# FINITE GROUPS OF LIE TYPE: RATIONAL POINTS

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## 1. $\mathbb{F}_q$-RATIONAL STRUCTURES

We now assume that k is an algebraic closure of a finite field $\mathbb{F}_q$. We want to study varieties and algebraic groups defined over a finite field. Most naively when studying polynomial equations in $n$-variables over $\mathbb{F}_q$, one might consider their solutions $\mathbb{F}_q^n$. As such the solution sets will be finite, and so rather lifeless from a geometrical point of view. To rectify this we should consider instead not just the solutions of our polynomials in $\mathbb{F}_q$, but in any finite extension of $\mathbb{F}_q$ (and hence in k our chosen algebraic closure). The set of all of these solutions is a much richer object for geometric study, containing within it the original set of $\mathbb{F}_q$-solutions. Somewhat better, this larger solution set will have some extra symmetries coming from the fact that its defining polynomials all had coefficients in $\mathbb{F}_q$. Before giving formal definitions, we give an example in the situation of real algebraic geometry.

**Example 1.1.** Consider the polynomial $f \in \mathbb{R}[x,y]$ given by $f(x,y) = x^2 + y^2 + 1$. Then the set $X = \{(x,y) \in \mathbb{R} : x^2 + y^2 + 1 = 0\}$ of solutions to the equation $f(x,y) = 0$ is of course empty, giving us very little information about $f$. However if we consider the solution set $X$ in $\mathbb{C}^2$, we get a copy of $\mathbb{C}^\times$. The extra structure we obtain on this complex solution locus is an involution coming from complex conjugation (which preserves the polynomial $f$). Indeed an isomorphism between $X$ and $\mathbb{C}^\times$ is given by $(x,y) \mapsto z = ix+y$, so that the action of complex conjugation becomes $z \mapsto -1/\bar{z}$ (which has no fixed points, reflecting the fact that there are no real solutions to $f(x,y) = 0$). If you instead consider the locus $Y = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 2\}$, then again the complex solutions are a copy of $\mathbb{C}^\times$, but now the involution is $z \mapsto 1/\bar{z}$, so that the fixed points are exactly the circle. Note that the involution is *not* a morphism of complex varieties. It is remarkable feature of finite fields that we will be able to find an analog of this involution which if a morphism of varieties.

In the context of $\mathbb{R}$ the lesson that that one should study the complex solutions rather than just the real solutions is easily learnt from, for example, trying to classify conics.

Formally, we have the following definition:

**Definition 1.2.** An affine variety $X$ over k is defined over $\mathbb{F}_q$ if there an $\mathbb{F}_q$-algebra $A_0$ such that $A \cong A_0 \otimes_{\mathbb{F}_q} \mathsf{k}$. Equivalently, $X$ is defined over $\mathbb{F}_q$ if it is the pullback via the map Spec(k) $\to$ Spec($\mathbb{F}_q$) of an affine variety over Spec($\mathbb{F}_q$).

Our approach, however, will be to view the $\mathbb{F}_q$-points of the variety as fixed points of an endomorphism on the more "geometric" structure of the k-points of

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the variety , that is, we wish to study varieties defined over $\mathbb{F}_q$ "by descent". Consider first the case of affine $n$-space over $\mathsf{k}$. There is a *standard Frobenius* map on the $\mathsf{k}$-points denoted $Fr\colon \mathsf{k}^n \to \mathsf{k}^n$, given by

$$(x_1, x_2, \dots, x_n) \mapsto (x_1^q, x_2^q, \dots, x_n^q).$$

Now $Fr$ is clearly a bijective morphism of $\mathsf{k}^n$ to itself, which however is *not* an isomorphism (the inverse uses the map $\lambda \mapsto \lambda^{1/q}$ which is not algebraic). Now if $V$ is an algebraic subset of $\mathsf{k}^n$, defined by the vanishing of polynomials whose coefficients lie in $\mathbb{F}_q$, clearly $V$ will be preserved by $Fr$, so there is an induced Frobenius map on $V$. Moreover, the fixed points of $Fr$ inside $V$, denoted $V^{Fr}$, are a finite set – they are precisely the intersection $V \cap \mathbb{F}_q^n$ – giving exactly the $\mathbb{F}_q$-points of the variety $V$.

We want, however, a more intrinsic way of describing the action of Frobenius map on an affine variety. Consider first the map $Fr^*$ induced by $Fr$ on the coordinate algebra. Clearly we have, for $A = \mathsf{k}[\mathbb{A}^n]$ we have

$$Fr^*(f)(X_1, X_2, \dots, X_n) = f(X_1^q, X_2^q, \dots, X_n^q)$$

so the pullback $Fr^*$ gives an algebra homomorphism between $A$ and its subalgebra of $q$-th powers $A^q$. Moreover, since any element of $\mathsf{k}$ lies in a finite extension of $\mathbb{F}_q$, it follows that given any polynomial $g$ in $A$, all of its coefficients lie in some finite extension of $\mathbb{F}_q$, and so for some $m$ we have $(Fr^*)^m(f) = f^{q^m}$.

In the general situation of an affine variety $X$ obtained from an $\mathbb{F}_q$-variety $X_0$, where $\mathsf{k}[X] = A$ and $\mathbb{F}_q[X_0] = A_0$ so that $A = A_0 \otimes_{\mathbb{F}_q} \mathsf{k}$, we get a corresponding homomorphism given by $a \otimes \lambda \mapsto a^q \otimes \lambda$. This suggests the following definition.

**Definition 1.3.** Let $A$ be the coordinate algebra of an affine variety $X$ defined over $\mathsf{k}$. A (*geometric*) *Frobenius* morphism is a morphism of $\mathsf{k}$-varieties $F\colon X \to X$ such that the pullback morphism $F^*\colon A \to A$ satisfies

(1) $F^*$ is injective and its image is precisely $A^q$.
(2) For any $f \in A$ there is an $m > 0$ such that $(F^*)^m(f) = f^{q^m}$

We call such a map a "Frobenius morphism".

Our first aim is to show that an affine variety equipped with a Frobenius morphism is in fact defined over $\mathbb{F}_q$. Equivalently, if $A$ is the coordinate algebra of an affine variety $X$ and $F\colon X \to X$ is as in the definition, we need to find an $\mathbb{F}_q$-subalgebra of $A$ from which all of $A$ can be obtained by base change. Note that by the first criterion for $F^*$ there is a well-defined map $\sigma\colon \mathbb{A} \to A$ given by $\sigma(f) = (F^*)^{-1}(f^q)$. Then it follows that $\sigma\colon A \to A$ is a bijective ring homomorphism such that $\sigma(\lambda f) = \lambda^q \sigma(f)$ for $\lambda \in \mathsf{k}$, $f \in A$. Moreover, by the second criterion it is clear that for each $f \in A$ there is some $m > 0$ such that $\sigma^m(f) = f$. The map $\sigma$ is called the *arithmetic Frobenius*, and if we have $A \cong A_0 \otimes_{\mathbb{F}_q} \mathsf{k}$ it is given simply by $a \otimes \lambda \mapsto a \otimes \lambda^q$. (It does *not* correspond to a morphism of $\mathsf{k}$-varieties). Then we have the following proposition:

**Proposition 1.4.** *Let $X$ be an affine variety with Frobenius morphism $F\colon X \to X$, and let $\sigma\colon \mathsf{k}[X] \to \mathsf{k}[X]$ be the associated arithmetic Frobenius. Then the fixed points of $\sigma$ on $\mathsf{k}[X]$ form a finitely generated $\mathbb{F}_q$-subalgebra $A^\sigma$ such that the canonical map*

$$A^\sigma \otimes_{\mathbb{F}_q} \mathsf{k} \to \mathsf{k}[X]$$

*is an isomorphism. Thus $X$ is a variety defined over $\mathbb{F}_q$.*

*Proof.* Set $A = \mathsf{k}[X]$. First note that the map $\sigma$ is locally finite a semi-linear map $A \to A$. Fix $f \in A$ and let $S$ be the $\mathsf{k}$-span of the functions $\sigma^k(f)$ for $k \geq 0$. Then by the paragraph preceeding the Proposition, there is some $m > 0$ with $\sigma^m(f) = f$, and hence the set $\{\sigma^k(f)\}_{k \in \mathbb{Z}_{\geq 0}}$ is actually finite. We claim moreover that $S$ is spanned by $S^\sigma$, the functions in $S$ fixed by $\sigma$.

Fix $\lambda_0, \lambda_2, \ldots, \lambda_{m-1} \in \mathbb{F}_{q^m}$ to be a basis of $\mathbb{F}_{q^m}$ as an $\mathbb{F}_q$-vector space. Then define

$$g_i = \sum_{k=0}^{m-1} \sigma^k(\lambda_i f) = \sum_{k=0}^{m-1} \lambda_i^{q^k} \sigma^k(f), \quad (0 \leq i \leq m-1).$$

Then each of the $g_i$ is fixed by $\sigma$: indeed using the fact that $\sigma^m(f) = f$, we see that

$$\begin{aligned} \sigma(g_i) &= \sum_{k=0}^{m-1} \sigma^{k+1}(\lambda_i f) \\ &= \sum_{k=1}^{m-1} \sigma^k(\lambda_i f) + \sigma^m(\lambda_i f) \\ &= g_i. \end{aligned}$$

since $\lambda_i \in \mathbb{F}_{q^m}$. But now since $\sigma^k(\lambda_i f) = \lambda_i^{q^k} \sigma(f)$, and the matrix $(\lambda_i^{q^j})_{0 \leq i \leq m-1}$ is invertible since the $\lambda_i$ are a basis for $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, it follows that $f$ lies in the span of the $\{g_i\}_{0 \leq i \leq m-1}$ as claimed. This shows that the canonical map $A^\sigma \otimes_{\mathbb{F}_q} \mathsf{k} \to \mathsf{k}[X]$ is surjective. A similar, but easier argument shows that it is injective, and hence an isomorphism. Applying the argument to a finite generating set $S$ for $A$, we may find a new generating set $T$ for $A$ contained in $A^\sigma$. It is easy to check that $T$ is a finite generating set for $A^\sigma$, so that $A^\sigma$ is also a finitely generated $\mathbb{F}_q$-algebra. □

Thus we see that a $\mathsf{k}$-variety equipped with a Frobenius morphism is the same as a variety defined over $\mathbb{F}_q$. Notice also that this description of a variety defined over a finite field, has the consequence that if we "twist" the Frobenius by an automorphism of finite order, then we obtain a new Frobenius, and consequently a new variety over $\mathbb{F}_q$. This is often very useful.

**Example 1.5.** Let $G = \mathrm{GL}_n(\mathsf{k})$, and let $F_n \colon G \to G$ be given by $(a_{ij}) \mapsto (a_{ij}^q)$. Then $G^{F_n} = \mathrm{GL}_n(\mathbb{F}_q)$. On the other hand, if we let $\rho \colon G \to G$ be the map $A \mapsto A^{-t}$, then $\rho$ commutes with $F_n$, so that $F_\rho = \rho \circ F_n$ is again a Frobenius morphism, and we get a new algebraic group defined over $\mathbb{F}_q$, the *unitary group*: we have $G^{F_\rho} = \mathrm{U}_n(\mathbb{F}_q) = \{g \in \mathrm{GL}_n(\mathbb{F}_q) : F(g)^t = g^{-1}\}$.

It is not difficult to show that given an affine variety $X$ over $\mathbb{F}_q$ we can find an embedding $\iota \colon X \to \mathbb{A}_n$ such that the Frobenius morphism on $X$ is the restriction of the standard Frobenius on $\mathbb{A}_n$. In fact more is true:

**Lemma 1.6.** *Let $G$ be an affine algebraic group defined over $\mathbb{F}_q$, with Frobenius morphism $F$. Then there is an embedding of algebraic groups $\varphi \colon G \to GL_n(\mathsf{k})$ such if $F_n$ denotes the standard Frobenius on $GL_n(\mathsf{k})$ sending $(a_{ij}) \mapsto (a_{ij}^q)$, then diagram*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & GL_n(\mathsf{k}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_n} \\ X & \xrightarrow{\varphi} & GL_n(\mathsf{k}) \end{array}$$

*commutes.*

*Proof.* This can be proved in the same fashion as one shows that any affine algebraic group can be embedded in $\mathrm{GL}_n(\mathsf{k})$. One just needs to show that it is possible to find a $G$-stable subspace $W$ of $\mathsf{k}[G]$ such that $W^\sigma$ generates $\mathsf{k}[G]$. □

Note that a consequence of this lemma is that $G^F$ is a finite set. Similarly, since any element of $\mathrm{GL}_n(\mathsf{k})$ clearly lies in $\mathrm{GL}_n$ of some finite field, every element of $G$ has finite order. The most important basic result in the study of affine algebraic groups defined over a finite field is the *Lang-Steinberg* theorem. We state it for a *generalized Frobenius morphism*, that is, an endomorphism $F\colon G \to G$ of the group, some power of which is a Frobenius morphism.

**Theorem 1.7.** *(Lang-Steinberg Theorem): Let G be a connected affine algebraic group over* $\mathsf{k}$*, and let F be a generalized Frobenius map. Then the map* $\mathcal{L}\colon G \to G$ *given by* $g \mapsto g^{-1}F(g)$ *is surjective.*

*Proof.* (P.Muller, see [M]). Note that $G$ acts on itself on the right via $g.x = g^1 xF(g)$. Moreover, we know that there is a closed orbit for this action, so we may pick $x \in G$ such that $G.x$ is closed in $G$. If we can show that $\dim(G.x) = \dim(G)$, it will follow that there is exactly one orbit, and hence for *any*, $h \in G$, the orbit of $1$ is the same as the orbit of $h$, and so there is some $g \in G$ with $g^{-1}F(g) = h$ as required.

Thus it remains to show that $\dim(G.x) = \dim(G)$. Considering the morphism $G \to G$ given by $g \mapsto g.x$ we see that it is enough to show that the equation $g^{-1}xF(g) = x$ has finitely many solutions. Let $f\colon G \to G$ be given by $f(g) = xF(g)x^{-1} = Ad_x \circ F(g)$. Let $m$ be a positive integer such that $F^m$ is a standard Frobenius morphism and $F^m(x) = x$. Let $r$ be the order of $xF(x)F^2(x)\dots F^{m-1}(x)$. Then it is easy to see that $f^{mr}(g) = F^{mr}(g)$ for any $g \in G$. But then $f^{mr}(g) = g$ has only finitely many solutions, and so certainly $f(g) = g$ has only finitely many solutions as required. □

*Remark* 1.8. In fact, it can be shown that $\mathcal{L}$ is a finite map. Note that the fibers of the map $\mathcal{L}$ are all $G^F$, so that $\mathcal{L}$ exhibits $G$ as a "$G^F$-fibration" over itself: $\mathcal{L}$ gives an inseparable isogeny between $G/G^F$ and $G$ so that they are isomorphic for the etale topology. This fact is used by Quillen to compute the group cohomology of $G^F$ (see [Q]).

*Remark* 1.9. Given a connected reductive algebraic group over $\mathsf{k}$ and a generalized Frobenius morphism $F\colon G \to G$, by considering the fixed points of $F$, we obtain a finite group, $G^F$. Such groups are called "finite groups of Lie type" or sometimes, "finite reductive groups". In the case where $G$ is the adjoint form of a simple affine algebraic group (*i.e.* the form with trivial center), the groups $G^F$ are finite simple groups (except perhaps for some very small fields (*e.g.* $\mathrm{PSL}_2(\mathbb{F}_3)$ is isomorphic to the alternating group $\mathrm{Alt}_4$). In fact, there are very few generalized Frobenius morphisms which are not already Frobenius morphisms (at least on a simple affine algebraic group).

The theorem is extremely useful in showing that a variety defined over $\mathbb{F}_q$ possesses an $\mathbb{F}_q$-point.

**Corollary 1.10.** *Let* $X$ *be an algebraic variety defined over* $\mathbb{F}_q$*, and let* $G$ *be a connected affine algebraic group defined over* $\mathbb{F}_q$ *acting on* $X$ *by an action defined over* $\mathbb{F}_q$*.*

(1) *If $G$ is connected, any $F$-stable $G$-orbit has a rational point.*
(2) *If $\mathcal{O}$ is a rational orbit with a rational point whose stabilizer $H$ is connected, then the set $\mathcal{O}^F$ of rational points of $\mathcal{O}$ forms a single $G^F$-orbit. Thus if $H$ is a closed connected subgroup of a group $G$ then $(G/H)^F = G^F/H^F$.*

*Proof.* Let $x \in X$ be a point of the rational orbit. Then $F(x) = h.x$ for some $g \in G$. Taking $g \in G$ with $g^{-1}F(g) = h^{-1}$, we immediately see that

$$F(g.x) = F(g)F(x) = F(g)h.x = g.x$$

and so $g.x$ is a rational point as required.

For the second part, if $y$ is a rational point in the orbit $\mathcal{O}$ with connected stablizer, and $z$ is another rational point, then we have $y = g.z$ for some $g \in G$. Then we see that $g^{-1}F(g)$ lies in the stabilizer of $y$. Thus by Lang's theorem applied to the stabilizer, we find an $h$ in the stabilizer such that $h^{-1}F(h) = g^{-1}F(g)$, so that $gh{-}1 \in G^F$, and $z = gh^{-1}y$ as required. The statement about cosets follows immediately. □

**Lemma 1.11.** *Let $G$ be a connected affine algebraic group and let $F$ be a Frobenius morphism. Then $G$ contains an $F$-stable Borel subgroup (this is sometimes expressed by saying that $G$ is "quasi-split"). Moreover, any rational Borel contains a rational torus.*

*Proof.* This follows because the variety of all Borel subgroups is a single $G$-orbit, and similarly the maximal tori in a Borel subgroup are a single orbit of the Borel (since a Borel subgroup is connected).

□

If $T$ is a maximal torus of $G$ defined over $\mathbb{F}_q$, then the Frobenius acts on the Weyl group of $T$, and since $T$ is connected we have $W^F = N_G(T)^F/T^F$. Finally, if $T \subset B$ is pair consisting of a rational torus in a rational Borel, we may construct a $BN$-pair for $G^F$. To prove this we use the following lemma on the number of rational points in a connected unipotent group.

**Lemma 1.12.** *If $U$ is a connected unipotent group with a Frobenius morphism $F\colon U \to U$, then the number of $\mathbb{F}_q$-points of $U$ is $q^{\dim(U)}$.*

*Proof.* One can show this by induction on the dimension of the group, using that one can embed $U$ into the group of upper triangular matrices of $GL_n(\mathsf{k})$ in such a way that the Frobenius on $U$ is intertwined with the standard Frobenius on $GL_n(\mathsf{k})$.

It also follows from the general fact that if $F$ is any Frobenius on an affine space of dimension $n$, then $F$ has $q^n$ fixed points. This is easy to see if you know the basic properties of etale cohomology. □

**Proposition 1.13.** *The pair $B^F$ and $N^F = N_G(T)^F$ form a $BN$-pair for $G^F$, with Weyl group $N_G(T)^F/T^F$.*

*Proof.* First observe that $F$ preserves $U$, $T = B \cap N$ and $N$, and that each coset $Tn$ in $N/T$ which is preserved by $F$ contains a rational point. Since we are assuming that $B$ and $T$ are defined over $\mathbb{F}_q$ all of these are clear by the previous lemma except the fact that $F(U) = U$. But this follows since $F(B) = B$, and $U$ can be characterized as the set of unipotent elements in $B$. Note that it follows that $\bar{F}$ induces a map on $W = N/T$, which is a permutation since $F(N) = N$, and so $\bar{F}\colon W \to W$ is surjective. Thus $F$ induces a permutation of the Bruhat decomposition.

The Bruhat decomposition for $G^F$ above certainly shows that $G^F$ is generated by $B^F$ and $N^F$. Since the simple reflections can be characterized as the elements $s$ of $W$ for which $B \sqcup BsB$ is a subgroup of $G$, it follows that $\bar{F}$ perserves $S$. Let $\mathcal{J}$ be the set of $\bar{F}$-orbits on $S$, and for each $J \in \mathcal{J}$ let $W_J$ be the subgroup generated by the $J$m. Then if $w_J$ be the longest element in $W_J$, $w_J$ is an element of order 2, and $W^{\bar{F}}$ is the subgroup of $W$ generated by $W$.

Clearly we have $G^F = \bigsqcup_{w \in W^{\bar{F}}} (B\dot{w}B)^F$, and by the previous lemma, we may assume that the representatives $\dot{w}$ are in $G^F$. We now use the sharp form of the Bruhat decomposition: if $g \in G^F$ we may write $g$ uniquely as $g = ut\dot{w}v$ where $u \in U$ and $v \in U_w$. Since $w \in W^F$, uniqueness forces each of $u, t, v$ to be in $G^F$, and we find that $\bigsqcup_{w \in W^F} B^F \dot{w} B^F$. Thus certainly $B^F$ and $N^F$ generate $G$. We have also already shown that $W^F = N^F/T^F$ is generated by elements of order 2, and for $J \in \mathcal{J}$, the inclusions $w_J B^F w_J \subset B^F w B^F \cup B w_J w B^F$ follow from

$$Bw_J B.BwB \subset \bigcup_{v \in W_J} BvwB,$$

by taking $F$-fixed points, since for $v \in W_J$ with $v \neq w_J$ we have $vw \notin W^F$ and so $(BvwB)^F = \emptyset$. The last property of a $BN$-pair is the most subtle: to see that $w_J B^F w_J \not\subset B^F$, we note that $w_J U_{w_J} w_J \not\subset B$, and apply Lemma 1.12 to see that it contains a rational point. □

The following Corollary is useful in that it gives an explicit formula for the order of the finite group $G^F$.

**Corollary 1.14.** *Let $G$ be a reductive algebraic group defined over $\mathbb{F}_q$, and let $B$ be a rational Borel subgroup containing $T$ a rational Borel. Then*

$$G^F = \bigsqcup_{w \in W^F} U^F T^F \dot{w} U_w^F.$$

*and hence $|G^F| = q^{|\Phi^+|} |T^F| \sum_{w \in W^F} q^{l(w)}$.*

*Proof.* The first part of the lemma is just the refined form of the Bruhat decomposition in the previous Proposition. To prove the formula about the order of $G^F$ we use Lemma 1.12. □

Now fix a pair $T \subset B$ consisting of a rational torus in a rational Borel, and consider the root data we attach to this pair. If $U_\alpha$ is a connected one-dimensional unipotent subgroup normalized by $T$, then $F(U_\alpha)$ is again normalized by $T$, and clearly a connected one-dimensional unipotent subgroup. Conversely, the identity component of the preimage of $U_\alpha$ under $F$ must also be such a subgroup, so that $F$ induces a permutation $\tau$ on the roots of $G$. Moreover, it is clear that $F$ induces a morphism on $X(T)$ and $Y(T)$, and one can check that such that $F^*(\alpha) = q\tau(\alpha)$ for $\alpha \in \Phi \subset X(T)$. The root datum of $G$ equipped with the action of $\tau$ is enough to characterize the group and its rational structure.

**Theorem 1.15.** *Let $G$ be a connected reductive algebraic group, and let $F$ be a Frobenius morphism on $G$. Then the datum $(X(T), Y(T), \Phi, \check{\Phi})$ along with the automorphism $\tau$ on $X(T)$ which permutes the roots, characterizes the pair $(G, F)$ up to isomorphism. Moreover, for any root datum, any $q$ a prime power, and an automorphism $\tau$ of $X(T)$ such that $\tau$ permutes the roots and $\tau^*$ permutes the coroots, there is a pair $(G, F)$ realizing this datum.*

*Proof.* To prove this is not much more elaborate than the proof of the classification of reductive groups: the Frobenius is an isogeny of $G$, and there is a generalization of the isogeny theorem (prove in [St99]) which classifies all the possibilities. To show that any possibilty actually occurs is straightforward given the existence theorem. □

**Example 1.16.** Not every isogeny of $G$ arises as a Frobenius morphism, although those that do not are very rare. Let $G = \mathrm{Sp}_4(\mathsf{k})$ where $\mathsf{k}$ is the algebraic closure of $\mathbb{F}_2$. Then if $q$ is an odd power of 2, say $q = 2^{2e+1}$, there is an isogeny $F$ of $G$ whose square is the Frobenius map associated to $q$. If $T$ is a maximal torus one can characterize the isogeny $F$ as follows. Let $\alpha$ and $\beta$ be the short and long root respectively. Then $F$ exchanges the root subgroups corresponding to $\alpha$ and $\beta$ respectively, so that if $\eta_\alpha \colon \mathbb{G}_a \to G$ parametrizes $U_\alpha$ and $\eta_\beta \colon \mathbb{G}_a \to G$ parametrizes $U_\beta$, we have

$$F(\eta_\alpha(\lambda)) = \eta_\beta(\lambda^{2^{e+1}}); \qquad F(\eta_\beta(\lambda)) = \eta_\alpha(\lambda^{2^e}).$$

The fixed points of $G$ under $F$ are exactly the groups discovered by Suzuki. See [St68], [St99] for more information about these "exceptional isogenies". They give rise to the all of the Ree and Suzuki groups.

Since we have established that $G^F$ has a $BN$-pair, I would like to mention one of the motivating applications of the notion of a $BN$-pair: it was noticed quite early that reductive groups defined over a not necessarily algebraically closed field $k$, such as the special linear groups $\mathrm{SL}_n(k)$ often possess a "large" normal subgroup which is simple modulo its center. Tits introduced the notion of a $BN$-pair to provide a unified framework in which to study this phenomenon. For groups which he called "isotropic", meaning that they have proper parabolic subgroups defined over $k$, he obtained a general answer in [T], using the theory of $BN$-pairs. He first shows $G(k)$ has a $BN$-pair whose construction we now sketch. Let $\mathsf{k}$ be a fixed algebraic closure of the field $k$, and $G$ the algebraic group with $k$-points $G(k)$.

Let $T$ be a maximal *split* torus in $G$, and let $\tilde{T}$ be a maximal torus of $G$ containing $T$. Thus we have $\Phi$ the roots of $G$ with repect to $\tilde{T}$. Let $\Sigma$ be the restriction of the roots $\Phi$ to $T$, and for each $\alpha \in \Sigma$ let $U_{(\alpha)}$ be the group generated by the $U_{\tilde{\alpha}}$ where $\tilde{\alpha} \in \Phi$ restricts to $\alpha$ (these subgroups turn out to be defined over $k$). Then making a choice of positive roots in $\Phi$, let $U^+$ be the subgroup generated by all the $U_{(\alpha)}$ for $\alpha \in \Sigma^+$. Let $H = C_G(T)$ and $N = N_G(T)$. One then sets $B^+ = H.U^+$ where $U^+$ is the subgroup generated by the $U_{(\alpha)}$ for $\alpha \in \Sigma$ positive.

**Theorem 1.17.** [T] *The group $G(k)$ has a $BN$-pair given by $B^+(k)$ and $N(k)$.*

Tits goes on to show that if we let $G(k)^0$ be the subgroup of $G(k)$ generated by the conjugates of $U^+(k)$ then any normal subgroup of $G(k)$ is either central or contains $G(k)^0$, so that in particular $G(k)^0$ modulo its center is a simple group. In [St68], Steinberg gives a beautiful unified construction of finite groups of Lie type as the fixed points of an endomorphism of a simple algebraic group over $\bar{\mathbb{F}}_q$ (thus including the exceptional isogenies), so that Tits technique may be applied to give a unified proof of the simplicity of these groups.

The non-trivial automorphisms all arise from diagram automorphisms for the associated Dynkin diagram – the case of $SL_n$ corresponds to a type $A$ diagram and its unique nontrivial automorphism corresponds to the rational structure associated to the unitary groups described above. The complete list of root systems with automorphisms is:

$$^2A_n, {}^2D_n, {}^3D_4, {}^2E_6.$$

We now address the question of conjugacy in the group $G^F$: We will only address the question of $G^F$-conjugacy classes of rational tori in $G$.

Fix a rational maximal torus $T_0 \subset B_0$, where $B_0$ is also defined over $\mathbb{F}_q$. Let $n \in N_G(T_0)$. Then by the Lang-Steinberg theorem, we can find a $g \in G$ such that $g^{-1}F(g) = n$. But then we see that

$$F(gT_0g^{-1}) = F(g)T_0F(g)^{-1} = gnT_0n^{-1}g^{-1} = gT_0g^{-1},$$

and so $gT_0g^{-1}$ is also a rational torus. However, $g$ is not determined by $n$, indeed if $\gamma \in G^F$, then $\mathcal{L}(\gamma g) = \mathcal{L}(g) = n$. Thus in fact $gT_0g^{-1}$ is determined up to $G^F$-conjugacy. Hence we obtain a map from $W = N_G(T_0)/T_0$ to the set of $G^F$-conjugacy classes of rational tori of $G$.

On the other hand, suppose that $T$ is some rational torus, then since maximal tori are conjugate, it follows that $T = gT_0g^{-1}$ for some $g \in G$. Since $F(T) = T$ it follows that $F(g)T_0F(g)^{-1} = gT_0g^{-1}$ and so $g^{-1}F(g) \in N_G(T_0)$. But again, $g$ is not unique – we could replace it with $gn$ for any $n \in N_G(T_0)$, yielding $n^{-1}gF(g)F(n) \in N_G(T_0)$, thus we actually obtain a map from rational maximal tori to $N_G(T_0)$ modulo the equivalence relation $g\tilde{g}'$ if there is an $n \in N_G(T_0)$ such that $g' = n^{-1}gF(n)$ (the orbits of a right action of $N_G(T_0)$ on itself – these orbits are called the $F$-twisted conjugacy classes of $N_G(T_0)$).

**Proposition 1.18.** *The above maps induce a bijection between the $G^F$-conjugacy classes of rational tori in $G$, and the $F$-twisted conjugacy classes of the Weyl group $W = W(T_0)$.*

*Proof.* By the above this we need only show that there is a bijection between the $F$-twisted conjugacy classes of $N_G(T_0)$ and the $F$-twisted conjugacy classes of $W$. The obvious quotient map is certainly surjective, so we need only show injectivity. To see this, it is enough to show that if $n_1, n_2 \in N_G(T_0)$, and $n_2 = n^{-1}n_1F(n)t$ for some $n \in N_G(T_0)$, $t \in T_0$, then there is an $m \in N_G(T_0)$ such that $n_2 = m^{-1}n_1F(m)$.

We wish to solve $m^{-1}n_1F(m) = n^{-1}n_1F(n)t$. In fact we show there is a solution of the form $m = t_1n$ for some $t_1 \in T_0$. Indeed $m^{-1}n_1F(m) = n^{-1}n_1F(n)t$ then becomes

$$n^{-1}t_1^{-1}n_1F(t_1)F(n) = n^{-1}n_1F(n)t$$

hence we must solve

$$t_1^{-1}n_1F(t_1)n_1^{-1} = n_1F(n)tF(n)n_1^{-1} = t_2 \in T_0,$$

for $t_1 \in T_0$. But if we set $F' \colon T_0 \to T_0$ to be given by $s \mapsto n_1F(s)n_1^{-1}$, then if we can show that $F'$ is a Frobenius morphism from $T_0$, we are done by the Lang-Steinberg theorem. However, let $g \in G$ be such that $g^{-1}F(g) = n_1$, and let $T = gT_0g^{-1}$ be the corresponding rational structure. Then considering the diagram

$$\begin{array}{ccc} T_0 & \xrightarrow{Ad_g} & T \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ T_0 & \xrightarrow{Ad_g} & T \end{array}$$

it is clear that $F'$ is just the pullback of the Frobenius structure $F$ on $T$ via the isomorphism given by conjugation by $g$. Hence we are done. $\square$

In the next section we will begin the investigation of the representation theory of finite groups of Lie type. In this course we will focus on the character theory for such groups, thus our objective will be to describe the characters of finite groups of Lie type as class functions on the group, and not to construct explicit representations of these groups. The story of the character theory of these groups is older than that of the representation theory – already in $1955$ Green determined explicitly the character table of $\mathrm{GL}_n(\mathbb{F}_q)$, but construction of representations comes later with [L], [DL].

There are two fundamental ideas in the study of the character theory of finite reductive groups: that of cuspidal forms, as introduced by Harish-Chandra, and that of unipotent representations, present already in Green's work, but made explicit in the work of Deligne-Lusztig [DL] and Springer [S]. The first of these ideas is easier to describe. The essential observation is that one should study reductive groups as a family, and seek to understand a group using groups of smaller rank, which one assumes understood by some kind of induction. In order to make this more precise, we need to select a good class of reductive subgroups of a reductive group $G$, and a suitable notion of induction for representations of these subgroups. For this we need to discuss parabolic subgroups in more detail.

Recall that if $B$ is a Borel containing a maximal torus $T$, then the parabolic subgroups containing $B$ are indexed by subsets of the simple roots. Let $I$ be such a subset, and let $\Phi_I \subset \Phi$ be the set of roots lying in the span of the simple roots in $I$, and similarly let $\Phi_I^+$ be the positive roots lying in this span.

**Proposition 1.19.** *Let $I \subset S$ and $P_I$ the corresponding parabolic subgroup of $G$.*

(1) *If $\alpha, \beta \in \Phi_I^+$, then $m\alpha + n\beta \in \Phi_I^+$ for positive integers $m, n$ if and only if $m\alpha + n\beta \in \Phi$.*
(2) *The unipotent radical of $P_I$ is $V_I = \prod_{\alpha \in \Phi^+ \setminus \Phi_I^+} U_\alpha$.*

*Proof.* The first part follows immediately from the fact that a positive root is a positive linear combination of simple roots. For the second part, it follows from part $(1)$ and Theorem **??** $(iv)$ that $V_I$ is a closed unipotent subgroup of $P_I$ normalized by $U$. Clearly it is normalized by $T$, our fixed maximal torus (which is maximal in $G$ and hence also in $P_I$). Thus to show that $V_I$ is normalized by $P_I$ it is enough to show that it is normalized by $W_I$, the Weyl group associated to $I$. But since $W_I$ is generated by simple reflections, we need only show that if $\alpha \in \Phi - \Phi_I^+$ and $s_\beta \in I$, then $s_\beta(\alpha) \in \Phi^+ - \Phi_I^+$. But since $l(s_\beta) = 1$ and $\beta \neq \alpha$, it follows that $s_\alpha(\beta) \in \Phi^+$, so that again by part $(1)$ we must have $s_\beta(\alpha) \in \Phi_I^+$. Thus $V_I$ is contained in $R_u(P_I)$.

Now the unipotent radical of $P_I$ is is normalized by $T$, and equals the product of the root subgroups it contains. Moreover it lies in every Borel subgroup of $P_I$. Thus there is a set of roots $\Psi \subset \Phi^+$ such that $R_u(P_I) = \prod_{\alpha \in \Psi} U_\alpha$, and since $V_I$ is a subgroup of $R_u(G)$ it follows that $\Phi^+ - \Phi_I^+ \subseteq \Psi$. But now if $\alpha \in \Phi_I$, we have $s_\alpha \in W_I$ so that any lift of $s_\alpha$ lies in $P_I$. But then since $U_{-\alpha} = s_\alpha U_\alpha s_\alpha = \nsubseteq B$, we see that $U_\alpha \nsubseteq s_\alpha B s_\alpha$ where $s_\alpha B s_\alpha$ is a Borel subgroup of $P_I$. Since $R_u(P_I)$ lies in every Borel subgroup of $P_I$, it follows that $\alpha \notin \Psi$, and hence $\Psi = \Phi^+ - \Phi_I^+$ as required. ∎

**Definition 1.20.** (*Borel-Tits*:) Let $G$ be an affine algebraic group, and $\pi \colon G \to G/R_u(G)$ the quotient map. If $L$ is a closed subgroup of $G$ such that $\pi$ induces an isomorphism $L \cong G/R_u(G)$ we say that $L$ is a *Levi subgroup* of $G$. In this case, since $L \cap R_u(G) = \{e\}$ it follows that $G = L \ltimes R_u(G)$ is a semi-direct product. Such an decomposition is called a Levi decomposition of $G$.

In general it is not necessarily the case that an algebraic group has a Levi decomposition. In characteristic zero the existence of Levi subgroups is a theorem of Mostow [Mo], but in characteristic $p$, the existence is not guaranteed. However, for parabolic subgroups of reductive groups, there is always a Levi decomposition.

**Example 1.21.** Consider $H$ a maximal closed unipotent subgroup of $\mathrm{GL}_3(\mathsf{k})$, so that $H$ is conjugate to the group of upper triangular matrices with 1s on the diagonal. Then there is an exact sequence

$$1 \to V \to \mathrm{Aut}(H) \to \mathrm{SL}_2 \to 1,$$

where $V = H/[H, H]$. This sequence does not split if $\mathrm{char}(\mathsf{k}) = 2$. This gives an example of an affine algebraic group without Levi decomposition.

**Lemma 1.22.** *Let $S$ be a torus in $G$. Then $L = Z_G(S)^0$ is a connected reductive group, and the Borel subgroups of $L$ are all of the form $B \cap L$ where $B$ is a Borel subgroup of $G$.*

*Proof.* (*sketch*:) We have already seen in Theorem **??** that $L$ is connected and reductive. To see that for any Borel $B$ of $G$, the group $B \cap L$ is a Borel subgroup of $L$, one first shows that $B \cap L$ is connected and solvable, and then proves that $L/B \cap L$ is complete. □

**Definition 1.23.** Let $I$ be a subset of $S$, and let $S_I = (\bigcap_{i \in I} \ker(\alpha_i))^0$, a subtorus of $T$. We define $L_I$ to be the centralizer of $S_I$ in $G$, $L_I = Z_G(S_I)$, a connected reductive algebraic group.

**Proposition 1.24.** *The group $L_I$ generated by the $U_\alpha$ for $\alpha \in \Phi_I$ is a reductive subgroup of $G$, with root datum given by $(X(T), Y(T), \Phi_I, \check{\Phi}_I)$. Moreover there is a semidirect product decomposition $P_I = R_u(P_I) \rtimes L_I$. Moreover $L_I$ is the unique closed subgroup containing $T$ which has the property that the quotient map induces an isomorphism $L_I \cong P_I/V_I$.*

*Proof.* Since $L_I$ contains $T$, and is reductive, hence we just need to determine the root datum of $L_I$ in $(X(T), Y(T))$. This is most easily done using the Lie algebra of the group since it is can be shown that the root subgroups of a reductive group are given by the decomposition of the adjoint representation with respect to $T$. One shows that the Lie algebra of $Z_G(S_I)$ is the subspace of $\mathrm{Lie}(G)$ which is fixed by the adjoint action of $S_I$. It then follows easily that $L_I$ has root system $\Phi_I$ as claimed.

Now $L_I$ normalizes the group $V_I$ (the argument is the same as the one used to show that $P_I$ normalized $V_I$), so that the product $P = L_I V_I$ is a subgroup of $G$. Since $V_I \cap L_I$ is a unipotent normal subgroup of $L_I$, and $L_I$ is reductive, it follows that $V_I \cap L_I$ is trivial, and hence $P = L_I \ltimes V_I$. But since $P$ contains $B$, it must be a parabolic subgroup, and using the Bruhat decomposition it is easy to see that $P = P_I$, showing that $P_I$ has a Levi decomposition.

To see uniqueness, note that if $L$ is any other such group, then $L$ is reductive, since $P_I/V_I$ is, and hence it is generated by $T$ and the $U_\alpha$ it contains. But now for $\alpha \in \Phi^+ - \Phi^+$ we have $U_\alpha \in V_I$, so that $U_\alpha \notin L$. Hence clearly $L$ is a subgroup of $L_I$, and so since $L \cong P_I/V_I \cong L_I$ we must have $L = L_I$. □

**Corollary 1.25.** *Let $I \subset S$. Two Levi subgroups of the parabolic subgroup $P_I$ are conjugate by an element of $R_u(P_I)$.*

*Proof.* Let $L$ and $L'$ be two Levi subgroups, and let $T$ and $T'$ be maximal tori of $P_I$ containing $L$ and $L'$ respectively. Then we know that $T$ and $T'$ are conjugate, hence there is some $g \in P_I$ with $gTg^{-1} = T'$. By uniqueness, we must have $gLg^{-1} = L'$ also. Writing $g = l.u$ for $l \in L$ and $u \in R_u(P_I)$ we see that $uLu^{-1} = L'$ as required. □

When dealing with finite reductive groups, we must (at least at first sight) restrict our attention to rational parabolic subgroups. However, the rationality questions here are straight-forward. Rational parabolics correspond exactly to the "parabolic" subgroups of $W^F$, and by the conjugacy of Levi subgroups just established and Corollary 1.10 any rational parabolic $P$ contains rational Levi subgroups, all of which are conjugate by $R_u(P)^F$.

*Remark* 1.26. Given a reductive group $G$, we say that $L$ is a Levi subgroup of $G$ if $L$ is a Levi subgroup of a parabolic subgroup of $G$. Since it is clear that the only Levi subgroup of a reductive group $G$ in the sense of Defintion 1.20 is $G$ itself, this ambiguity of terminology is usually not problematic. It is not hard to show that any Levi subgroup of $G$ is the centralizer of a torus in $G$, and conversely, thus the class of Levi subgroups of $G$ is easy to describe intrinsically. We wish to make two observations about Levi subgroups.

(1) It is *not* the case that the identity component of the centralizer of a semisimple element is necesarily a Levi subgroup of $G$. Since any semisimple element lies in a maximal torus, the rank of the centralizer is the same as the rank of $G$ (this is of course also true of any Levi subgroup). Thus the study of centralizers of semisimple elements is essentially equivalent to the study of maximal rank subgroups of $G$. An example of such a subgroup which is not a Levi subgroup is given by the following: Consider $G = \mathrm{Sp}_4(\mathsf{k})$, and write $\mathsf{k}^4$ as the sum of two orthogonal hyperbolic planes $H_1 \oplus H_2$. Then if $s \in G$ acts as $1$ on $H_1$ and $-1$ on $H_2$, then clearly $s$ is a semisimple element, and it is easy to see that $Z_G(s)^0$ is isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2$, which is not a Levi subgroup of $G$.
(2) Two Levi subgroups $L_1$ and $L_2$ may be conjugate in $G$ without there being parabolic subgroups $P_1$ and $P_2$ containing $L_1$ and $L_2$ respectively where $P_1$ and $P_2$ are conjugate. Indeed this is already easy to see in the case of $GL_3$: there is a single conjugacy classes of Levi subgroups of rank $1$, but two conjugacy classes of parabolics containing them.

## 2. Harish-Chandra theory

Let $G$ be a reductive algebraic group defined over $\mathbb{F}_q$. We wish to study the character theory of $G^F$ using "induction" (in more than one sense) on the rank of $G$, or algebraic group. The idea is to understand the character theory of some family of reductive subgroups of $G$, and then by some form of induction obtain characters of $G$ itself which we can hope to decompose into irreducible characters. The family of reductive subgroups we take are the rational Levi subgroups of rational parabolics of $G$.

Of course, given a character $\chi$ of $L^F$ for some rational Levi subgroup of $G$, one could consider the character of $\mathrm{Ind}_{L^F}^{G^F}(\chi)$, however this turns out to be too large a representation to study. Instead we make use of a rational parabolic subgroup $P$ which contains $L$. Since we know that $L$ is a quotient of $P$, we may "lift" a character of $L^F$ to a character of $P^F$, and *then* induce the result to $G$. The characters we obtain in this way are much smaller and easier to decompose. Of course it is not clear from this description that the character we obtain will be independent of the auxiliary choice of the parabolic $P$, an issue we will return to later. Notice that this is similar to the case of algebraic groups over $\mathbb{C}$, where one lifts characters of a torus to a Borel subgroup, and only then induces in order to obtain the simple highest weight representations (the "smallness" of the induced representation here corresponds loosely to the fact space of global sections of a line bundle over $G/B$ is finite dimensional since $G/B$ is proper) . The difference here is that one must consider all parabolic subgroups, not just the Borel subgroups.

**Definition 2.1.** Let $L$ be a rational Levi subgroup of $G$ such that there is a rational parabolic $P$ containing $L$. Define $I_{L\subset P}^G\colon \mathrm{Rep}(L^F) \to \mathrm{Rep}(G^F)$, by setting

$$I_{L\subset P}^G(\chi) = \mathrm{Ind}_{P^F}^{G^F}\tilde{\chi}$$

where $\tilde{\chi}$ is the representation of $P^F$ obtained via the isomorphism $P^F/R_u(P)^F \cong L^F$. The operation $I_{L\subset P}^G$ is called *parabolic induction*.

There is another way to describe parabolic induction, which fits into a more general context[1]. Suppose that $G$ and$H$ are finite groups, and that $M$ is a $G$-$H$ bimodule. Then we obtain a functor $R_M\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ by setting $I_M(V) = M \otimes_{\mathbb{C}[H]} V$. When $H$ is a subgroup of $G$, we recover standard induction by taking $M = \mathbb{C}[G]$ (and so for example the exactness of induction comes from the fact that $\mathbb{C}[G]$ is a free $\mathbb{C}[H]$-module, by Lagrange's theorem).

Parabolic induction fits into this formalism: we set $M$ to be the $G^F$-$L^F$ module $\mathbb{C}[G^F/U^F]$ where $L^F$ acts on the right because $L^F$ normalizes $U^F$. It is easy to see that $I_M = I_{L\subset P}^G$. It is clear that these induction functors $I_M$ have adjoints given by the dual module $\mathrm{Hom}(M, \mathbb{C})$.

The first result we wish to establish about parabolic induction is a form of "transitivity". This is essential if we wish to study reductive groups compatibly. We need a preparatory lemma in order to make the necessary statement.

**Lemma 2.2.**
(1) *Let $Q$ and $P$ be parabolic subgroups of $G$, such that $Q \subset P$. Then for any Levi subgroup $M$ of $Q$ there is a unique Levi subgroup $L$ of $P$ such that $M \subset L$.*
(2) *For a Levi subgroup $L$ of a parabolic $P$ of $G$, the following are equivalent.*
   (a) *$M$ is a Levi of a parabolic subgroup of $L$.*
   (b) *$M$ is a Levi of a parabolic subgroup of $G$, and $M \subset L$.*

*Proof.* To prove the first claim, pick a maximal torus of $G$ contained in $M$. Then there is a unique Levi subgroup of $P$ which contains $T$, and it is determined by the root subgroups it contains, and similarly for $M$. Then result then follows from the description of the Levi subgroups in Proposition 1.24.

The second part follows from the characterization of Levi subgroups in terms of tori: let $S_L = Z(L)^0$, a torus in $G$, so that $L = Z_G(S_L)$. Similarly, let $S_M = Z(M)^0$ a

[1]This bimodule point of view was introduced in this context by Broué.

torus in $M$. Then since $S_M$ and $S_L$ commute, the generate a torus $S = S_L S_M$ in $G$. It is clear that $Z_G(S) \subset Z_G(S_L) = L$, but then if $l \in L$, we have $l \in Z_G(S_L)$, so that $l \in Z_L(S) \iff l \in Z_L(S_M) = M$. Hence $M$ is a Levi subgroup of $G$ and we see that $(a)$ implies $(b)$. For the converse, let $S_M$ be as before. Then since $M \subset L$, we see $S_M \subset L$, and so $M$ is the centralizer of a torus in $L$, and hence a Levi subgroup of a parabolic of $L$ as required. □

**Proposition 2.3.** *Let $G$ be a reductive algebraic group defined over $\mathbb{F}_q$. Suppose that $P$ is a rational parabolic subgroup of $G$, and $Q$ is a rational parabolic contained in $P$. Let $L$ be a rational Levi for $P$ and $M$ a rational Levi for $Q$ contained in $L$. Then we have*

$$I^G_{L\subset P} \circ I^L_{M\subset Q\cap L} = I^G_{M\subset Q}$$

*Proof.* The functor $I^G_{L\subset P}$ has a left adjoint $R^G_{L\subset P}$ given by $W \mapsto W_{U^F}$, where $W_U$ is the coinvariants of $W$, (that is the largest quotient of $V$ on which $U$ acts trivially) thought of as a $L^F \cong P^F/U^F$-representation. Since we work with representations of finite groups in characteristic zero, the coinvariants are naturally isomorphic to the invariants, so we may also define $R^G_{L\subset P}(W) = V^{U^F}$.

Since adjoints are unique up to isomorphism when they exist, we may prove the transitivity of parabolic induction via the corresponding statement for parabolic restriction, which is somewhat simpler:

$$R^L_{M\subset L\cap V} \circ R^G_{L\subset P}(W) = (W^{U^F})^{(L\cap V)^F} = W^{V^F},$$

since $V = U.(V \cap L)$.

□

We end this section by calculating what the character of a parabolically induced representation is:

**Lemma 2.4.** *Let $\chi\colon L^F \to \mathbb{C}$ be the character of a representation $\rho\colon L^F \to V$. Then the character of $R^G_L(V)$ is*

$$I^G_L(\chi)(g) = |P^F|^{-1} \sum_{h\in G^F} \chi(\pi(h^{-1}gh))$$

*Proof.* Let $e = |L^F|^{-1}\sum_{l\in L^F} l^{-1} \otimes l$, and write $M = \mathbb{C}[G^F/U^F]$. Then $e$ is an idempotent in the group algebra of $L^F \times (L^F)^{\text{op}}$. Considering its action on $M \otimes_{\mathbb{C}} V$ it is easy to see that its kernel is generated by elements of the form $mh\otimes v - m\otimes hv$. Since the quotient by this kernel is exactly $\mathbb{M} \otimes_{\mathbb{C}[L^F]} V$ we see that

$$\begin{aligned} I^G_L(\chi)(g) &= tr(ge, M \otimes_{\mathbb{C}} V) \\ &= |L^F|^{-1} \sum_{l\in L^F} tr((g, l^{-1}) \otimes l, M \otimes_{\mathbb{C}} V) \\ &= |L^F|^{-1} \sum_{l\in L^F} tr((g, l^{-1}), M)\chi(l) \end{aligned}$$

Now we calculate $tr((g, l), \mathbb{C}[G^F/U^F]$. This is a permutation representation of $G^F \times L^F$, so that its character at $(g, l)$ is just the number of fixed points of $(g, l)$ on $G^F/U^F$. But $(g, l)(hU^F) = hU^F$ if and only if $ghl^{-1}U^F = hU^F$, that is if $ghl^{-1} = hv$ for some $v \in U^F$, if and only if $h^{-1}gh = vl \in lU^F$. Thus we see that

$$tr((g, l), M) = |\{hU^F \in G^F/U^F : h^{-1}ghU^F = lU^F\}|.$$

Writing $\pi\colon P^F \to L^F$ for the quotient map, and noting that the set $\{h \in G^F : h^{-1}gh \in lU^F\}$ is a $U^F$-bundle over the fixed point set of $(g, l)$, we see that

$$I_L^G(\chi)(g) = |P^F|^{-1} \sum_{h \in G} \chi(\pi(h^{-1}gh)).$$

□

Up to this point, it is arguable that one should consider the reductive quotients of rational parabolic subgroups rather than Levi subgroups of $G$, since this is exactly how we have constructed representation of $G^F$ from $L^F$. However, the fact that although the choice of a rational parabolic containing a rational Levi is not unique, the resulting induction functor is, makes a stronger case for considering subgroups rather than subquotients of $G$. To see this independence of the parabolic, we need to establish a formula relating parabolic induction from different parabolics.

**Proposition 2.5.** *(Mackey formula): Let $P$ and $Q$ be rational parabolics of $G$, and $L$, $M$ be Levi subgroups of $P$ and $Q$ respectively. Then we have*

$$R_{L\subset P}^G \circ I_{M\subset Q}^G = \sum_x I_{L\cap xMx^{-1}\subset L\cap xQx^{-1}}^L \circ R_{L\cap xMx^{-1}\subset P\cap xMx^{-1}}^{xMx^{-1}} \circ ad(x),$$

*where $x$ runs over a set of representatives of $L^F\backslash S(L,M)^F/M^F$, where*

$$S(L,M) = \{x \in G : L \cap xMx^{-1} \text{ contains a maximal torus of } G\}.$$

*Proof.* The proof of this proposition essentially involves studying the geometry of the intersection of parabolic subgroups of $G$. We will establish the result over the following few Lemmas. □

We first describe the $P$-$Q$ double cosets of $G$ in terms of the Weyl group.

**Lemma 2.6.** *Let $(W,S)$ be a Coxeter system, and let $I$, $J$ be subsets of $S$. The each double coset $W_IwW_J$ contains a unique element of minimal length $w$, which moreover is $I$-$J$ reduced in the sense that if $u \in W_I$ and $v \in W_J$, then $l(uwv) = l(u) + l(w) + l(v)$.*

*Proof.* One shows that each double coset contains an $I$-$J$ reduced element using the exchange condition for Coxeter groups (see Lemma **??**), which is then necessarily the minimal length representative.

□

**Lemma 2.7.** *Let $P_I$ and $P_J$ be parabolic subgroups of $G$ containing a common Borel. Then we have a natural bijection:*

$$W_I\backslash W/W_J \cong P_I\backslash G/P_J.$$

*Proof.* Since both of $P_I$ and $P_J$ contain $B$, the $P_I$-$P_J$ double cosets are a union of $B$ double cosets. Since $P_I = \bigsqcup_{w\in W_I} BwB$ and $P_J = \bigsqcup_{v\in W_J} BvB$, we have

$$P_IwP_J = BW_IBwBW_JB$$

and so using the $BN$-pair axioms, for example, it is easy to see that there is a surjection $W_I\backslash W/W_J \to P_I\backslash G/P_J$. We must show that this map is injective. To do this we note that if we choose $w$ to be the element of minimal length in $W_I\backslash W/W_J$ then it follows from the $BN$-axioms (in refined form) that

$$P_IwP_J = \bigsqcup_{x\in W_I, y\in W_J} BxwyB; \qquad l(xwy) = l(x) + l(w) + l(y).$$

hence the element $w$ (and thus the coset $W_I w W_J$) can be recovered from $P_I w P_J$, showing injectivity. $\square$

This combinatorial description of the double cosets lets us describe the variety $P\backslash G/Q$ of double cosets for a pair of rational parabolics $P$ and $Q$ in terms of the associated Levi subgroups. Recall the definition of $S(L,M)$ in the statement of the Mackey formula.

**Lemma 2.8.** *The natural map from the variety $L\backslash S(L,M)/M$ to $P\backslash G/Q$ is an isomorphism. Moreover, by taking fixed points of Frobenius it induces a bijection*

$$L^F\backslash S(L,M)^F/M^F \cong P^F\backslash G^F/Q^F.$$

*Proof.* Since we may conjugate the pair $M \subset Q$ by an element of $G$ so that $P$ and $gQg^{-1}$ contain a common Borel, and $gMg^{-1}$ and $L$ contain common maximal torus, and the double cosets of $P$ and $gQg^{-1}$ are translates of those of $P$ and $Q$, we may assume that $P$ and $Q$ already contain a common Borel and $L$ and $M$ a common maximal torus. Thus fixing a pair $T \subset B$ of a maximal torus lying in a Borel, we may assume that $P = P_I$ and $Q = P_J$ for some $I, J \subset S$.

Now consider the map $S(L,M) \to P_I\backslash G/P_J$. Let $P_I g P_J$ be a double coset in $P_I\backslash G/P_J$. It follows from the Bruhat decomposition that $P_I \cap gP_J g^{-1}$ contains a maximal torus, $T'$ say. Let $L'$ and $M'$ be the unique Levi subgroups of $P_I$ and $gP_J g^{-1}$ containing $T'$. Since Levi subgroups of a parabolic are all conjugate under the action of the unipotent radical, we may find $u \in R_u(P_I)$ and $v \in R_u(P_J)$ such that $uLu^{-1} = L'$ and $gvMv^{-1}g^{-1} = M'$. It follows that $u^{-1}gv \in S(L,M)$, and so the map $L\backslash S(L,M)/M \to P_I\backslash G/P_J$ is surjective.

Now let $x \in S(L,M)$. Then $xMx^{-1} \cap L$ contains a maximal torus $T'$. Since such tori are all conjugate in a reductive groups we may find $l \in L$ and $m \in M$ so that $lTl^{-1} = T'$ and $xmTm^{-1}x^{-1} = T'$. Thus we may replace $x$ by $l^{-1}xm$ to obtain a representative of the double coset $LxM$ which lies in $N_G(T)$. Thus given $x$ and $y$ in $S(L,M)$ such that $P_I x P_J = P_I y P_J$ we may suppose that $x$ and $y$ lie in $N_G(T)$. Then it follows from the identification of double cosets that $xT$ and $yT$ are in the same $W_I$-$W_J$ double coset, and hence the same $L$-$M$ double coset.

To check the identification of $F$-fixed points, one just needs to note that the relevant centralizers $P \cap xQx^{-1}$ and $L \cap xMx^{-1}$ are connected, and then apply Lang's theorem. $\square$

*Proof of Mackey formula*: Let $U = R_u(P_I)$ and $V = R_u(P_J)$. Note that the functor given by induction and then restriction is given by tensoring with the $L^F$-$M^F$ bimodule given by $\mathbb{C}[U^F\backslash G^F] \otimes_{\mathbb{C}[G^F]} \mathbb{C}[G^F/V^F]$, which is clearly isomorphic to $\mathbb{C}[U^F\backslash G^F/V^F]$. On the level of sets of $F$-fixed points, we have

$$U^F\backslash G^F/V^F = \bigsqcup_{x \in L^F\backslash S(L,M)^F/M^F} U^F\backslash P^F x Q^F/V^F.$$

**Lemma 2.9.** *For $x \in S(L,M)$ the map*

$$(l(L \cap xVx^{-1}), (xMx^{-1} \cap U).xmx^{-1}) \mapsto UlxmV,$$

*gives an isomorphism*

$$L/(L \cap xVx^{-1}) \times_{L \cap xMx^{-1}} (xMx^{-1} \cap U)/xMx^{-1} \to U\backslash PxQ/V.$$

*inducing a corresponding bijection on $F$-fixed points.*

*Proof.* It is easy to see that the map is well-defined, and that the relevant stabilizers are connected (to see that the stabilizer in $U \times V$ of a point $PxQ$ is connected, recall that it must be a product of the root subgroups it contains, while the action of $L \cap xMx^{-1}$ is free). Proving surjectivity follows from the fact that every element of $PxQ$ can be written in the form $ulxmv$ where the elements in the factorization belong to the obvious groups. To see injectivity one uses the fact that for any parabolics $P$ and $Q$ we have:

$$P \cap Q = (L \cap M)(L \cap V)(M \cap U)(U \cap V)$$

an isomorphism of varieties. □

We can now conclude the proof of the theorem: taking the union over all $P$-$Q$ double cosets we find that $U^F \backslash G^F / V^F$ is isomorphic to

$$\bigsqcup_{x \in L^F \backslash S(L,M)^F / M^F} L^F/(L^F \cap xV^F x^{-1}) \times_{L^F \cap xM^F x^{-1}} (xM^F x^{-1} \cap U^F)/xM^F x^{-1}$$

compatibly with the $L^F$-$M^F$ actions. But each of the sets in this union corresponds to a composition of functors

$$I^L_{L \cap xMx^{-1} \subset L \cap xQx^{-1}} \circ R^{xMx^{-1}}_{L \cap xMx^{-1} \subset P \cap xMx^{-1}} \circ ad(x).$$

and so the formula follows.

The first important application of the Mackey formula is to show that parabolic induction does not depend on the choice of parabolic subgroup, as we mentioned before.

**Proposition 2.10.** *Let $G$ be a reductive algebraic group over $\mathbb{F}_q$. Let $L$ be a rational Levi subgroup, and $P_1, P_2$ be rational parabolics with $L$ as a Levi subgroup. Then $R^G_{L \subset P_1}$ is isomorphic to $R^G_{L \subset P_2}$.*

*Proof.* The proof uses induction on the *semisimple rank* of the group $G$, that is, the rank of the derived subgroup $[G,G] \cong G/R_u(G)$. Crucially, the semisimple rank of a rational Levi subgroup of $G$ is strictly smaller than that of $G$. We use the Mackey formula for the pairs $L \subset P_1$ and $L \subset P_2$. Thus we find that

$$\mathrm{Hom}_{G^F}(I^G_{L \subset P_1}(\pi), I^G_{L \subset P_2}(\pi)) = \mathrm{Hom}_{L^F}(R^G_{L \subset P_2} \circ I^G_{L \subset P_1}(\pi), \pi)$$

and hence using the Mackey formula, this is equal to

$$\bigoplus_{x \in L^F \backslash S(L,L)^F / L^F} \mathrm{Hom}_{L^F}(I^L_{L \cap xLx^{-1} \subset L \cap xP_2x^{-1}} \circ R^{xLx^{-1}}_{L \cap xLx^{-1} \subset P_1 \cap xLx^{-1}} \circ ad(x)(\pi), \pi)$$

which is just

$$\bigoplus_{x \in L^F \backslash S(L,L)^F / L^F} \mathrm{Hom}_{L^F \cap xL^F x^{-1}}(R^{xLx^{-1}}_{L \cap xLx^{-1} \subset P_1 \cap xLx^{-1}} \circ ad(x)(\pi), R^L_{L \cap xLx^{-1} \subset L \cap xP_2x^{-1}}(\pi))$$

since the functors $I$ and $R$ are in fact two-sided adjoints. Now, by induction, we know that this last expression is independent of the parabolics chosen, and hence there is an isomorphism

$$R^{xLx^{-1}}_{L \cap xLx^{-1} \subset P_1 \cap xLx^{-1}} \cong R^{xLx^{-1}}_{L \cap xLx^{-1} \subset P_2 \cap xLx^{-1}}$$

Thus each of the summands above is isomorphic to

$$\mathrm{Hom}_{L^F \cap xL^F x^{-1}}(R^{xLx^{-1}}_{L \cap xLx^{-1} \subset P_2 \cap xLx^{-1}} \circ ad(x)(\pi), R^L_{L \cap xLx^{-1} \subset L \cap xP_2x^{-1}}(\pi)),$$

but these last two representation are both equal to the $R_u(L \cap xP_2x^{-1})$-invariants in $\pi$, and hence the result follows. $\square$

We can now formulate the strategy of Harish-Chandra. We will call a rational Levi subgroup of $G$ which is the Levi of some rational parabolic a *cuspidal subgroup*.

**Definition 2.11.** A representation $\pi$ of $G^F$ is said to be *cuspidal*, if for every cuspidal subgroup $L$ we have $R_L^G(\pi) = 0$. By adjointness and the transitivity of parabolic induction, we see that every representation of $G^F$ is obtained by parabolic induction from a cuspidal representation of a cuspidal subgroup. Write $\text{Rep}(G^F)^0$ for the full subcategory of cuspidal representations of $G^F$. A *cuspidal pair* is a pair $(L, \delta)$ where $L$ is a cuspidal subgroup of $G$ and $\delta$ is a cuspidal representation of $L^F$.

The Harish-Chandra philosophy splits up the classification of the simple representations of $G^F$ into two parts

1. Classify $\text{Rep}(G^F)^0$ for all finite reductive groups $G^F$.
2. For $\delta$ a cuspidal representation of a cuspidal subgroup $L$ of $G$, calculate $I_L^G(\delta)$.

In fact, we can already do a little better: given a representation $\pi$ of $G^F$, the cuspidal pair $(L, \delta)$ such that $\pi$ occurs in $I_L^G(\delta)$ is actually uniquely determined up to conjugacy, so that we obtain a partition of $\text{Rep}(G^F)$ according to the cuspidal pairs of $G^F$.

**Proposition 2.12.** *Let $(L, \delta)$ and $(M, \eta)$ be cuspidal pairs.*

1. *We have*
$$(I_M^G(\eta), I_L^G(\delta)) = |L^F \backslash \{x \in G^F : L = xMx^{-1} : \eta \circ ad(x) = \delta\}|$$
2. *$I_L^G(\delta)$ and $I_M^G(\eta)$ are either disjoint (that is, have no irreducible constituent in common) or are isomorphic. In the latter case, the cuspidal pairs are $G^F$ conjugate.*

*Proof.* To prove the first part, notice that

$$\begin{aligned}(I_M^G(\eta), I_L^G(\delta)) &= (R_L^G I_M^G(\eta), \delta) \\ &= \sum_{x \in L^F \backslash S(L,M)^F / M^F} (I_{L \cap xMx^{-1}}^L \circ R_{L \cap xMx^{-1}}^{xMx^{-1}} \circ ad(x)(\eta), \delta).\end{aligned}$$

Now since $\eta$ is cuspidal, we have $R_{L \cap xMx^{-1}}^{xMx^{-1}} \circ ad(x)(\eta) = 0$ unless $L \supseteq xMx^{-1}$, in which case the restriction is the identity functor. Thus we find that

$$\begin{aligned}(I_L^G(\delta), I_M^G(\eta)) &= \sum_{L^F \backslash \{x \in G^F, xMx^{-1} \subseteq L\} / M^F} (I_{L \cap xMx^{-1}}^L ad(x)(\eta), \delta) \\ &= \sum_{L^F \backslash \{x \in G^F, xMx^{-1} \subset L\} / M^F} (\eta, ad(x^{-1}) \circ R_{L \cap xMx^{-1}}^L(\delta)).\end{aligned}$$

and so since $\delta$ is cuspidal, each term in this summation is zero unless $L \cap xMx^{-1} = L$, whence $L = xMx^{-1}$, in which case the restriction is again the identity functor. Now since $\delta$ is irreducible, it follows that the terms in this sum are zero unless $ad(x)(\eta)$ is isomorphic to $\delta$, when they are equal to $1$. Part $(1)$ follows immediately. Part $(2)$ is an immediate consequence of part $(1)$. $\square$

Notice that this last Proposition also shows us that

$$\dim(\mathrm{End}_{G^F}(I_L^G(\delta)) = |L^F \backslash \{x \in N_G(L)^F : ad(x)(\delta) \cong \delta\}|.$$

Let $W_G(\delta)$ be the set occuring on the right-hand side of the last equality. It turns out that it is much more intimately connected with $I_L^G(\delta)$. Indeed Howlett and Lehrer [HL] have shown that the algebra $\mathrm{End}_{G^F}(I_L^G(\delta))$ is a twisted version of the group algebra of $W_G(\delta)$. Notice that a complete understanding of $\mathrm{End}_{G^F}(I_L^G(\delta))$ yields at least a parametrization of the irreducible $G^F$ representations occuring in $I_L^G(\delta)$ – they are in bijective correspondence with the simple representations of the endomorphism algebra. Thus at least if we seek only a parametrization of the irreducibles of $G^F$, part $(2)$ of the Harish-Chandra program is quite satisfactory. The question of determining cuspidal representations, however, is much harder. We will next attempt to get fairly complete information in the case where $G = GL_n(\mathsf{k})$.

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