(H, x) \mapsto C_n\widetilde{X^H}(x)$
- $H^*(X; \mathscr{M})$ is the cohomology of $\mathrm{Hom}_{\Pi_G X}(C_*\tilde{X}, \mathscr{M})$.

Fact: This definition of cohomology with local coefficients agrees with the one in the paper by Moerdijk and Svensson.

A spectral sequence

Theorem (S.)

There's a spectral sequence with

$$E_2^{u,v} = \mathrm{Ext}^u_{\Pi_G X}(\mathscr{H}_v \tilde{X}, \mathscr{M}) \Longrightarrow H_G^{u+v}(X; \mathscr{M})$$

where $\mathscr{H}_*$ is levelwise homology.

$\mathrm{Ext}_{\Pi_G X}(\mathscr{M}_1, \mathscr{M}_2)$ is computed by replacing $\mathscr{M}_2$ by an injective resolution $\mathscr{I}^*$, then computing the homology of $\mathrm{Hom}_{\Pi_G X}(\mathscr{M}_1, \mathscr{I}^*)$.

The tricky part of the proof is showing that $C_n\tilde{X}$ is a projective object in the functor category $[\Pi_G X^{\mathrm{op}}, R\text{-mod}]$.

In the nonequivariant case (due to Cartan-Eilenberg), the projectivity comes from observing that $C_*\tilde{X}$ is a projective $\pi_1 X$-module.

A Calculation

Consider the exact sequence of groups $SO(2) \longrightarrow O(2) \longrightarrow \mathbb{Z}/2$. This induces a C_p-fibration $B_{C_p}\text{det} : B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$.

With an appropriate choice of coefficients and ground ring, we can use the equivariant Cartan-Eilenberg spectral sequence to compute this.

Choose odd primes $p \neq q$. Let

- $R = \mathbb{F}_q$;
- $M = A_q := \mathbb{F}_q\mathscr{B}_G(-, G/G)$, the represented Mackey functor;
- $G = C_p$.

$H^*_{C_p}(B_{C_p}O(2); A_q)$

Recall $R = \mathbb{F}_q$, $A_q = \mathbb{F}_q\mathscr{B}_G(-, G/G)$, $G = C_p$.
We are considering the G-fibration $B_{C_p}\det : B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$.

Facts

- $B_{C_p}\mathbb{Z}/2$ has as a model $\mathbb{R}P^\infty$ with the trivial C_p action. Its universal cover $\widetilde{B_{C_p}\mathbb{Z}}/2$ is the constant functor at S^∞.
- The preimage $B_{C_p}\det^{-1}(x)$ for any $x \in B_{C_p}\mathbb{Z}/2$ is homotopy equivalent to $B_{C_p}SO(2) \simeq \mathbb{C}P(\mathscr{U})$.
- We can use Lewis' computation of $H^*_{C_p}(\mathbb{C}P(\mathscr{U}))$ to calculate the local coefficient system $h^*_{C_p}(B_{C_p}\det; A_q)$.

$H^*_{C_p}(B_{C_p}O(2); A_q)$

This puts us in a good position to use our equivariant Cartan-Eilenberg spectral sequence, which has E_2 page

$$E_2^{u,v} = \mathrm{Ext}^u_{\Pi}(\mathscr{H}_v(\underline{S^\infty}), h^*_{C_p}(B_{C_p}\mathrm{det}; A_q))$$

where $\Pi = \Pi B_{C_p}\mathbb{Z}/2$.

Our assumptions on p and q imply that both the equivariant Cartan-Eilenberg spectral sequence and the Serre spectral sequence collapse with no extension problems. This enables us to explicitly describe $H^*_{C_p}(B_{C_p}O(2); A_q)$ as a subalgebra of $H^*_{C_p}(B_{C_p}SO(2); A_q)$.

Future directions

- Now that the machinery has been set up, the calculation of $RO(G)$-graded Bredon cohomology should also be within reach.
- It would be nice to have a multiplicative structure on the spectral sequence.
- The calculation of the cohomology of more classifying spaces should lead to a better understanding of equivariant bundle theory.