ON CHARACTER SHEAVES.

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1. THE DEFINITION OF CHARACTER SHEAVES.

Fix k an algebraically closed field of characteristic p. We begin by recalling the definition of character sheaves, in the manner of [MV] (most of the results we state here are either from or in the form given by [MV]). Let G be a connected algebraic group, and H a connected subgroup, and X a G-variety. Let denote by $D_G(X)$ the G-equivariant derived category of ℓ-adic constructible sheaves on X. The forgetful functor from $D_G(X)$ to $D_H(X)$ has a right adjoint Γ_H^G defined by the following diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\nu"] \arrow[d, "a"'] & G/H \times X \arrow[d, "p"] \\ X & X \end{tikzcd}$$

where p is the projection map, a is given by $a(g,x) = g^{-1}.x$, and ν is given by $\nu(g,x) = (gH, x)$. Then if $F \in D_H(X)$, there is a unique $\tilde{F} \in D_G(G/H \times X)$ such that $a^*(F) = \nu^*(\tilde{F})$. Then one sets $\Gamma_H^G(F) = p_*(\tilde{F})$. Alternatively one can use the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "p"'] & G \times_H X \arrow[d, "b"] \\ X & X \end{tikzcd}$$

where b is induced by the action map, and p is the projection. In this setting we have $\Gamma_H^G(F) = b_*(\tilde{F})$ where $p^*(F) = \mu^*(\tilde{F})$.

Remark 1.1. In the language of stacks, there is a natural map between quotient stacks $p\colon X/H \to X/G$. The forgetful functor and Γ_H^G are the associated pull-back and push-forward operations. Note that Γ_H^G does not preserve perverse sheaves in general.

Remark 1.2. Given a morphism $f\colon X \to Y$ between varieties, we set $f^0 = f^![\dim(Y) - \dim(X)]$ (in both the equivariant or nonequivariant contexts). If Z is an H-variety and $Y = G \times_H Z$, with $i\colon Z \to Y$ the obvious embedding, then i^0 gives an equivalence between $D_G(Y)$ and $D_H(Z)$ with inverse $\Gamma_H^G \circ i_*$ (see for example [MV] for a simple proof).

We now use this functor to define character sheaves on G, a connected reductive algebraic group over k. Fix a pair $T_0 \subset B_0$ of a maximal torus of G and a Borel

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subgroup containing it, and let $n = \dim(G/B_0)$. Then we have a corresponding Bruhat decomposition

$$G = \bigsqcup_{w \in W} B_0 \dot{w} B_0,$$

where $W = N_G(T_0)/T_0$ and $\dot{w}$ is a choice of representative of $w \in W$ in $N_G(T_0)$. Now the map $i_w \colon T_0 \to B_0\dot{w}B_0$ given by $t \mapsto \dot{w}t$ is an embedding, and moreover the refined Bruhat decomposition shows that $B\dot{w}B = U_0^w T U_0$ where U^w is the product of the root subgroups U_α of $U_0 = R_u(B_0)$ for which $\dot{w}U_\alpha\dot{w}^{-1}$ does not lie in U_0. It follows that i_w^0 provides a bijection between B-equivariant local systems on $B\dot{w}B$ and one-dimensional local systems $\mathcal{L}$ on T fixed by w. Let j_w denote the locally closed embedding of $B\dot{w}B$ into G (thus j_w is an affine embedding).

Definition 1.3. (*Lusztig* [LC1]): Given a tame[1] local system $\mathcal{L}$ on T for which $w^*(\mathcal{L}) \cong \mathcal{L}$ we may view $\mathcal{L}$ as a B-equivariant local system on $B\dot{w}B$, and define a complex $K_w^{\mathcal{L}} = \Gamma_{B_0}^G((j_w)_!(\mathcal{L}))[n]$. The irreducible perverse constituents of $K_w^{\mathcal{L}}$ as $\mathcal{L}$ runs over one-dimensional local systems in T and w runs over the Weyl group, are defined to be the character sheaves of G.

If we let $\mathcal{C}_{B_0}$ be the full subcategory of $D_{B_0}(G)$ (where B_0 acts by conjugation on G), consisting of complexes constructible with respect to the Bruhat stratification of G, then it is not hard to show that the character sheaves on G are just the irreducible constituents of all sheaves $\Gamma_{B_0}^G(F)$ where $F \in \mathcal{C}_{B_0}$. It is easy to see that if $\Gamma^{U_0} = \Gamma_{\{1\}}^{U_0}$, then $\Gamma^{U_0} = \pi^*\pi_*$ where $\pi \colon G \to G/U_0$, and moreover Γ^{U_0} takes character sheaves to $\mathcal{C}_{B_0}$.

2. AN ALTERNATIVE DEFINITION OF CHARACTER SHEAVES

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We now give a rephrasing of the definition of character sheaves, as found in the works of Mirkovic-Vilonen [MV] and Ginzburg [G], further developed by Grojnowski [Gr] in his thesis. Let U_0 be the unipotent radical of B_0, and consider $Y = (G/U_0 \times G/U_0)/T_0$, where T_0 acts on the right on both factors: $(gU, hU).t = (gtU, htU)$. More invariantly, we can view Y as the *horocycle space*

$$\{(B, xU) : x \in G, B \in \mathcal{B}, U = R_u(B)\},$$

where $\mathcal{B}$ is the flag variety of G, with the isomorphism given by $(gU, hU)T \mapsto (gBg^{-1}, hg^{-1}(gUg^{-1}))$. Note that this identification also shows that $Y = G \times_{B_0} G/U_0$ (where B_0 acts by conjugation on G), so there is an equivalence of categories $D_G(Y) \cong D_B(G/U)$, where if $i_Y : G/U \to Y$ denotes the embedding of G/U_0 in Y, the equivalences are given by $\beta = \Gamma_{B_0}^G \circ (i_Y)_!$ and $\alpha = i_Y^0$.

We have a transform

$$\begin{tikzcd}
& G \times \mathcal{B} \arrow[dl, "q"'] \arrow[dr, "r"] & \\
G & & Y
\end{tikzcd}$$

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[1] To be precise, we take the local systems of the from $\omega^*(\mathcal{E})$, where $\omega \colon T_0 \to \mathbb{G}_m$ is a character of T_0, and $\mathcal{E}$ is a one-dimensional local system which occurs as a constituent of $\rho_n(\bar{\mathbb{Q}}_l)$ where ρ_n is the map $t \mapsto t^n$ from $\mathbb{G}_m$ to itself with n relatively prime to the characteristic of k

where $r(g, {}^xB_0) = (gU_0, xU_0)T$, and $p(g, {}^xB) = g$. We define two functors $R\colon D(G) \to D(Y)$ and $\check{R}\colon D(Y) \to D(G)$, where $R = r_!q^0$ and $\check{R} = q_!r^0$ (the functor R is known as the Radon transform or horocycle transform – Ginzburg in [G] considers a similar situation where he calls $\check{R}$ the *Harish-Chandra* functor). These functors (with our normalizations) are adjoints. Since the diagram defining $\check{R}$ is G-equivariant, it is not hard to see that $\Gamma_B^G \circ \pi^0 = \check{R} \circ \beta$ (up to shifts).

Notice that T acts freely on Y, via $t \cdot (gU, hU)T = (gtU, hU)T$, so that if $\mathcal{L}$ is a local system on T, we may speak of $\mathcal{L}$-monodromic objects in $D(Y)$. This allows us to give an alternative definition of character sheaves:

Definition 2.1. ([MV], [G]) Let $D_G^{\mathcal{L}}(Y)$ be the full subcategory of $\mathcal{L}$-monodromic equivariant sheaves on Y. Let $\hat{G}_{\mathcal{L}}$ be the irreducible constituents of $\check{R}(A)$ where A is a complex in $D_B^{\mathcal{L}}(Y)$. We write $\mathcal{M}_G^{\mathcal{L}}(G)$ for the full subcategory of G-equivariant perverse sheaves on G whose simple constituents lie in $\hat{G}_{\mathcal{L}}$. The character sheaves on G are the set $\hat{G}$ where

$$\hat{G} = \bigcup_{\mathcal{L}} \hat{G}_{\mathcal{L}}.$$

Lemma 2.2. *The two definitions of character sheaves coincide.*

Proof. Recall that Lusztig's definition can be phrased as follows: the character sheaves on G are the irreducible perverse constituents of the sheaves $\Gamma_{B_0}^G(F)$ where $F \in \mathcal{C}_{B_0}$. Now the category $\mathcal{C}_{B_0}$ is equivalent to the category $\mathcal{C}'_{B_0}$ of B_0-equivariant sheaves on G/U_0 constructible with respect to the Bruhat stratification, via the functor π^0. The lemma then follows immediately from the observation made above that $\Gamma_{B_0}^G \circ \pi^0 = \check{R} \circ \beta$ (up to shift). ☐

Thus our second definition is easily seen to be equivalent to the original definition. We discribe it here because it allows for a simpler discussion of the rough classification of character sheaves.

We write $\check{G}_{\mathcal{L}}$ for the irreducible perverse sheaves occuring as constituents of $\check{R}(F)$ for $F \in D_G^{\mathcal{L}}(Y)$. Since $G \times T$ acts on Y with finitely many orbits the category $\mathcal{M}_G^{\mathcal{L}}(Y)$ of G-equivariant $\mathcal{L}$-monodromic perverse sheaves on Y has finitely many objects. It follows that $\check{G}_{\mathcal{L}}$ also has finitely many objects.

3. CLASSIFICATION OF CHARACTER SHEAVES: CENTRAL CHARACTERS

We wish to discuss the classification of character sheaves on the group G. Our discussion follows [MV], [G] and [Gr]. The first invariant of a character sheaf is it's "central character" – a W-orbit of a local system on T_0.

Recall the functors R and $\check{R}$ from the previous section. We begin with two straight-forward but very useful results on their composition. Given a local system $\mathcal{L}$ we write $W'_{\mathcal{L}} = \{w \in W : w^*(\mathcal{L}) = \mathcal{L}\}$. This is a subgroup of W which is not in general a Coxeter group, however if $W_{\mathcal{L}}$ denotes the group generated by reflections in W perserving $\mathcal{L}$, then $W_{\mathcal{L}}$ is a normal subgroup which is naturally a Coxeter group, and the quotient is a cyclic group.

Proposition 3.1. *Let $\mathcal{L}$ be a tame local system on T_0. The composition of functors $R \circ \check{R}$ restricts to a functor*

$$R \circ \check{R}\colon D_G^{\mathcal{L}}(Y) \to \bigoplus_{w \in W/W'_{\mathcal{L}}} D_G^{w^*(\mathcal{L})}(Y).$$

Proof. This is proved, for example, in [Gr]. It suffices to prove the proposition for monodromic sheaves in the non-equivariant setting. Let $Z = G \times \mathcal{B} \times \mathcal{B}$, and let q_i be the map given by $(g, h_1U_0, h_2U_0) \mapsto (gh_iU_0, gh_iU_0)T_0$ for $i = 1, 2$. Then $R \circ \check{R}$ is given by $(q_1)_!(q_2)^0$. Partitioning Z into pieces Z_w according to the G-orbits in $\mathcal{B} \times \mathcal{B}$ and using a long exact sequence, one reduces to showing that if q_{iw} are the restrictions of q_i to the pieces of the partition, then $(q_{wi})_!(q_{w2})^0\colon D^{\mathcal{L}}(Z_w) \to D^{w^*(\mathcal{L})}(Z_w)$. This is shown using the refined Bruhat decomposition. □

It follows from this proposition that if $A \in \check{G}_{\mathcal{L}}$ we may recover the W-orbit of $\mathcal{L}$ from A – simply any simple perverse constituent of $R(A)$ and find the W-orbit of its monodromy. This invariant of A is called its *central character*.

It turns out that the composition $\check{R} \circ R$ can be precisely described in terms of convolution on G: Let $m\colon G \to G$ be the multiplication map, and let $\star$ be the convolution operation on G given by

$$F_1 \star F_2 = m_!(F_1 \boxtimes F_2).$$

Let $\mathcal{U}$ be the unipotent cone in G. Then if $\tilde{\mathcal{U}} = \{(x, B) \in G \times \mathcal{B} : x \in R_u(B)\}$ the map $\mu\colon \tilde{\mathcal{U}} \to \mathcal{U}$ is a resolution of singularities, and [BM] $\mu_!(\mathbb{Q}_l[\dim(\mathcal{U})]$ is a semisimple perverse sheaf on $\mathcal{U}$. Moreover it is known that δ_e, the skyscraper sheaf at the identity, is a constituent of $\mathcal{S}$, so that we may write $\mathcal{S} = \delta_e \oplus \mathcal{S}'$ for $\mathcal{S}'$ a semisimple perverse sheaf.

Proposition 3.2. ([MV]*) The composition of functors* $\check{R} \circ R$ *is given on* G *by*

$$\check{R} \circ R(F) = F \star \mathcal{S} = F \oplus (F \star \mathcal{S}')$$

hence in particular, the identity functor is a summand of $\check{R} \circ R$.

Proof. This is a diagram chase using base change, the projection formula. □

Remark 3.3. Mirkovic and Vilonen [MV] also show that $\check{R} \circ R = \Gamma^G_{B_0} \circ \Gamma^{U_0}$. The previous proposition is the crucial ingredient in their proof that character sheaves in characteristic zero can be characterized by their characteristic varieties. For us it will yield the partition of character sheaves with a fixed central character into two-sided cells.

4. UNIPOTENT CHARACTER SHEAVES

We now specialize to the case where $\mathcal{L}$ is trivial, that is, to the case of unipotent character sheaves. By defining parabolic induction and restriction functors on character sheaves one can reduce most calculations to the case of trivial central character, so this is not an enormous loss of generality. In this case, the functor R takes $\hat{G}_1$ to $D^{\mathbb{Q}_l}_G(Y) \cong D_G(\mathcal{B} \times \mathcal{B})$. Now since G acts with finitely many orbits on $\mathcal{B} \times \mathcal{B}$, the simple G-equivariant perverse objects are simply the middle extension of the constant local systems on the orbits. Using convolution on $\mathcal{B} \times \mathcal{B}$, Lusztig decomposes these simple objects into *two-sided cells*, and then within that parition, into *left cells* and *right cells*.

We describe Lusztig's partition of the simple objects into cells, which can then be used to give a partition of the unipotent character sheaves into two-sided cells – an analogous partition exists for the character sheaves with any central character, where the group W must be replaced by the group $W'_{\mathcal{L}}$.

For this we use mixed complexes – that is, we assume that our group G arises from a split form G_0 over $\mathbb{F}_q$ for some sufficiently large q, and work with mixed complexes over G_0. Then the perverse cohomologies ${}^pH^i(A)$ of a sheaf A have a weight filtration with subquotients ${}^pH^i_j$ pure perverse sheaves of weight j. Then for any mixed complex A we define

$$\chi_v(A) = \sum_{i,j} (-1)^i \{{}^pH^i_j(A)\} v^j,$$

an object in $K(G)\otimes\mathbb{Z}[v, v^{-1}]$ where $K(G)$ is the Grothendieck group of the category of perverse sheaves on G (we use $\{A\}$ to denote the element of $K(G)$ given by a perverse sheaf A).

Similarly we can define $\chi_v(A)$ for A a perverse sheaf on $\mathcal{B}\times\mathcal{B}$. Then convolution on G and on $\mathcal{B}\times\mathcal{B}$ give an algebra structure on $K(G)\otimes\mathbb{Z}[v, v^{-1}]$ and $K(\mathcal{B}\times\mathcal{B})\otimes\mathbb{Z}[v, v^{-1}]$ respectively. Let $K_G(\mathcal{B}\times\mathcal{B})$ and $K_G(G)$ be the Grothedieck groups of the categories of equivariant perverse sheaves on G and $\mathcal{B}\times\mathcal{B}$ respectively.

It can be shown that the algebra $K_G(\mathcal{B}\times\mathcal{B})\otimes\mathbb{Z}[v, v^{-1}]$ is isomorphic to the Hecke algebra: that is, an algebra $\mathcal{H}$ with $\mathbb{Z}[v, v^{-1}]$-basis $\{T_w : w \in W\}$ where the multiplication is given by

$$T_sT_w = \begin{cases} T_{sw}, & l(sw) < l(w); \\ T_{sw} + (v - v^{-1})T_w, & l(sw) > l(w). \end{cases}$$

(see [S] for a slightly different formulation of this result – the essential ingredients are the Bruhat decomposition and and SL_2 calculation which reduces to the fact that $\mathbb{P}^1$ has Poincare polynomial $1 + v^2$)). The basis of $\mathcal{H}$ given by the classes of simple G-equivariant perverse sheaves $\{A\}$ is the Kazhdan-Lusztig basis. Lusztig uses the structure of this algebra equipped with this basis to define a partition of the Weyl group (an idea which goes back to the seminal paper with Kazhdan [KL]).

Definition 4.1. Let A be an algebra over a commutative ring R with R-basis B. Then a *based ideal* in A is an ideal spanned by a subset of the basis B. We may define a preorder on B by setting $x \preceq y$ if x lies in every based ideal containing y. The equivalence classes of this preorder are called *cells*. By taking left/right/two-sided ideals, (we do not assume that A is commutative) we obtain left/right/two-sided cells respectively. Note that clearly each two-sided cell is a union of left cells (and of right cells).

In his study of the representations of finite groups of Lie type, Lusztig investigated the structure of these cells extensively in the case where A is a Hecke algebra and the basis is the Kazhdan-Lusztig basis[2]. In [LC3] he showed that, given $A \in \hat{G}_{\mathcal{L}}$ one can attach to it a two-sided cell in $W'_{\mathcal{L}}$ (recall that in the unipotent case this is just W).

We give here a characterization of this assignment which was first observed by Grojnowski in his thesis [Gr]. A detailed proof in a more general context is also available in [LD9].

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[2] Since this basis is naturally indexed by the corresponding Weyl group, the cells are often thought of as a partition of the Weyl group.

Theorem 4.2. *Let $A \in \hat{G}_{\mathbb{Q}_l}$ be a unipotent character sheaf. Then there is a simple perverse sheaf $A' \in \mathcal{M}_G^{\mathbb{Q}_l}(Y)$ such that A is a constituent of $\check{R}(A')$ and A' is a constituent of $R(A)$. Moreover, the two-sided cell of A' is uniquely determined.*

Proof. To show that such an A' exists, one uses the calculation of $\check{R} \circ R$ in the previous section. For details see [Gr] and [LD9]. □

Remark 4.3. It is not hard to show that the functor R respects convolution. Moreover, if A_1, A_2 are in $D_G(G)$, then $A_1 \star A_2 \cong A_2 \star A_1$, that is, convolution on the group is commutative. Moreover it can be checked that for $A \in D_G(G)$ then $R(A) \star B = B \star R(A)$ for B in $D_G(Y)$. Thus the functor R is in some sense central. Of course we could have defined cells just in terms of convolution of sheaves, and ignored the Hecke algebra, however describing the finer structure of two-sided cells at the moment requires the use of Hecke algebra.

Remark 4.4. It can be shown that the sets $\hat{G}_{\mathcal{L}}$ and $\hat{G}_{w^*(\mathcal{L})}$ are equal, and moreover that there is a well-defined map from $\hat{G}$ to the Weyl group orbits of pairs $(\mathcal{L}, \mathbf{c})$ where $\mathcal{L}$ is a tame local system and $\mathbf{c}$ is a two-sided cell of the group $W'_{\mathcal{L}}$. The fibers of this map might be called *L-packets* of character sheaves. Lusztig's original characterization of this partition into L-packets used considerably more detailed information about the character sheaves. On the other hand, Lusztig goes on to precisely describe the elements of each two-sided cell, a much more subtle result than anything we have stated here. In the next section, we will sketch this description.

5. ASYMPTOTIC ALGEBRAS

In this section we discuss some of the work of Lusztig on cells in Weyl groups. He observed [L87a], [L87b] that one could attach an "asymptotic algebra" to a Hecke algebra, which has a much simpler structure than the original Hecke algebra (although its existence and structure is not at all evident). In the case of finite Weyl groups, for example, it is an integral form of the group algebra $\mathbb{Q}[W]$ of the Weyl group. We describe this algebra for the Hecke algebra of the previous section (modifications for the Hecke algebras of the groups $W'_{\mathcal{L}}$ can also be defined.

Let $\{c_w : w \in W\}$ be the Kazhdan-Lusztig basis of $\mathcal{H}$. Notice that the definition of two-sided cells on an algebra with basis immediately implies that there is a partial order on the set of two sided cells. The span of all two-sided cells strictly less than or equal to a given cell $\mathbf{c}$ is clearly a two-sided ideal, so that we may take the associated graded with respect to this partial order to obtain an algebra $\bigoplus_{\mathbf{c}} \mathcal{H}_{\mathbf{c}}$. It turns out that it is not the subquotients $\mathcal{H}_{\mathbf{c}}$ but certain "limits" of them which are most important in Lusztig's theory:

Fix a two-sided cell $\mathbf{c}$. Let L be the $\mathbb{Z}[v^{-1}]$ lattice spanned by the elements $\{c_w : w \in \mathbf{c}\}$. Clearly, for each $w \in \mathbf{c}$ there is a positive integer n minimal with respect to the requirement that $v^{-n} c_w L \subseteq L$. Let $a(w)$ denote this integer. We rescale the Kazhdan-Lusztig basis using the function a, setting $\hat{c}_w = v^{-a(w)} c_w$, and let L' be the $\mathbb{Z}[v^{-1}]$-span of the elements $\hat{c}_w$.

Proposition 5.1. *The $\mathbb{Z}[v^{-1}]$-lattice L' is preserved by multiplication in $\mathcal{H}_{\mathbf{c}}$ (although it does not contain a unit). Thus we may define a (possibly non-unital) ring $J_{\mathbf{c}} = L'/v^{-1}L'$, with basis $t_w = \hat{c}_w + v^{-1}L'$.*

Proof. See [L87a]. One shows that in fact the function a is constant on a two-sided cell of W. Note that the subquotient $\mathcal{H}_{\mathbf{c}}$ is not clearly unital. It should be mentioned that the definitions in [L87a] are somewhat different to the ones we use here – the formulation we use, and the fact that it is equivalent to the original definitions, can be found in [L95] (see also [L03] for a general discussion of cells in Hecke algebras). □

We now give a rough description of the rings $J_{\mathbf{c}}$. The first remarkable fact is that they are unital. Lusztig shows [L87a] that there is a subset $\mathcal{D} \subset W$ of involutions (the identity is also in $\mathcal{D}$) such that the elements $\{t_d : d \in \mathcal{D}\}$ are orthogonal idempotents in J, and $\sum_{d \in \mathcal{D} \cap \mathbf{c}} t_d$ is an identity element for $J_{\mathbf{c}}$. Moreover, each left cell of W contains exactly one element of $\mathcal{D}$. He also shows that once we extend scalars to $\mathbb{Q}$, the ring $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ is isomorphic to the group algebra of W (indeed this allows him to give an explicit version of the Tits deformation theorem, showing the the Hecke algebra is isomorphic to the group algebra of the Weyl group for generic parameters). The algebra J is known as the *asymptotic Hecke algebra*.

To each two-sided cell $\mathbf{c}$ Lusztig [L84, Chapter 4] attaches a "small" finite group[3] $\mathcal{G}_{\mathbf{c}}$. In [L87b] he conjectures that the ring $J_{\mathbf{c}}$ (together with its basis $\{t_w : w \in \mathbf{c}\}$) is completely determined by $\mathcal{G}_{\mathbf{c}}$ and certain finite $\mathcal{G}_{\mathbf{c}}$-sets as follows: Given a finite $\mathcal{G}_{\mathbf{c}}$-set X we may consider the category of equivariant vector bundles on X. The corresponding category on $X \times X$ has a convolution product, and by taking Grothendieck groups one obtains a ring $K_{\mathcal{G}_{\mathbf{c}}}(X \times X)$ with a distinguished basis given by the irreducible objects. Lusztig conjectured [L87b] that the ring $J_{\mathbf{c}}$ is isomorphic to $K_{\mathcal{G}_{\mathbf{c}}}(X \times X)$ for certain finite sets X (Lusztig has informed me that in unpublished work he demonstrated the analogous statement for the intersection of the canonical left cell with its transpose).

The parametrization of character sheaves attached to a two-sided cell $\mathbf{c}$ can also be given in terms of the group $\mathcal{G}_{\mathbf{c}}$. Here one takes $\mathcal{G}_{\mathbf{c}}$-equivariant vector bundles on $\mathcal{G}_{\mathbf{c}}$ itself, with convolution given by the group multiplication. The simple objects in this category are then in bijection with the character sheaves associated to the two-sided cell. (Note that the ring $K_{\mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}})$ has an identity element given by the trivial representation of $\mathcal{G}_{\mathbf{c}}$ on the identity element of $\mathcal{G}_{\mathbf{c}}$).

Of course, since convolution operations exist on the level of sheaves, it is natural to conjecture that the algebra isomorphisms above are shadows of equivalences of tensor categories, and indeed Lusztig observes this in [L97] (in the context of affine Hecke algebras, where there is a similar theory of cells). Many of his conjectures are now theorems of Bezrukavnikov and Ostrik. A weak version of these conjectures for finite Hecke algebras are proved recently in [BFO].

Finally we note that in [L87b], Lusztig observes that if X is a $\mathcal{G}_{\mathbf{c}}$-set then there is a natural algebra homomorphism $K_{\mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}}) \to K_{\mathcal{G}_{\mathbf{c}}}(X \times X)$ whose image is precisely the center of $K_{\mathcal{G}_{\mathbf{c}}}(X \times X)$. This is remnicent of the fact R is central with respect to convolution, and [BFO] has made a precise conjecture on the level of categories which "explains" the parametrization of character sheaves in a given two-sided cell in terms of the Drinfeld double of the finite group $\mathcal{G}_{\mathbf{c}}$[4] (I do not

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[3] "small" really means small: the group is either a product of $\mathbb{Z}/2\mathbb{Z}$s (this is alwayst the case for a classical group) or a symmetric group on at most 5 letters.

[4] I confess to being woefully ignorant in the world of finite tensor categories – I mention this conjecture as it seems important and potentially enlightening, but beg forgiveness if I misrepresent it.

know if they are aware of the obervation of Lusztig that I am referring to). Some evidence can be found in [L04] where Lusztig shows that the convolution of unipotent almost characters on the $\mathbb{F}_q$-points of G (the almost characters are, up to a scalar factor, the functions obtained by taking traces of Frobenius on the stalks of character sheaves) has "leading coefficients" given by the structure constants of the corresponding basis of $K_{\mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}})$.

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