THE GENERAL LINEAR GROUP

KEVIN MCGERTY

Let $\mathbb{F}_q$ be a finite field with q elements, and k a fixed algebraic closure. An irreducible polynomials in $\mathbb{F}_q[t]$ will be assumed to be monic, and the set of such polynomials (excluding the polynomial t) will be denoted Φ_q. This set can thus alternatively be viewed as the set of F-orbits on k^*, where F is the Frobenius map, $x \mapsto x^q$.

Let GL_n be the $\mathbb{F}_q$-variety given by the general linear group[1] as a k-variety with the standard Frobenius F, so that $\mathrm{GL}_n^F = \mathrm{GL}_n(\mathbb{F}_q)$ is the group of $\mathbb{F}_q$-points. Then $\mathrm{GL}_n(\mathbb{F}_q)$ is a finite group of order

$$|\mathrm{GL}_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1}(q^n - q^i) = q^{n(n-1)/2}\prod_{i=1}^{n}(q^i - 1)$$

We let V_n be the canonical n-dimensional vector space as a k-variety, and let V_n^F be its $\mathbb{F}_q$-rational points, thus V_n^F is a $\mathbb{F}_q$-vector space with an action of $\mathrm{GL}_n(\mathbb{F}_q)$. We let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis vectors of V_n, so that $e_i \in V_n^F$.

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of weakly decreasing positive integers (*i.e.* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$). We write $|\lambda| = \sum_{i=1}^r \lambda_i$ and say that λ is a partition of $|\lambda|$. Write $\emptyset$ for the empty partition (so that $|\emptyset| = 0$), and let $\mathcal{P}$ denote the set of all partitions. For a partition λ we let

$$r_j(\lambda) = |\{i : 1 \leq i \leq r, \lambda_i = j\}|.$$

Set $m_j = \sum_{j \geq i} r_j(\lambda)$, then the partition $\lambda^t = (m_1, m_2, \ldots)$ is called the dual partition, and has $|\lambda| = |\lambda'|$. We will need some standard notation attached to partition, which we now record. Set

$$n(\lambda) = \sum_{i\geq 1} m_i(m_i - 1)/2;$$

$$z_\lambda = \prod_{i \geq 1} r_i(\lambda)! i^{r_i(\lambda)}.$$

Define polynomials ϕ_λ and $m_\lambda \in \mathbb{Z}[t]$ by

$$m_\lambda = \prod_{i=1}^{r}(t^{\lambda_i} - 1) = \prod_{i\geq 1}(t^i - 1)^{r_i(\lambda)};$$

$$\phi_\lambda = \prod_{i=1}^{r} \phi_{r_i(\lambda)}$$

Date: November, 2006.

[1]Clearly, for GL_n one can define a scheme over $\mathbb{Z}$, from which we can base change to any field k, to obtain the k-variety $\mathrm{GL}_m(\mathsf{k})$

where

$$\phi_r = \prod_{i=1}^{r}(1-t^i).$$

A *composition* is an ordered sequence of positive integers $\Lambda = (n_1, n_2, \dots, n_r)$ (and as for partitions we say that Λ is a composition of $n_1 + n_2 + \dots n_r$). Each composition of n yields a partition of n by reordering the elements of the sequence.

Finally, we will write Π_d for the functor of restriction of scalars from $\mathbb{F}_{q^d}$ to $\mathbb{F}_q$, so that if X is a $\mathbb{F}_{q^d}$-variety, $\Pi_d(X)$ is a $\mathbb{F}_q$-variety and $\Pi_d(X)(\mathbb{F}_q)$ can be identified with $X(\mathbb{F}_{q^d})$ uniquely up to the action of the Galois group Gal$(\mathbb{F}_{q^d}, \mathbb{F}_q)$.

If $x \in \mathrm{GL}_n(\mathbb{F}_q)$ lies in the conjugacy class $\mathbf{c}$, we write $Z_{\mathrm{GL}_n}(\mathbf{c})$ for the conjugacy class of subgroups containing $Z_{\mathrm{GL}_n(\mathbb{F}_q)}(x)$, the centralizer of x in $\mathrm{GL}_n(\mathbb{F}_q)$. We now describe the semisimple and unipotent conjugacy classes in $\mathrm{GL}_n(\mathbb{F}_q)$.

A semisimple conjugacy class is determined completely by its characteristic polynomial $\varphi \in \mathbb{F}_q[t]$. We may write $\varphi = \prod_{i=1}^{r} f_i^{n_i}$ where the f_i are distinct elements of Φ_q. Thus we may index semisimple conjugacy classes by functions $\rho\colon \Phi_q \to \mathbb{N}$ such that $\sum_{f\in\Phi} \rho(f)\deg(f) = n$ (the function being given by the integers n_i). Let $d_i = \deg(f_i)$. It is easy to see that $Z_{\mathrm{GL}_n}(x)$ is isomorphic as a k-variety to $\prod_i \mathrm{GL}_{n_i}^{d_i}$ and indeed as a $\mathbb{F}_q$-variety to

$$\prod_i \Pi_{d_i}(\mathrm{GL}_{n_i}).$$

where GL_{n_i} is viewed as a $\mathbb{F}_{q^{d_i}}$-variety via the Frobenius F^{d_i}. A unipotent conjugacy class is determined by its Jordan form, which is determined by the partition of n corresponding to the size of its blocks. It can be shown that the centralizer in $\mathrm{GL}_n(\mathbb{F}_q)$ of a unipotent element x corresponding to the partition λ has order $a_\lambda(q)$ where $a_\lambda \in \mathsf{k}[t]$ is given by

$$a_\lambda(t) = t^{|\lambda|+2n(\lambda)}\phi_\lambda(t^{-1}).$$

By the Jordan decomposition of elements of an algebraic group, we can combine the observations above to get a parametrization of the conjugacy classes of G_n:

Lemma 0.1. *There is a bijection between the conjugacy classes of $GL_n(\mathbb{F}_q)$ and the functions $\nu\colon \Phi_q \to \mathcal{P}$ such that $\sum_{f\in\Phi} |\nu(f)| deg(f) = n$.*

Next we describe the rational tori in GL_n. As we have already seen, although all tori are conjugate over k, this is no longer true over $\mathbb{F}_q$: in general the conjugacy classes are indexed by F-twisted conjugacy classes in the Weyl group. If we use the standard diagonal maximal torus T for GL_n, the action of the Frobenius map on the Weyl group $W = N(T)/T$ is trivial, so that the $\mathrm{GL}_n(\mathbb{F}_q)$ conjugacy classes of rational tori are indexed by conjugacy classes in W, the symmetric group on n letters. Since the conjugacy class of an element of the symmetric group is given by its cycle type, these are again indexed by partitions of n. We now make this correspondence explicit. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition, and split up the standard basis of V_n according to the numbers λ_i, so that we get a direct sum decomposition of $V_n = \bigoplus_{i=1}^{r} V_n^i$. Letting $T_d = \Pi_d(\mathrm{GL}_1)$ we get an action of the torus T_{λ_i} on V_n^i by identifying V_n^i with $\Pi_{\lambda_i}(V_1)$. This gives an action of

$$\prod_{i=1}^{r} T_{\lambda_i}$$

on $\mathbf{V}_n$, and so determines a $\mathbb{F}_q$-torus T_λ in $\mathbf{G}_n$. Any rational maximal torus is conjugate to some T_λ, and the conjugacy classes of the tori T_λ are all mutually distinct. The Weyl group W_λ of T_λ is $N_{\mathrm{GL}_n(\mathsf{k})}(T_\lambda)/T_\lambda$, a finite group. In fact, the elements of W_λ can be represented by elements of G_n, so that $W_\lambda = N_{G_n}(T_\lambda)$. W_λ is isomorphic to the centralizer of an element of cycle type λ in S_n the symmetric group on n letters, and so it has order z_λ.

The first result that hints in the direction of a correspondence between characters of rational tori and the representation of G_n is the following numerical calculation.

Lemma 0.2. $\sum_{\lambda,|\lambda|=n} |W_\lambda \backslash T_\lambda|$ *is the number of conjugacy classes of* G_n.

Proof. By construction we have $T_\lambda \cong \prod_i \mathsf{k}^*_{\lambda_i}$, thus we may write $t \in T_\lambda$ as a sequence $(x_1, x_2, \ldots, x_r)$ where $x_i \in \mathsf{k}^*_{\lambda_i}$. We define a map $\rho_t \colon \Phi \to \mathcal{P}$ as follows: for $f \in \Phi$ let $\rho_t(f)$ be such that

$$r_j(\rho_t(f)) = |\{i : 1 \leq i \leq r, x_i \text{ is a root of } f \text{ and } \lambda_i/\deg(f) = j\}|$$

(thus the partition is empty unless some x_i is a root of f). Clearly we have

$$\sum_{f\in\Phi} |\rho_t(f)|\deg(f) = n.$$

Since the map $t \mapsto \rho_t$ obviously descends to a map on the W_λ orbits $W_\lambda\backslash T_\lambda$, and it is easy to check that any function $\rho \colon \Phi \to \mathcal{P}$ satisfying $\sum_{f\in\Phi} |\rho(f)|\deg(f) = n$ arises as ρ_t for some t in some T_λ, we have a bijection between the disjoint union of the $\bigsqcup_{\lambda,|\lambda|=n} W_\lambda\backslash T_\lambda$ and functions $\rho \colon \Phi \to \mathcal{P}$ satisfying $\sum_{f\in\Phi} |\rho(f)|\deg(f) = n$. Since the set of such functions also indexes the conjugacy classes in G_n we are done. □

Recall that given a torus T defined over $\mathbb{F}_q$, there is a unique $\mathbb{F}_q$-split subtorus T_s of T. Define the k-*rank* of T to be the dimension of T_s. A torus $\mathbf{T}$ is said to be *anisotropic* if $T_s = \{e\}$. For any reductive algebraic group G over $\mathbb{F}_q$, a maximal torus T is called *minisotropic* if $\mathbb{F}_q$-rank of T is minimal. In that case T_s lies in the center of GL_n (and so for a semisimple group, minisotropic implies anisotropic). Thus for $\mathrm{GL}_n(\mathsf{k})$ a minisotropic torus T has T_s of dimension 1. It is easy then to check that $T_{(n)}$ is the unique conjugacy class of minisotropic tori in $\mathrm{GL}_n(\mathsf{k})$.

The conjugacy classes of parabolic subgroups of GL_n are indexed by compositions in the following way: let $\Lambda = (n_1, n_2, \ldots, n_r)$ be a composition of n, and let $\mathbf{V}^i$ be the span of the basis vectors $\{e_1, e_2, \ldots, e_{n_1+n_2+\ldots+n_i}\}$ so that

$$0 = V^0 \subset V^1 \subset V^2 \ldots \subset V^r = V.$$

If P_Λ denotes the stabilizer of this flag, P_Λ is a parabolic subgroup of $\mathrm{GL}_n(\mathsf{k})$, and moreover each parabolic of $\mathrm{GL}_n(\mathsf{k})$ is conjugate to exactly one P_Λ. Let A_Λ be the $\mathbb{F}_q$-torus of P_Λ consisting of the diagonal matrices which act by scalar multiplication on the quotients V^i/V^{i-1} $(1 \leq i \leq r)$. Let M_Λ be the centralizer of A_Λ in P_Λ, a Levi subgroup isomorphic to $\prod_i \mathrm{GL}_{n_i}$. Two parabolics in GL_n are *associated* if their Levi subgroups are conjugate in GL_n, so we see immediately that two parabolics P_Λ and $P_{\Lambda'}$ are associated if and only if Λ and Λ' yield the same partition of n. Since a partition of n is also a composition of n, we may consider the parabolics P_λ. These give representatives of the association classes of parabolics in GL_n, sometimes called the *cuspidal parabolics*.

1. Representations of GL_n and Hopf algebras

In this section we review the construction of a Hopf algebra from the representations of $\mathrm{GL}_n(\mathbb{F}_q)$, first given by Zelevinsky, inspired by his work with Bernstein on p-adic groups. A general reference for Hopf algebras is [MM].

Definition 1.1. Let R be a commutative ring. A Hopf algebra over R is a datum (H, m, e, Δ, c, S) such that

(1) (H, m, e) is an associative algebra (H, m, e) where $m\colon H \otimes_R H \to H$ is a multiplication and $e\colon R \to H$ gives the unit.
(2) The maps $\Delta\colon H \to H \otimes_R H$ and $c\colon H \to R$ satisfy the dual axioms to those of multiplication and unit, and the map $\Delta\colon H \to H \otimes_R H$ is an algebra homomorphism.
(3) The map $S\colon H \to H$ is an antipode, that is, a map $S\colon H \to H$ such that the composition $c \circ e$ agrees with either composition $m \circ (S \otimes id) \circ \Delta$ or $m \circ (id \otimes S) \circ \Delta$.

Example 1.2. We have the following examples of a Hopf algebra.

(1) Let G be a finite group. Then $\mathbb{C}[G]$ the group algebra of G is a Hopf algebra, with the comultiplication given by $\Delta(g) = g \otimes g$, and $S(g) = g^{-1}$.
(2) For an algebraic group G, the ring of regular functions on G is a Hopf algebra, where the comultiplication is the map on functions given by multiplication $G \times G \to G$. The counit is given by the trivial representation, and the anitpode is given by pull-back via the inverse map $\iota\colon G \to G$. (Thus the category of affine algebraic groups over k is just the category of k-Hopf algebras).
(3) The universal enveloping algebra of a Lie algebra $U(\mathfrak{g})$ is a Hopf algebra, where the comultiplication is given by the unique extension of the Lie algebra map $x \mapsto x \otimes 1 + 1 \otimes x$ from $\mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, and S is similarly determined by the condition that $S(x) = -x$ for $x \in \mathfrak{g}$. The counit is again given by the trivial representation.

A graded Hopf algebra will here mean an $\mathbb{N}$-graded Hopf algebra, such that all the structure maps are compatible with the grading. We say that a Hopf algebra $H = \bigoplus_{n \in \mathbb{N}} H_n$ is *connected* if the maps $e\colon R \to H_0$ and $c\colon H_0 \to R$ are inverses of each other (where we have identified $c\colon H \to R$ with its restriction to H_0 by viewing H_0 as a quotient of H). All Hopf algebras we will consider from now on are connected graded Hopf algebras. If H is such an algebra the kernel of the counit is $\bigoplus_{n>0} H_n$, and the axiom for the counit shows that $\Delta(x) = 1 \otimes x + x \otimes 1 + y$ where $y \in I \otimes I$. We set $\Delta_+(x) = y$. The antipode for a connected graded Hopf algebra is uniquely determined by the other structure, thus it is not necessary to specify the antipode for connected graded coalgebras (what the antipode actually *is* however, is still interesting to determine). A *bialgebra* is an algebra which is also compatibly a coalgebra (that is, a Hopf algebra without antipode).

Definition 1.3. Given connected graded bialgebras A and B over a commutative ring R, let $G(A, B)$ be the set of morphisms of R-modules from $f\colon A \to B$, such that f_0 is the identity map of R. The we may equip $G(A, B)$ with a product: for $f, g \in G(A, B)$ we set $f \star g$ to be the composition:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{m} B$$

Lemma 1.4. *The set $G(A,B)$ is a group under the operation $\star$. Thus a graded connected bialgebra A has a unique antipode: the inverse of the identity map in $G(A,A)$.*

Proof. That $\star$ is associative follows from the definitions, so it remains to check that inverses exist. Let $f \in G(A,B)$. We define f^{-1} inductively on the graded pieces of A, where $(f^{-1})_0$ is just the identity on R. Suppose that f^{-1} is define on $\bigoplus_{k=0}^{n-1} A_k$. Then for $x \in A_n$ we have

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \Delta_+(x)$$

and now since $\Delta_+(x) = \sum_i x_i \otimes y_i$ where $0 < \deg(x_i), \deg(y_i) < n$, it follows that f^{-1} has already been defined for these terms in $\Delta(x)$. Thus we may set

$$f^{-1}(x) = x - f(x) - \sum_i f(x_i) f^{-1}(y_i),$$

so that f^{-1} exists and is unique. □

For us, the most important example of a connected graded Hopf algebra is the following:

Example 1.5. The ring of symmetric functions: let $\{x_i : i \geq 1\}$ be a set of indeterminates. Then for each $n \geq 1$ we may consider the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. It is acted on by the symmetric group S_n by permutation of the variables. The resulting ring of invariants $\Lambda_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ is also a polynomial ring on the elementary symmetric functions. There is a natural map $\mathbb{Z}[x_1, x_2, \ldots, x_{n+1}] \to \mathbb{Z}[x_1, x_2, \ldots, x_n]$ given by taking the quotient by the ideal generated by x_{n+1}. This is compatible with the action of S_n where we think of S_n as a subgroup of S_{n+1} in the obvious way. Thus it passes to a map on the invariants. We set

$$\Lambda = \lim \Lambda_n$$

The ring Λ is called the ring of symmetric functions. It is graded by degree, since the quotients preserves degree It can be made into a Hopf algebra as follows: $\Lambda \otimes \Lambda$ can be viewed as the algebra of functions $f(x,y)$ in variables x and y symmetric in each separately. Then we obtain a map $\Lambda \to \Lambda \otimes \Lambda$ by letting $\Delta(f)(x,y) = f(x,y)$, where we identify the variables some way.

Thus if $p_n = \sum_{i \geq 1} x_i^n$, then $\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$, while $\Delta(e_n) = \sum_{k=0}^n e_k \otimes e_{n-k}$.

Our interest in Hopf algebras comes from the fact that we may build a Hopf algebra from the character rings of $\mathrm{GL}_n(\mathbb{F}_q)$. More precisely, we have the following definition.

Definition 1.6. Let $R_n = K(\mathrm{Rep}(\mathrm{GL}_n(\mathbb{F}_q))$ be the Grothendieck group of the category of representations of $\mathrm{GL}_n(\mathbb{F}_q)$, a free $\mathbb{Z}$-module with a basis naturally indexed by the isoclasses of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$. Let $R(q) = \bigoplus_{n \geq 0} R_n$, (where $R_0 = \mathbb{Z}$ by definition, or rather, we consider $\mathrm{GL}_0(\mathbb{F}_q)$ to be the trivial group).

The abelian group R is naturally a Hopf algebra in the following way: given χ_1 and χ_2 in R_n and R_m respectively, we may form $\chi_1 \otimes \chi_2$ a representation of $GL_{n_1}(\mathbb{F}_q) \times GL_{n_2}(\mathbb{F}_q)$. Now this is a standard Levi of $\mathrm{GL}_{n_1+n_2}(\mathbb{F}_q)$, and so we may parabolically induce it to a representation of $\mathrm{GL}_{n_1+n_2}(\mathbb{F}_q)$ to obtain an element of $R_{n_1+n_2}$. Extending linearly we obtain a graded mulitplication $m \colon R \otimes R \to R$. In an analogous way, we obtain a graded comultiplication using parabolic restriction from $\mathrm{GL}_{n_1+n_2}(\mathbb{F}_q) \to \mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q)$. The antipode S is given by duality on representations, the counit and unit by the obvious maps since $R_0 \cong \mathbb{Z}$. If we extend scalars to the complex numbers, then the trace map, there is a natural isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} R \cong \bigoplus C[\mathrm{GL}_n(\mathbb{F}_q)]^{\mathrm{GL}_n(\mathbb{F}_q)}$ between R and the direct sum of the rings of invariant functions on $\mathrm{GL}_n(\mathbb{F}_q)$ (where the groups act on themselves via conjugation).

Theorem 1.7. *The datum (R, m, e, Δ, c, S) is a Hopf algebra.*

Proof. Most of the axioms are easy to check: the fact that (H, m, e) is an algebra is just a restatement of the fact that parabolic induction is transitive, and similiarly the coassociativity of Δ is the transitivity of parabolic restriction. The only difficult axiom is the condition that Δ be an algebra homomorphism.

To see this we note that there is a natural pairing on R given on each R_n by

$$(\chi_1, \chi_2) = \dim(\mathrm{Hom}(\chi_1, \chi_2)).$$

This pairing is nondegenerate, and by adjointness of induction and restriction, our multiplication and comultiplication are adjoints of each other under this pairing. Using this pairing, the statement that Δ is an algebra homomorphism becomes exactly the statement of the Mackey formula. □

Example 1.8. If, instead of $\mathrm{GL}_n(\mathbb{F}_q)$, we use the sequence of $\{S_n\}_{n\geq 1}$ of symmetric groups, this time with the normal induction and restriction functors, then we obtain a Hopf algebra $\mathcal{S} = \bigoplus_{n\geq 1} K(\mathrm{Rep}(S_n))$. This algebra therefore has a basis indexed by the isoclasses of irreducible representations. These can be naturally labeled by partitions, as we now describe. Let λ be a partition of n, and let S_λ be the subgroup of S_n which preserves the sets $\{1, 2, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \dots$. Then let $\mu_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} 1$, the permutation representation of S_n on S_λ-cosets, and let $\nu_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} \varepsilon$, where ε is the sign representation of S_λ. It can be shown that $\mathrm{Hom}_{S_n}(\nu_\lambda, \mu_{\lambda^t})$ is one-dimensional, so that the image of ν_λ in μ_{λ^t} under any nonzero such homomorphism is an irreducible. This representation is denoted χ_λ. The set of irreducible representations of S_n is thus $\{\chi_\lambda : \lambda \text{ a partition of } n\}$.

In fact, this Hopf algebra is none other than the Hopf algebra of symmetric functions. To describe the isomorphism we need to introduce some symmetric functions. First suppose that r is a fixed positive integer, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$. Then we may consider the skew-symmetric function $a_\lambda = \sum_{\sigma \in S_r} \varepsilon(\sigma) x^{\sigma(\lambda)}$, where $x^c = x_1^{c_1} \dots x_r^{c_r}$ for any composition $(c_1, c_2, \dots, c_r)$. Now let $\delta = (r-1, r-2, \dots, 1)$, and define

$$s_\lambda = a_{\lambda+\delta}/a_\delta.$$

As the ratio of two skew-symmetric functions, is symmetric. It can be seen to lie in $\mathbb{Z}[x_1, x_2, \dots, x_r]$ by using a Vandermonde determinant argument. Indeed examining this argument more carefully, we see that multiplication by a_δ gives a linear isomorphism from Λ_n to the space of skew-symmetric polynomials. Now $a_\mu \neq 0$ if and only if $\mu = w(\lambda+\delta)$ for some λ, and $a_{w(\lambda+\delta)} = \varepsilon(\sigma) a_\lambda$, thus it follows that the $a_{\lambda+\delta}$ are a basis for the space of skew-symmetric polynomials, and hence the s_λ form a basis of Λ_r.

Finally, note that if $\ell(\lambda) < r$ (that is, λ has less than r nonzero parts) then $s_\lambda(x_1, x_2, \dots, x_r, 0) = s_\lambda(x_1, x_2, \dots, x_r)$, so that for any $\lambda \in \mathcal{P}$ there is a well defined element $s_\lambda \in \Lambda$. These are the Schur functions. It can be shown that there is an isomorphism of Hopf algebras $\mathcal{S} \to \Lambda$ which sends the irreducible representation χ_λ to the Schur function s_λ.

Definition 1.9. Let H be a (connected graded) Hopf algebra. An element p of I is *primitive* if $\Delta(p) = p \otimes 1 + 1 \otimes p$, or equivalently $\Delta_+(p) = 0$. The subspace of primitive elements is denoted P.

Proposition 1.10. *Let H be a connected graded Hopf algebra over a field of characteristic zero, and let I be the augmentation ideal $I = \bigoplus_{n>0} R_n$, and $I^2 = m(I \otimes I)$. If $I = P \oplus I^2$, then H is isomorphic as an algebra to the symmetric algebra on the subspace P.*

Proof. This is proved in [Z, Appendix 1]. One begins by showing that the multiplication and comultiplication are commutative. □

Zelevinksy in [Z] also introduces a certain class of Hopf algebras, which he calls PSH-algebras (for Positive-Self-adjoint-Hopf-algebras – a shameless attempt to get them called Zelevinksy algebras?) To define these, let R be an ordered ring.

A *positive* R-module to be a free R-algebra with a specified R-basis Ω. Clearly if A and B are positive R-module, then $A \otimes B$ is a positive R-module, with basis the tensor product of the bases of A and B. For a positive R-module A we let A^+ be the subset of positive linear combinations of elements of Ω_A. Finally, a morphism of positive R-modules is a morphism of R-modules $\phi\colon A \to B$ which takes A^+ into B^+.

Given a positive R-module, we obtain an R-valued nondegenerate bilinear symmetric form $(\cdot, \cdot)$ by declaring the basis Ω_A to be orthonormal and extending linearly.

Definition 1.11. A PSH-algebra is a positive $\mathbb{Z}$-module which is graded Hopf algebra H where all the structure morphisms are positive, and the multiplication and comultiplication are adjoint with respect to the bilinear forms on H and $H \otimes H$.

It turns out that a PSH-algebra is automatically commutative and cocommutative, as the following easy lemma shows.

Lemma 1.12. *If H is a PSH-algebra, then $I = P_H \oplus I_H^2$.*

Proof. In fact, we see show that P_H is the orthogonal complement of I^2 in I. Indeed we have for $x \in I$ and $y \in I \otimes I$,

$$(x, m(y)) = (\Delta(x), y) = (\Delta_+(x), y), x \in I, y \in I \otimes I.$$

thus $x \in I$ is orthogonal to I^2 if and only if $\Delta_+(x) = 0$, that is, $x \in P$. But the pairing $(\cdot, \cdot)$ is positive definite, and so this implies that $P_H \oplus I_H^2 = I$ as required. □

Of course, for us, the crucial examples of PSH-algebras are the algebras $R(q)$, and Λ, viewed as a Hopf algebra built from the representations of the symmetric group. In fact, Zelevinsky proves a structure theorems for PSH-algebras analogous to Proposition 1.10. We say that a primitive element of a PSH-algebra is *irreducible* if it belongs to the basis Ω_H. Given a collection $\{R_\alpha\}_{\alpha in A}$ of PSH-algebras, when A is finite, the tensor product $\bigotimes_{\alpha \in A} R_\alpha$ is clearly a PSH-algebra, while in the infinite case, we may define the (restricted) tensor product to be the colimit,

$$\bigotimes_{\alpha \in A} R_\alpha = \mathrm{colim}_{|B|<\infty} \bigotimes_{\beta \in B} R_\beta,$$

since the family of finite tensor products is clearly an inductive system.

Theorem 1.13. *Let H be a PSH-algebra. Let Ω_P be the set of irreducible primitive elements in H. Then for $\rho \in \Omega_P$, set*

$$\Omega(\rho) = \{\omega \in \Omega : \exists n \geq 0, (\omega, \rho^n) \neq 0\}; \qquad H(\rho) = \bigoplus_{\omega \in \Omega(\rho)} \mathbb{Z}\omega.$$

There is an isomorphism $\bigotimes_{\Omega_P} H(\rho) \to H$.

Proof. A crucial lemma in the proof of this result is to show that if $\pi = \rho_1\rho_2 \dots \rho_r$ and $\pi' = \rho'_1\rho'_2 \dots \rho'_s$ where the ρ_i and ρ'_j are in Ω_P, then $(\pi, \pi') = 0$ unless the ρ_i and ρ'_j are equal up to permutation. This is easy to check using the primitiveness of the ρ_i. □

This theorem shows that the study of PSH-algebras reduces to the study of PSH-algebras where there is exactly one irreducible primitive element. Let R be such an algebra and let ρ be the irreducible primitive element. The previous theorem shows that each basis element of R has nonzero inner product with some power of ρ. This shows that the nonzero graded pieces are in degrees which are a multiple of the $\deg(\rho)$, thus one can assume without loss of generality that $\deg(\rho) = 1$. It turns out R is essentially unique, as the following theorem (also due to Zelevinksy) shows.

Theorem 1.14. *Let R be the PSH-algebra built from the representations of the symmetric groups. Then R has exactly on irreducible primitive element ρ, and exactly one nontrivial automorphism as a PSH-algebra. If R' is any other PSH-algebra with a single irreducible primitive element of degree 1, then there are exactly two isomorphisms $R \cong R'$.*

Proof. To prove the theorem, one studies an abstract PSH-algebra H with a single irreducible primitive element, and shows the following:

(1) $\rho^2 = x_2 + y_2$ for unique elements $x_2, y_2 \in \Omega$.
(2) For each $n \geq 0$ there are unique $x_n, y_n \in \Omega_n$ such that $x_2^*(y_n) = y_2^*(x_n) = 0$ (where x^* is the adjoint operator to multiplication by x).
(3) For $0 \leq k \leq n$ we have $x_k^*(x_n) = x_{n-k}$ and $y_k^*(y_n) = y_{n-k}$. Moreover is $\omega \in \Omega$ is distinct from $\{x_k : 0 \leq k \leq n\}$ then $\omega^*(x_n) = 0$, and similarly for y_n.
(4) For each $n \geq 1$ we have
$$\Delta(x_n) = \sum_{k=0}^{n} x_k \otimes x_{n-k}, \quad \Delta(y_k) = \sum_{k=0} y_k \otimes y_{n-k}.$$
(5) H has a unique nontrivial automorphism t given by $t(x_n) = y_n$ and $t(y_n) = x_n$ for all $n \geq 1$.

One can moreover show that there is a unique primitive element $z_n \in H_n$ such that $(z_n, x_n) = 1$, and the subgroup of primitive elements in H_n is $\mathbb{Z} \cdot z_n$.

It is easy to check that R is an example of such a PSH-algebra: the primitive irreducible is clearly the trivial representation of the trivial group. The antipode is given by tensoring with the sign representation. The x_n and y_n are the trivial and sign representations of S_n, and z_n is $|S_n|/|Z_{S_n}(C_{(n)})|\mathbf{1}_{(n)} = n\mathbf{1}_{(n)}$, where $\mathbf{1}_\lambda$ denotes the characteristic function of the conjugacy class with cycle type λ. □

We may apply these result to the PSH-algebra $R(q)$ to find the following.

Theorem 1.15. *Let $\mathcal{C}_n$ be set of isomorphism classes of irreducible cuspidal representations of the group $GL_n(\mathbb{F}_q)$, and let $\mathcal{C} = \bigsqcup_{n\geq 1} \mathcal{C}_n$. Then we have*

$$R(q) \cong \bigotimes_{\rho\in\mathcal{C}} R(\rho),$$

where $R(\rho)$ is the subspace spanned by irreducibles which are constituents of ρ^n for some n. Moreover, if $\rho \in R(q)_n$, then $R(q)$ is isomorphic, up to a shift of grading by the degree of ρ, to R, the PSH-algebra of representations of the symmetric groups.

Notice that this theorem has given us a parametrization of the irreducible representations in terms of the cuspidal ones (of course it gives much more that this besides): every irreducible of $\mathrm{GL}_n(\mathbb{F}_q)$ parabolically restricts irreducibly to uniquely determined Levi, and on each factor GL_r say, of that Levi, we have $r = md$ and it yields an irreducible representation which has associated to it a cuspidal representation of GL_d, and an irreducible representation χ_λ of some symmetric group, that is, a partition λ.

Remark 1.16. It therefore follows from this result that the endomorphism algebras of the representations $I_L^G(\rho)$ for a cuspidal representation ρ of a Levi subgroup of GL_n are isomorphic to the group algebra of the group $W_G(\rho)$ – the twisting of the group algebras which can happen for arbitrary finite reductive groups does not occur in the case of $\mathrm{GL}_n(\mathbb{F}_q)$.

We want to do more than just parametrize the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ however, we also want to compute their characters. To do this we introduce the algebra of class functions on the general linear groups.

Let $\mathcal{C}_n$ be the space of complex-valued functions on $\mathrm{GL}_n(\mathbb{F}_q)$ which are constant on conjugacy classes, and let $\mathcal{C}(q) = \bigoplus_{n\geq 0} \mathcal{C}_n$ be the direct sum of these spaces. Clearly if we set $R_{\mathbb{C}}(q) = \mathbb{C} \otimes_{\mathbb{Z}} R(q)$, then $R_{\mathbb{C}}(q) \cong \mathcal{C}(q)$ via the map which assigns to a representation its character, a class function on the group. Thus $\mathcal{C}(q)$ is a Hopf algebra. Moreover, $\mathcal{C}(q)$ has a basis given by the characteristic functions of the conjugacy classes. Given a function $\underline{\lambda}\colon \Phi_q \to \mathcal{P}$, we denote the characteristic function of the associated conjugacy class by $\chi_{\underline{\lambda}}$.

Given a conjugacy class in $\mathrm{GL}_n(\mathbb{F}_q)$ indexed by $\underline{\lambda}$, let $V_{\underline{\lambda}}$ be the isomorphism class of $\mathbb{F}_q[t]$-module given by sending t to an element of the conjugacy class of λ. For any three conjugacy classes $\underline{\lambda}$, $\underline{\mu}$ and $\underline{\nu}$, we let $g^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}}$ be the number of submodules $W \subset V_{\underline{\lambda}}$ such that $W \cong V_{\underline{\mu}}$ and $V_{\underline{\lambda}}/W \cong V_{\underline{\nu}}$ as $\mathbb{F}_q[t]$-modules. Let $a_{\underline{\lambda}}$ be the order of the centralizer of an element of the conjugacy class of $\underline{\lambda}$.

Lemma 1.17. *We have*

(1)
$$\chi_{\underline{\mu}}.\chi_{\underline{\nu}} = \sum_{\underline{\lambda}} g^{\underline{\lambda}}_{\underline{\nu}\underline{\mu}} \chi_{\underline{\lambda}}.$$

(2)
$$\Delta(\chi_{\underline{\lambda}}) = \sum_{\underline{\mu},\underline{\nu}} a_{\underline{\lambda}}^{-1} a_{\underline{\mu}} a_{\underline{\nu}} g^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}} \chi_{\underline{\nu}} \otimes \chi_{\underline{\mu}}.$$

Proof. The first of these equations follows immediately from the formula for the character of an parabolically induced representation. The second is a consequence of the first, since comultiplication is the adjoint of multipliction, and we have $(\chi_{\underline{\mu}}, \chi_{\underline{\nu}}) = \delta_{\underline{\mu},\underline{\nu}} a_{\underline{\mu}}^{-1}$ □

Definition 1.18. Let $\mathcal{H}(q)$ be the subalgebra of C consisting of those functions which vanish outside the set of unipotent elements. We have a projection map $p\colon C \to \mathcal{H}(q)$ given by restricting $f \in C_n$ to the set of unipotent elements of $\mathrm{GL}_n(\mathbb{F}_q)$. The algebra $\mathcal{H}(q)$ is called the *Hall algebra*. Let χ_λ be the characteristic function of the unipotent conjugacy class with Jordan blocks of size λ. Thus $\{\chi_\lambda : \lambda \in \mathcal{P}\}$ is a basis of $\mathcal{H}(q)$.

Lemma 1.19. *$\mathcal{H}(q)$ is a Hopf subalgebra of C, and the map p is a morphism of Hopf algebras, so $\mathcal{H}(q)$ is also a quotient of C.*

Proof. Both of these statements are consequences of the previous lemma. □

We now describe the structure of $\mathcal{H}(q)$ more explicitly. To do this, note that if ρ is any one-dimensional representation of $\mathrm{GL}_1(\mathbb{F}_q)$, then ρ is an irreducible cuspidal representation, and so we have the PSH-subalgebra of $R(q)$ corresponding to it $R(\rho)$. Let ι be the trivial representation of $\mathrm{GL}_1(\mathbb{F}_q)$, and let $R(\iota)$ be the PSH-algebra attached to it. Then we know that $R(\iota)$ is isomorphic to R, the algebra of representations of the symmetric groups (this isomorphism can be normalized by requiring the trivial representations of the groups S_n and $\mathrm{GL}_n(\mathbb{F}_q)$ to correspond).

This lemma gives another decomposition of the PSH-algebra $R(q)$. Indeed let $f \in \Phi_q$ and $\lambda \in \mathcal{P}$. Then define $\pi_\lambda(f) = \chi_{\underline{\mu}}$ where $\underline{\mu}(g) = \lambda$ if $g = f$ and $\underline{\mu}(g) = 0$ otherwise.

Proposition 1.20. *Let $p_\iota\colon R_{\mathbb{C}} \to \mathcal{H}(q)$ be the map given by*

$$R_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} R \to \mathbb{C} \otimes_{\mathbb{Z}} R(\iota) \to \mathcal{H}(q),$$

where the first map is given by the isomorphism above, and the second is given by p. Then p_ι is an isomorphism of Hopf algebras.

Proof. To prove the theorem we show the following:

(1) For $n \geq 1$ we have $p_\iota(x_n) = \sum_{\lambda \in \mathcal{P}_n} \chi_\lambda$.
(2) $p_\iota(y_n) = q^{n(n-1)/2}\chi_{(1^n)}$.
(3) $p_\iota(z_n) = \sum_{\lambda \in \mathcal{P}_n} (1-q)(1-q^2)\dots(1-q^{r(\lambda)-1})\chi_\lambda$.

Note that since $\mathbb{C} \otimes_{\mathbb{Z}} R$ is just a polynomial algebra on its primitive elements, to show that p_ι is injective, it is enough to show that $p_\iota(z_n)$ is nonzero for each n, so that the injectivity of p_ι follows from (3). Moreover, since $R_{\mathbb{C}}$ and $\mathcal{H}(q)$ have the same graded dimensions, the injectivity of p_ι in fact implies that it is an isomorphism. Thus we are reduced to proving (1), (2) and (3).

Since $p(x_n)$ is the trivial representation of $\mathrm{GL}_n(\mathbb{F}_q)$, the equation (1) is immediate. Denote by $\tilde{x}_n$ $\tilde{y}_n$ and $\tilde{z}_n$ the right hand sides of equations (1), (2) and (3) respectively. We need to use the q-binomial theorem: For each $n \geq 0$ we set $[n] = (q^n-1)/(q-1)$ and $[n]! = [n][n-1]\dots[1]$. Then the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix} = [n]!/[n-k]![k]! \qquad (0 \leq k \leq n).$$

and where we set $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if k lies outside this range. We claim that

$$\sum_{k=0}^{n} (-1)^k y_k x_{n-k} = 0; \qquad \sum_{k=0}^{n} (n-k) y_k x_{n-k} = z_n.$$

and that the corresponding formulas hold for $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{z}_n$ respectively. Indeed by examining the structure constants of $R_{\mathbb{C}}$ with respect to the basis $\mathbf{1}_\lambda$, and similarly in $\mathcal{H}(q)$ with the basis χ_λ, we see that

$$y_k x_{n-k} = \sum_{\lambda \in \mathcal{P}_n} \binom{\ell(\lambda)}{k} \mathbf{1}_\lambda$$

and similarly that

$$\tilde{y}_k \tilde{x}_{n-k} = \sum_{\lambda \in \mathcal{P}_n} q^{k(k-1)/2} \begin{bmatrix} \ell(\lambda) \\ k \end{bmatrix} \chi_\lambda.$$

Substituting these into the previous equations, we see that the identities can be deduced from

$$\prod_{i=0}^{n-1} (x + q^i y) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^k,$$

and it's derivative with respect to x. □

Remark 1.21. The algebra $\mathcal{H}(q)$ contains a natural integral form $\mathcal{H}_{\mathbb{Z}}(q)$ given by the $\mathbb{Z}$-valued class functions on the unipotent classes. Since it is clear from the above that the structure constants of $\mathcal{H}_{\mathbb{Z}}(q)$ with respect to the basis $\{\chi_\lambda\}_{\lambda \in \mathcal{P}_n}$ are nonnegative integers, $\mathcal{H}(q)$ is a PSH-algebra with basis $\{\chi_\lambda : \lambda \in \mathcal{P}\}$. Since the unipotent radical of a Borel subgroup intersects every unipotent conjugacy class, it is clear that the only primitive irreducible is the characteristic function of the identity element of $\mathrm{GL}_1(\mathbb{F}_q)$. It follows that $\mathcal{H}_{\mathbb{Z}}(q)$ is isomorphic as a PSH-algebra to R, and hence certainly it follows $\mathcal{H}(q)$ is isomorphic to $R \otimes_{\mathbb{Z}} \mathbb{C}$.

Notice that this is *not* the isomorphism we have described above: the inner products on $R \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathcal{H}_{\mathbb{Z}}(q)$ are not intertwined by the map p_ι.

We denote the characteristic functions of unipotent conjugacy classes by χ_λ where λ is the partition of n given by the size of the Jordan blocks of an element of the conjugacy class, and similarly denote the structure constants by $g^\lambda_{\mu,\nu}$. (This is consistent with our previous notation if we identify $\mathcal{P}$ with the functions which take $1 \in \Phi_q$ to λ and every other element of Φ_q to the emptyset).

We now want to show that the Hopf algebra $R(q)$ has another decomposition as a tensor product of Hopf algebras. This we show by examining the structure constants $g^{\underline{\lambda}}_{\underline{\mu},\underline{\nu}}$, $a_{\underline{\lambda}}$.

Lemma 1.22. *Let $\underline{\lambda}, \underline{\nu}, \underline{\nu}$ are functions from $\Phi_q \to \mathcal{P}$ with finite support. Then we have*

$$a_{\underline{\lambda}} = \prod_{f \in \Phi_q} a_{\underline{\lambda}(f)}(q^{\deg(f)}); \qquad g^{\underline{\lambda}}_{\underline{\mu},\underline{\nu}} = \prod_{f \in \Phi_q} g^{\underline{\lambda}(f)}_{\underline{\mu}(f),\underline{\nu}(f)}(q^{\deg(f)}).$$

Moreover

Proof. We only show the first of these identities, the second being a similar, but slightly more elaborate argument. Take $x \in C_{\underline{\lambda}}$ the conjugacy class given by $\underline{\lambda}$. Then we wish to calculate $|Z_{\mathrm{GL}_n}(x)^F|$. But is $x = su = us$ is the Jordan decomposition of x, then if $g \in Z_{\mathrm{GL}_n}(x)$, we must have $g \in Z_{\mathrm{GL}_n}(s)$. But we have already seen that the $\mathbb{F}_q$ points of this group are of the form $\prod_{f:\underline{\lambda}(f)\neq\emptyset} \mathrm{GL}_{|\underline{\lambda}(f)|}(\mathbb{F}_{q^{\deg(f)}})$, where the product corresponds to the decomposition of $\mathbb{F}_q^n$ according to the F-orbits of eigenvalues of s (that is, the $f \in \Phi_q$ for which $\underline{\lambda}(f) \neq \emptyset$). The unipotent u preserves each of these summands of V, and the result follows from the definition of $a_\lambda(q)$.

□

Let $f \in \Phi_q$. We let $\mathcal{H}(f)$ be the subspace of $\mathcal{C}$ spanned by the functions $\chi_{\underline{\lambda}}$ where $\underline{\lambda}(f) \in \mathcal{P}$, but $\underline{\lambda}(f_1) = \emptyset$ for all $f_1 \neq f$. Write λ_f for the function $\Phi_q \to \mathcal{P}$ given by $\lambda_f(g) = \lambda$ if $f = g$ and $\emptyset$ otherwise.

Corollary 1.23. *The subspaces $\mathcal{H}(f)$ are sub-Hopf algebras. Moreover they are mutually orthogonal for the natural inner product on $\mathcal{C}(q)$, and the projection map is a homomorphism of Hopf algebras. Moreover $\mathcal{C} = \bigotimes_{f \in \Phi_q} \mathcal{C}(f)$ (a restricted tensor product).*

Proof. The orthogonality is clear, while the remains claims follow from the previous lemma. □

Remark 1.24. The proof of the Lemma along with the proof of Proposition 1.20 also shows that the algebras $C(f)$ can be identified explicitly with $R_{\mathbb{C}}$. If $l = n/\deg(f)$, then the map, which we denote by c_f, is given by:

$$x_n \mapsto \sum_{\mu \in \mathcal{P}_l} \chi_{\mu_f}; \qquad y_n \mapsto q^{\deg(f)l(l-1)/2} \chi_{(1^l)_f}.$$

and

$$z_n \mapsto \sum_{\lambda \in \mathcal{P}_l} (1-q^l)(1-q^{2l}) \dots (1-q^{(r(\lambda)-1)l}) \chi_{\lambda_f}.$$

2. The construction of Cuspidal Representations

We are finally ready to construct cuspidal characters for GL_n. The isomorphism $R(q) \otimes_{\mathbb{Z}} \mathbb{C} \cong C(q)$ along with the respective tensor product decompositions of these Hopf algebras, shows that the number of irreducible cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ must be the same as the number of irreducible monic polynomials of degree n in $\mathbb{F}_q[t]$. Denote this set of polynomials by $\Phi_q(n)$. The space of primitive elements $P = \bigoplus_n P_n$ of $\mathcal{C}(q)$ has a basis given by $\{c_f(z_n) : n \geq 1, f \in \Phi_q\}$. Let B_C be this basis. Similarly, we know that the primitive elements of $R(q)$ are free on the set

$$\{r_\rho(z_n) : \rho \textit{ irreducible cuspidal}, n \geq 1\},$$

where $r\rho \colon R \to R(\rho)$ is an isomorphism (determined up to the automorphism given by the antipode[2]. We let B_R denote the basis of the space of primitive elements of $R \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathcal{C}(q)$ that this yields. Let $L_q(n)$ be the group of complex characters $\eta \colon \mathbb{F}_{q^n}^* \to \mathbb{C}$. Then via the norm map (or rather its transpose), we may take the limit of the groups $\Theta_q(n)$, which we denote by L_q. Let Γ denote the Galois group of k over $\mathbb{F}_q$. Clearly, Γ acts compatibly on this system, and we let Θ_q denote the set of Galois orbits on L, and $\Theta_q(n)$ the Galois orbits on $L_q(n)$. For $\theta \in L$, let $d(\theta)$ be the size of the Γ-orbit of θ, so that $\Theta_q(n) = \{\phi \in \Theta_q : d(\phi) | n\}$. Note that we may similiarly view $\Phi_q(n)$ as the set of Frobenius orbits on $\mathbb{F}_{q^n}^*$.

The main result about cuspidal representations is the following:

Theorem 2.1. *There is a natural degree preserving bijection $\phi \to \rho(\phi)$ between the sets Θ_q and the set of irreducible cuspidal representations $R(q)^0$. If $\phi \in \Theta_q$ contains $\xi \in L$, and $d = d(\phi)$, then we write $\tilde{p}_n(\phi) = \tilde{p}_n(\xi) = 0$ if d does not divide n, and otherwise set $\tilde{p}_{ld}(\phi) = r_{\rho(\phi)}(z_l)$. The set $\{\tilde{p}_n(\phi) : \phi \in \Theta_q(n)\}$ forms a basis of P_n.*

[2]One can normalize this using the notion of a *nondegenerate* representation, due to Gelfand and Graev.

Moreover, the bases B_C and B_R are related as follows:

$$\tilde{p}_n(\phi) = (-1)^{l(d-1)} \sum_{x \in \mathbb{F}_{q^n}^*} (\xi, x)_n \tilde{p}_n(x),$$

where $\tilde{p}_n(x) = c_f(z_{n/d})$, ($f$ the minimal polynomial of x and $d = \deg(f)$).

Thus we must produce the bijection ρ. To do this we use a beautiful idea of Green and Brauer, which produces *complex* characters of a finite group from modular representations of the group.

Definition 2.2. Let G be a finite group. We say that a subgroup H is *elementary* if, for some prime p, we have $H = CP$ where C is a cyclic group of order prime to p, and P is a p-group. For any finite group G, the subring of the space of complex-valued class functions on G generated by the irreducible complex characters of G is called the *character ring* of G, denoted $\mathrm{Ch}(G)$.

Theorem 2.3. *Let G be a finite group, and ψ a complex-valued class function on G. Then $\psi \in Ch(G)$ if and only if $Res_H^G(\psi) \in Ch(H)$ for every elementary subgroup H of G.*

Thus the elementary subgroups of G detect the character ring of G. We use this to define a lifting, on the level of characters from characters of G in characteristic p to characteristic zero.

Definition 2.4. Let k be, as usual a choice of algebraic closure of $\mathbb{F}_p$. Pick an isomorphism $\theta\colon \mathsf{k}^* \to \mu_p$, between the multiplicative group of k and the group of roots of unity of $\mathbb{C}$ of order prime to p. (Of course here we could restrict ourselves to the algebraic closure of $\mathbb{Q}$.) Let $\phi\colon G \to \mathrm{GL}_n(\mathsf{k})$ be a representation of G on a k-vector space. Then for any symmetric polynomial $f \in \mathbb{Z}[x_1, x_2, \dots, x_n]^{S_n}$ we define a $\mathbb{C}$-valued function $B_\theta(\phi, f)$ on G by setting

$$B_\theta(\phi, f)(g) = f(\theta(x_1), \theta(x_2) \dots, \theta(x_n))$$

where $\{x_i : 1 \leq i \leq n\}$ are the eigenvalues of $\phi(g)$ counted with multiplicity.

Theorem 2.5. *Let θ, f and ϕ be as above, then $B_\theta(\phi, f)$ is in the character ring of G.*

Proof. Notice that since the ring of symmetric polynomials is a polynomial algebra in the elementary symmetric polynomials, by considering tensor products, it is enough to prove the theorem for $f = e_r$ an elementary symmetric polynomial. By replacing ϕ by its exterior powers, it is then enough to prove the theorem for $f = e_1$.

Suppose first that G has order g, prime to p. Since G is finite, we may assume that in fact $\phi\colon G \to \mathrm{GL}_n(\mathbb{F}_q)$ for some q a power of p. But then q is a unit in $\mathbb{Z}/g\mathbb{Z}$, and so g divides $q^r - 1$ where $r = \varphi(g)$ (where φ is Euler's totient function). Thus it follows that the eigenvalues of $x \in G$ all lie in $\mathbb{F}_{q^r}^*$, and hence it is only the restriction of θ to $\mathbb{F}_{q^r}$ which matters. Since the group of characters of $\mathbb{F}_{q^r}^*$ is cyclic, if we prove the theorem for a generator θ_0 of the character group, we will be done (since if f is symmetric, then $f(x_1^k, x_2^k, \dots, x_n^k)$ is also).

Let $\mathfrak{o}$ be the ring of integers of the cyclotomic field generated by the $q^r - 1$-th roots of unity in $\mathbb{C}$. Then let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ which contains p, so that $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_{q^r}$. The group U of roots of unity in $\mathfrak{o}$ maps isomorphically via the residue map to $\mathbb{F}_{q^r}^*$, thus we may choose θ_0 to be the inverse of this isomorphism. Our assumption that p does not divide the order of G implies that the representation

ϕ can be lifted to a representation $\phi_0 \colon G \to \mathrm{GL}_n(\mathfrak{o})$, so that $\phi(x) = \overline{\phi_0(x)}$ for each $x \in G$, where $\overline{m}$ denotes the image of $m \in \mathrm{GL}_n(\mathfrak{o})$ in $\mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}) = \mathrm{GL}_n(\mathbb{F}_q)$. It is then clear that the character of ϕ_0 is $B_{\theta_0}(\phi, e_1)$, and so we are done in this case.

Now let G be an arbitrary finite group. By the previous theorem of Brauer, it is enough to prove the theorem for an elementary group, say $G = CP$ where C is a cyclic group of order prime to l and P is a l-group (where l is prime). Then it is clear that G is the direct product PK of a p-group P and a group K of order prime to p. If $x \in G$, then we may write $x = yz$ where $y \in P$ and $z \in K$. Then since the matrices $\rho(y)$ and $\rho(z)$ can be simultaneously diagonalized, the eigenvalues of x and z are identical, and so $B_\theta(\phi, f)(x) = B_\theta(\phi, f)(z)$. But $B_\theta(\phi, f)_{|K}$ is in the character ring of K, by the above, since the order of K is coprime to p, hence $B_\theta(\phi, f)$ is a character of G as required. □

We now apply this to our situation.

Definition 2.6. Let $\theta \in L$ be a multiplicative character of k. Then for each $n \geq 1$ we define a class function on $\mathrm{GL}_n(\mathsf{k})$ by setting

$$b_n(\theta)(g) = \sum_{i=1}^{n} \theta(x_i)$$

where x_i are the eigenvalues of g (counted with multiplicity). Thus in particular $b_n(\theta)(1) = n\iota_n$, that is, n copies of the trivial representation.

Theorem 2.7. *The functions $b_n(\theta)$ are (virtual) characters of $GL_n(\mathbb{F}_q)$, that is, $b_n(\theta)$ is an integral combination of irreducible complex characters of $GL_n(\mathbb{F}_q)$.*

Proof. This is a simple application of the previous theorem in the case where $G = \mathrm{GL}_n(\mathbb{F}_q)$, and ϕ is the vector representation of G, and $f = e_1$. □

We record the essential properties of the characters $b_n(\theta)$. Let $\xi_n(\theta)$ be the restriction of θ to $\mathbb{F}_{q^n}$, and let $d_n(\theta)$ be the size of the Γ-orbit of $\xi_n(\theta)$. For $\theta, \theta' \in L$, , let

$$\delta_n(\theta, \theta') = \begin{cases} 1 & \text{if } \Gamma.\theta = \Gamma.\theta'; \\ 0 & \text{otherwise.} \end{cases}$$

and set

$$c_n(\theta, \theta') = d_n(\theta)^{-1} n.\delta_n(\theta, \theta'); \qquad c_n(\theta) = \delta_n(\theta, 1).$$

Proposition 2.8. *Let $\theta \in L$, and let $n \geq 1$. Then we have*

(1)
$$\Delta(b_n(\theta)) = \sum_{1 \leq k \leq n} (b_k(\theta) \otimes \iota_{n-k} + \iota_{n-k} \otimes b_k(\theta)).$$

(2) *Let $f \in \Phi_q$ and $d = \deg(f)$, so that $x_1, x_2, \ldots, x_d$ are the roots of f. Then if $r_f \colon \mathcal{C}(q) \to \mathcal{C}(f)$ denotes the orthogonal projection, we have*
$$r_f(b_{ld}(\theta)) = l \sum_{1 \leq i \leq d} (\xi_{ld}(\theta), x_i)_{ld} c_f(x_n)$$

(3) *For $\theta, \theta' \in L$ we have*
$$(b_n(\theta), b_n(\theta')) = \sum_{1 \leq i \leq n} c_i(\theta, \theta') + \sum_{i+j \leq n} c_i(\theta) c_j(\theta'),$$

Proof. The first two properties follow from the definitions. For the last property, one uses an finite field analogue of the Weyl integration formula, due to Kawanaka:

Let G be a reductive algebraic group defined over $\mathbb{F}_q$, and let $T_1, T_2, \dots, T_s$ be representatives of the rational conjugacy classes of rational maximal tori in G. Then if $u(g), v(g)$ are class functions on G^F which depend only on the semisimple part of g, we have

$$(u, v)_G = \prod_{i=1}^{s} |W_i|^{-1}(u, v)_{T_i}$$

where W_i is the Weyl group of T_i, that is $W_i = N_G(T_i)^F / T_i^F$, □

The crucial theorem is then the following. For $\theta \in L$ and $n \geq 1$ define $p_n(\theta) \in \mathcal{C}_n$ by

$$p_n(\theta) = \sum_{0 \leq i \leq n-1} (-1)^i \varepsilon_i b_{n-i}(\theta),$$

(here ε_i is the image of y_i under the map $R \to R(\iota)$ where ι is the trivial representation of GL_1).

Remark 2.9. The motivation for the definition of $p_n(\theta)$ comes from the work of Lusztig [L] on Brauer lifting at the level of representations. There he constructs a complex of characteristic zero representations $\{X_i : 1 \leq i \leq n\}$ such that the Brauer lift of the vector representation of $\mathrm{GL}_n(\mathbb{F}_q)$ is

$$X_1 - X_2 + \ldots + (-1)^{n-1} X_n,$$

and moreover the X_i are irreducible if $q > 2$ or $i > 1$, the representation X_n is cuspidal.

Theorem 2.10. *The function $p_n(\theta)$ is a primitive element of R_n. Moreover we have*

$$(p_n(\theta), p_n(\theta')) = c_n(\theta, \theta'),$$

and

$$p_n(\theta) = \sum_{x \in \mathbb{F}_{q^n}^*} (\xi_n(\theta), x) \tilde{p}_n(x).$$

The previous theorem contains enough information to allow us to compute the character table of $\mathrm{GL}_n(\mathbb{F}_q)$. Indeed, if ρ is cuspidal, the irreducible representations in $R(\rho)$ are $r_\rho(s_\lambda)$ where $r_\rho \colon R \to R(\rho)$ is an isomorphism from the PSH-algebra R to $R(\rho)$, and $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the irreducible representations of the symmetric groups. Identifying R with Λ, the ring of symmetric functions, these are the Schur functions discussed earlier. The elements $\tilde{p}_n(\theta)$ correspond to the basis of the submodule of primitive element of R, which in terms of symmetric functions are the power symmetric functions.

To give the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is to give the change of basis matrix between the basis $\{\chi_{\underline{\lambda}}\}_{\underline{\lambda} \colon \Phi_q \to \mathcal{P}}\}$ of characteristic functions of conjugacy classes, and $\{\rho_{\underline{\mu}}\}$ the basis of irreducible characters of $\mathcal{C}(q)$ (where $\underline{\mu}$ runs over partition valued functions on Θ_q the set of Galois orbits on $\mathsf{k}^\times$ – this set naturally indexes the irreducible representations of the groups $\mathrm{GL}_n(\mathbb{F}_q)$ via our description of the cuspidal irreducible representations, indeed each irreducible is of the form $\prod_{\theta \in \Theta_q} r_{\rho(\theta)}(s_{\underline{\mu}(\theta)})$.

Now we have two auxilliary bases of $\mathcal{C}(q)$: Given a function $\underline{\lambda}\colon \Phi_q \to \mathcal{P}$, we may define $p_{\underline{\lambda}} = \prod_{f\in\Phi_q} r_f(z_{\underline{\lambda}(f)})$ where if ν is a partition, we set $z_\nu = \prod_i z_{\nu_i}$. Similarly we can define, for $\mu\colon \Theta_q \to \mathcal{P}$ an element $p_{\underline{\mu}} = \prod_{\theta\in\Theta_q} r_{\rho(\theta)}(z_{\underline{\mu}(\theta)})$. Then each of the sets $\{p_{\underline{\lambda}}\}$ and $\{p_{\underline{\mu}}\}$ are bases of $\mathcal{C}(q)$. Moreover, the previous theorem allows us to compute the change of basis matrix between then via the pairing between $\mathsf{k}^\times$ and its group of multiplicative characers.

The change of basis matrix between $s_{\underline{\mu}}$ and $p_{\underline{\mu}}$ is given (in terms of Λ) by the change of basis matrix between the power symmetric functions and the power symmetric functions, or (in terms of R) by the character tables of the symmetric groups, which is explicitly known. On the other hand, the change of basis matrix between $\{\chi_{\underline{\lambda}}\}$ and $\{p_{\underline{\mu}}\}$ involves the Hall algebra. Indeed in showing that $\mathcal{H}(q)$ is isomorphic to $R\otimes_{\mathbb{Z}}\mathbb{C}$, we established explicitly the image of the primitive elements $p_\iota(z_n)$, and it is the change of basis matrix between the elements $p_\iota(z_\lambda)$ and the χ_λ which controls this last piece of the computation of the character table.

Definition 2.11. For $\lambda, \mu \in \mathcal{P}$ with $|\lambda| = |\mu|$ we set

$$p_\iota(z_\mu) = \sum_\lambda Q(\lambda,\mu)\chi_\lambda.$$

It turns out that these numbers $Q(\lambda,\mu)$ are polynomials in q. These polynomials are known as *Green polynomials*, and turn out to be pivotal in the development of a more general theory of characters for reductive groups. They are characterized by the following conditions:

Definition 2.12. For each ordered pair (λ,μ) of partitions $|\lambda| = |\mu| \leq n$, there is a unique integer $Q(\lambda,\mu)$ such that

(1) $Q(\lambda,\mu)$ is a polynomial function in q with integral coefficients;
(2) $Q(\lambda,\mu+\nu) = \sum_{\rho,\sigma} g^\lambda_{\rho,\sigma} Q(\rho,\mu)Q(\sigma,\nu)$;
(3) $\sum_\mu z_\mu^{-1} Q(\lambda,\mu) = 1$ for all λ;
(4) (*orthogonality relations*): We have

$$\sum_\lambda a_\lambda^{-1} Q(\lambda,\rho)Q(\lambda,\sigma) = m_\rho^{-1} z_\rho \delta_{\rho,\sigma}.$$

(5) If λ is a partition of k and $r = \ell(\lambda)$, then $Q(\{k\},\lambda) = 1$, $Q(\lambda,\{k\}) = \phi_{r-1}(q)$ and $Q(\{1^k\},\lambda) = (-1)^r m_\lambda^{-1}\phi_k(q)$.

$Q(\lambda,\mu)$ is already uniquely determined by the second and third of these conditions.

One can show that once the Green polynomials are known, the characters of $\mathrm{GL}_n(\mathbb{F}_q)$ are determined by their values on the regular semisimple set.

3. Intersection Cohomology and characters of finite reductive groups

We begin with a discussion of the case of a torus T. Since a torus is just a product of $\mathbb{G}_m$s, there is little lost in assuming that $T = \mathbb{G}_m$, equipped with the standard Frobenius structure $F\colon T \to T$ given by $x \mapsto x^q$. The finite group T^F is simply a cyclic group with $q-1$ elements, so its characteristic zero representation theory is elementary. We wish, however, to give a "geometric" account of it. In order to do this we need to use the theory of etale cohomology (see [Mil] for an introduction to this theory – a more standard reference is SGA4.5).

For each $m \in \mathbb{N}$ coprime to p, the characteristic of k, we have a map $\pi_m : \mathbb{G}_m \to \mathbb{G}_m$ given by $\pi_m(t) = t^m$. Then π_m exhibits $\mathbb{G}_m$ as a Galois cover of itself, with group $\mu_m(\mathsf{k})$ the m-th roots of unity in k (this is a cyclic group of order m, because of our assumption on m). Let l be a prime different from p, and consider the constant sheaf in the etale topology on $\mathbb{G}_m$. Then since π_m is a Galois covering, $(\pi_m)_!(\mathbb{Q}_l)$ is a local system on $\mathbb{G}_m$[3], which splits as a direct sum of one-dimensional local systems, according to the characters of the group $\mu_m(\mathsf{k})$:

$$\pi_n(\mathbb{Q}_l) = \bigoplus_{\rho \in \hat{\mu_m(\mathsf{k})}} \mathcal{E}_\rho.$$

Let $\mathcal{S}$ be the collection of one-dimensional local systems obtained in this way (for all m coprime to p). Now if a local system $\mathcal{E}_\rho \in \mathcal{S}$ is F-stable, that is, if $F^*(\mathcal{E}_\rho)$ is isomorphic to $\mathcal{E}_\rho$ itself, then, since the local system is one-dimensional, this isomorphism is unique up to scalar. Picking such an isomorphism $\phi\colon F^*(\mathcal{E}_\rho) \to \mathcal{E}_\rho$, we obtain a $\mathbb{Q}_l$-valued function $\chi_{\mathcal{E}_\rho,\phi}$ on the fixed points of F,

$$\chi_{\mathcal{E}_\rho,\phi}(x) = \sum_{i \in \mathbb{N}} (-1)^i tr(\phi_x, \mathcal{H}^i_x(\mathcal{E}_\rho)),$$

where $\mathcal{H}^i_x(\mathcal{E}_\rho)$ denotes the stalk cohomology of $\mathcal{E}_\rho$ at $x \in T^F$.

It can then be shown that the collection of functions $\chi_{\mathcal{E}_\rho,\phi}$ as $\mathcal{E}_\rho$ runs over the F-stable local systems in $\mathcal{S}$ gives, up to a scalar multiple, the irreducible characters of T^F. (This is not a hard result, we state it simply because it gives a geometric rephrasing of the character theory of T – note for example, that there is an obvious analog of the set of local systems $\mathcal{S}$ over $\mathbb{C}$, whereas there is no clear analog of the finite group T^F).

We would like a similar geometric picture for the character theory of $G = \mathrm{GL}_n(\mathbb{F}_q)$. Now the character values of irreducible representations are computable by the theory developed by Green, and by examining the answers, one can see two things:

- On the regular semisimple set, the character values of can be attached to local systems in a way similar to, but more elaborate than, the case of the torus.
- The character values on the entire group are determined by the values on the regular semisimple set.

We first describe the local systems: the character values on regular semisimple elements are combinations of terms which correspond to local systems on the torus, and character values of the symmetric group. However, if G^{rss} is the regular semisimple set, these ingredients arise naturally from the geometry of G^{rs}. Let T denote the standard maximal torus in G. Given a regular semisimple element $x \in G$, clearly x has n distinct eigenlines. Picking an ordering of these lines, we get an element of T, by taking the diagonal matrix with i-th entry the i-th eigenvalue of x. Clearly this element of T is conjugate to x in G.

[3] here "local system" means etale local system, that is, each stalk is finite dimensional, and every point has an etale neighborhood on which the sheaf is constant

Let $\tilde{G}^{rs}$ be the set of pair $\{(x,t) \in G^{rs} \times T : t = gxg^{-1}, \text{ some } g \in G\}$. Then there are natural maps $p_1 \colon \tilde{G}^{rs} \to G^{rs}$ and $p_2 \colon \tilde{G}^{rs} \to T$. Moreover, the map p_1 is a principal bundle with structure group S_n the symmetric group (it acts by permuting ordering of the eigenlines), and the map p_2 is a fibration with fiber G/T (since the centralizer of a regular semisimple element in G is a maximal torus) and so in particular a smooth map. Now given a local system $\mathcal{E}$ on T, we may pull it back via p_2 to obtain a local system on $\tilde{G}^{rs}$, and then since p_1 is an S_n-bundle, we may push the resulting local system forward to G^{rs} to obtain a local system $(p_1)_! p_2^*(\mathcal{E})$ on G^{rs}. Now S_n acts by endomorphism on this local system, fixing the base G^{rs}. This action allows us to decompose $(p_1)_! p_2^*(\mathcal{E})$ according to the representations of the symmetric group:

$$(p_1)_! p_2^*(\mathcal{E}) = \bigoplus_{\lambda \in \mathcal{P}_n} \mathcal{E}_\lambda \otimes V_\lambda,$$

where each $\mathcal{E}_\lambda$ is a one-dimesional local system. Let $\pi_m \colon T \to T$ be the map $t \mapsto t^m$. Then just as for $\mathbb{G}_m$, we can define a family of local systems $\mathcal{S}$ on T given by taking the local systems occuring in $\pi_n(\mathbb{Q}_l)$ for g.c.d.$(n,p) = 1$. Then using the procedure just described we get a collection of local systems $\{\mathcal{E}_\lambda : \mathcal{E} \in \mathcal{S}, \lambda \in \mathcal{P}_n\}$.

Just as for the case of a torus, it can be shown that if we consider the elements of this set which are F-stable, then taking traces of Frobenius, we obtain the values of the irreducible characters at the regular semisimple elements (again up to a scalar factor).

However, so far we have only obtained the character values on the regular semisimple elements – the question of what the values of the characters on other conjugacy classes is considerably deeper. The remarkable phenomenon here is that the rest of the character values are in some sense contained in the singularities of these local system around the complement of the regular semisimple elements. To understand this precisely, a new geometric construction is required – that of intersection cohomology.

Suppose that X is a (compact) variety. Then if X is singular, the (co)homology of X (etale cohomology if we are in positive characteristic, but ordinary, say, singular cohomology when we work with $\mathbb{C}$) is less well behaved than it is for smooth varieties: indeed for smooth varieties the cup product on cohomology groups, gives a duality map for groups in complementary dimensions, and corresponds geometrically to an intersection pairing on cycles (this is essentially a consequence of Poincare duality). Seeking invariants of singular spaces which are better behaved than ordinary homology theories, Goresky and MacPherson introduced Intersection Homology. In their initial definition of these groups, one introduces a stratification of X – that is, a partition of X into locally closed pieces such that the closure of a stratum is a union of other strata (and, normally, some more technical conditions). The the intersection homology groups (loosely) are defined by studying a complex who's groups consist of cycles in X which intersect the singular locus "not too perversely".

Shortly thereafter, Deligne pointed out that their construction could be viewed as providing X with a complex of sheaves whose hypercohomology gives the Intersection (co)homology groups. This complex is *constructible*. A sheaf $\mathcal{F}$ if said to be constructible if there is a stratification of the variety X such that when we restrict $\mathcal{F}$ to a stratum, the resulting sheaf is locally constant. A complex of sheaves is said to be constructible if each of its cohomology sheaves are. Constructible complexes of sheaves arise naturally when one studies the push-forward of local systems via maps which are not principal bundles.

Deligne's observation is enlightening because it emphasizes that one only needs to understand the intersection cohomology complex in the derived category, that is, up to quasi-isomorphism. Indeed he went on to give a construction of a complex which is quasi-isomorphic to the sheaf given by Goresky and MacPherson, starting with the constant sheaf on the dense open stratum (which is often taken to be the smooth locus of X) and extending to all of X stratum by stratum of some stratification of X (in the end the complex thus obtained does not depend on the choice of stratification, at least up to quasi-isomorphism). This construction has the advantage that it makes sense for *any* local system on the dense open stratum, not just the constant local system. Moreover, it makes sense in the context of etale sheaves, so that we may define intersection cohomology groups for varieties over k. This gives a canonical extension of any local system $\mathcal{E}$ on a dense open subvariety of X to an object $IC(\mathcal{E})$ of $D^b_c(X)$, the derived category of constructible sheaves on X (for more details on this, and much more, see [BBD]).

The first suggestion that intesection cohomology sheaves might be important in the study of the character theory of $\mathrm{GL}_n(\mathbb{F}_q)$ is the following beautiful theorem of Lusztig. Recall that the irreducible constituents of $I_T^{GL_n}(1)$ are indexed by partitions of n (they are just the n-th degree part of $R(\iota)$ in the notation of the last section). These representations are called the *unipotent representations* of $GL_n(\mathbb{F}_q)$; we denote them as $\{E_\lambda : \lambda \in \mathcal{P}_n\}$. Let X be the unipotent variety in G, so that $X = \bigsqcup_{\lambda\in\mathcal{P}_n}$ (in fact, this gives an example of a stratification of a variety). Let $n(\lambda) = \sum_{i\geq 0}(i-1)\lambda_i$ be the dimension of X_λ, and $IC(X_\lambda)$ the intersection cohomology sheaf on $\bar{X}_\lambda$ (extended by zero to all of X).

Theorem 3.1. *(Lusztig* [L81]*). Let X be the variety of unipotent elements, so that $X = X_\lambda$, and let $u \in X(\mathbb{F}_q)$. Then*

$$Tr(u, E_\lambda) = q^{n(\lambda)} \sum_{i\in\mathbb{Z}} q^{i/2} \dim(\mathcal{H}^i_u(IC(X_\lambda)),$$

moreover $\mathcal{H}^i_u(IC(X_\lambda)) = 0$ if i is odd.

Finally, we give a description of all whole of the character table of $\mathrm{GL}_n(\mathbb{F}_q)$. Given a local system $\mathcal{E}$ on G^{rs} in our collection $\mathcal{L}$ such that there is an isomorphism $F^*(\mathcal{E}) \to \mathcal{E}$. Then we may consider $A = IC(\mathcal{E})$, a constructible complex on G. It can be shown that ϕ extends uniquely to an isomorphism $\phi\colon F^*(A) \to A$, and one can thus define a function $\chi_{A,\phi}$ on G^F just as we did for local systems on the torus.

Theorem 3.2. *(Lusztig): Let $\mathcal{L}^F$ be the collection of local systems which are F-stable. The (up to scalars) the functions $\chi_{A,\phi}$ as above are the irreducible characters of G^F.*

Remark 3.3. The proof of the theorem is contained in the works of Lusztig on character sheaves.

References

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux Pervers*. Asterisque.

[B] A. Borel, *Linear algebraic groups*. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991. xii+288 pp. ISBN: 0-387-97370-2

[DL] P. Deligne, G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DM] F. Digne, J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, 21. Cambridge University Press, Cambridge, 1991. iv+159 pp. ISBN: 0-521-40117-8; 0-521-40648-X

[G] J.A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.

[HL] R.B. Howlett, G.I. Lehrer, *Induced cuspidal representations and generalised Hecke rings*. Invent. Math. **58** (1980), no. 1, 37–64.

[H] J. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975. xiv+247 pp.

[L] G. Lusztig, *The discrete series of GL_n over a finite field*. Annals of Mathematics Studies, No. 81. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. v+99 pp.

[L81] G. Lusztig *Green polynomials and singularities of unipotent classes*, Adv. Math. **42** (1981), 169–178.

[Mil] J.S. Milne, *Lectures on Etale Cohomology*, notes available online at http://www.jmilne.org/math/.

[MM] J.W. Milnor, J.C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** 1965 211–264.

[Mo] G. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. (**78**) no. 1, (1956), 200–221.

[M] P. Muller, *Algebraic groups over finite fields, a quick proof of Lang's theorem*, Proc. Amer. Math. Soc. **137**, no.2, 369–370.

[Q] D. Quillen, *Cohomology of groups*, Actes du Congrs International des Mathmaticiens (Nice, 1970), Tome 2, pp. 47–51. Gauthier-Villars, Paris, 1971.

[S] T. A. Springer, *Linear algebraic groups*, Second edition. Progress in Mathematics, 9. BirkhŁuser Boston, Inc., Boston, MA, 1998. xiv+334 pp. ISBN: 0-8176-4021-5

[St68] R. Steinberg, *Endomorphisms of algebraic groups*, Memoirs of the American Mathematical Society, No.**80** American Mathematical Society, Providence, R.I. (1968) 108 pp.

[St99] R. Steinberg, *The isomorphism and isogeny theorems for reductive algebraic groups*, J. Algebra **216** (1999), no. 1, 366–383. 20G15

[T] J. Tits, *Algebraic and abstract simple groups*. Ann. of Math. (2) **80** (1964) 313–329.

[Z] A. Zelevinsky, *Representations of Finite Classical Groups*, Lecture Notes in Mathematics 869, Springer-Verlag, (1981).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO.