# RESEARCH OUTLINE

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## 1. INTRODUCTION

Representation theory is a broad and synthetic field of mathematics, bound together perhaps more by its objects of study than by the techniques which are used, as these can come from subjects as diverse as partial differential equations and category theory. Moreover, as an expression of basic symmetries, it also provides a framework for studying structures which arise all over mathematics (a modern example being say the cohomology of the Hilbert schemes of points on algebraic surfaces)

My own research focuses on objects arising in Lie theory – algebraic groups, quantum groups and related algebras such as Hecke algebras, studying their representation theory, the geometry of associated space *e.g.* Schubert varieties and quiver varieties, and relations between them.

## 2. QUANTUM GROUPS

The representation theory of reductive algebraic groups is both a classical subject and one of active investigation. Over an algebraically closed field of characteristic zero we have a reasonably complete picture – the category of representations is semisimple, the simple objects are parametrized by highest weights, and their characters are given by Weyl's formula. Indeed the story is an algebraic version of the even more classical theory of representations of compact Lie groups. Nevertheless, Lusztig's discovery [7] of canonical bases only eighteen years ago showed that this theory still contained unexpected structure.

Given that Chevalley's classification of connected simply-connected simple algebraic groups over an algebraically closed field is, remarkably, independent of characteristic, it is natural to seek a description of the category of representations of an algebraic group over a field $\mathsf{k}$ of characteristic $p > 0$. Here, immediately, many properties familiar from characteristic zero collapse, as even elementary calculations with the simplest example $SL_2$ demonstrate. The category is no longer semisimple, and while the simple objects are again classified by highest weights, their character is no longer easy to obtain – even a formula for their dimension is not known in complete generality.

In the 1980s a remarkable observation of Lusztig introduced a new protagonist to this story. He showed that it is possible to define a form of the Drinfeld-Jimbo quantum group over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and that when the deformation parameter $v$ in this integral form is specialized to $\zeta$ a root of unity, the representations of the resulting algebra $\mathbf{U}_\zeta$ behave much like those of an algebraic group in positive characteristic. This was all the more remarkable for the fact that the quantum group at a root of unity is defined over the ring of integers of a cyclotomic field, and so is a characteristic zero object. If we take $\zeta$ to be a $p$-th root of unity, a field

$\mathsf{k}$ of characteristic $p$ is naturally a $\mathbb{Z}[\zeta]$-algebra via the map $\zeta \mapsto 1$, and in this way the quantum group over $\mathbb{Z}[\zeta]$ can be thought of as an integral lift of the characteristic $p$ situation (it is, however the analogue of the algebra of distributions, or hyperalgebra, $\mathcal{U}$ of the group, rather than the group itself).

One of the fundamental discoveries made by Lusztig was that there is an analogue of the Frobenius morphism for $\mathbf{U}_\zeta$. Its definition is somewhat more subtle than in the classical characteristic $p$ setting, as we now recall.

Let $\mathcal{U}_{\mathbb{Z}[\zeta]}$ be the Kostant-Chevalley integral form of the classical enveloping algebra with scalars extended from $\mathbb{Z}$ to $\mathbb{Z}[\zeta]$. For any positive integer $\ell$ coprime to the lacing number of $\mathbf{U}$, Lusztig constructed a map

$$Fr\colon \mathbf{U}_\zeta \to \mathcal{U}_{\mathbb{Z}[\zeta]},$$

which he called the quantum Frobenius map[1]. When $\ell = p$, if we base change $Fr$ we obtain the classical Frobenius map (or rather its transpose on the hyperalgebra).

2.1. **Recent work.** My recent work has been focused on aspects of this quantum version of Frobenius. The starting point was to understand the action of the quantum Frobenius in the context of $q$-Schur algebras. Introduced by Dipper and James, these are a family of quotients $S_q(n,d)$ of the quantum group $\mathbf{U}(\mathfrak{sl}_n)$, for any positive integer $d$. Moreover, the algebras $S_q(n,d)$ naturally form an inverse system so that the quantum group $\mathbf{U}(\mathfrak{sl}_n)$ embeds into the limit of this system.

Given any ring $R$ and an invertible element $\varepsilon$ we may base change $S_q(n,d)$ to obtain an algebra $S_R(n,d)$. (The specialization where $v$ is sent to 1 is referred to as a Schur algebra). I have observed [11] that the map $Fr$ descends to give a map from the $q$-Schur algebra at a root of unity to the Schur algebra:

$$F_d\colon S_{\mathbb{Z}[\zeta]}(n,\ell d) \to \mathbb{Z}[\zeta]\otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,d).$$

(In fact my proof works in the context of a "generalized $q$-Schur algebra", and indeed even for affine analogues of these algebras).

In [1], Beilinson, Lusztig and MacPherson gave a construction of these algebras using the geometry of finite fields. Once it is known that the quantum Frobenius descends to the $q$-Schur algebra, it is reasonable to ask for a construction of the map $F_d$ in the context of the geometry of finite fields. This appears, *a priori* to be somewhat unlikely, as the parameter $v^2 = q$ is supposed to be a prime power, and thus rather far from a root of unity. However it turns out that one can give such an interpretation, using the construction of $S_q(n,d)$ for the fields $\mathbb{F}_q$ and $\mathbb{F}_{q^\ell}$. Moreover using the limit construction mentioned above, this produces a construction of the map $Fr$ for $\mathfrak{sl}_n$.

Returning to the context of an arbitrary quantum group, there is a modification of a quantum group, also due to Lusztig, which is better to work with when dealing with questions of integral structures (indeed Lusztig has recently used this form to develop the theory of Chevalley groups over any commutative unital ring [9]). For modified quantum groups, the map $Fr$ is constructed by first producing its retriction to the "plus part" $\mathbf{U}^+$, and then extending by using an analogue of the triangular decomposition of $\mathbf{U}$. Thus the critical part of the construction of $Fr$ is demonstrating its existence on $\mathbf{U}^+$. Using the Hall algebra construction of

[1] Thus $\ell$ must be odd for types $B$ and $C$ and coprime to 3 for $G_2$. Lusztig later [8] extended his map to a kind of exceptional isogeny in the case where $\ell$ is not coprime to the lacing number.

$\mathbf{U}^+$, there is a version of the same technique used for $q$-Schur algebras [12] that constructs the map $F_d$ for $q$-Schur algebras yields the map $Fr_{|\mathbf{U}^+}$, and so we get a geometric construction of (at least half) of the quantum Frobenius in general.

Finally, we note that it is possible [11] to give a simple but useful extension of Lusztig's work and define a right inverse $c$ to $Fr$ (that is, a map $c$ so that $Fr \circ c$ is the identity on the modified form of the classical enveloping algebra). This map is intimately related to the notion of Frobenius splitting in algebraic geometry, as recent work of Kumar and Littelmann shows. I have also shown that the contraction map $c$ descends to a map $c_d$ on $q$-Schur algebras, giving a right inverse of the map $F_d$. The descent of $Fr$ to the $q$-Schur algebra gives some (modest) evidence for a compatibility between the quantum Frobenius and the canonical basis of the modified quantum group (the $q$-Schur algebras are quotients of the modified quantum group by ideals which are spanned by a subset of the canonical basis).

2.2. **Research aims: Geometrization of quantum Frobenius**: Once it is known that $Fr$ can be interpreted in the context of the geometry of finite fields, it is natural to ask for a "faisceaux-fonctions" lift, and obtain a "categorification" of $Fr$ on the level of perverse sheaves on the moduli of quiver representations. In work in progress, the use of the arithmetic of finite fields is replaced by the action of a cyclic group (to some extend this is of course anticipated by [8]) and localization with respect to a $\mathbb{C}^\times$-action on the category of sheaves after the work of Goresky-Kottwitz-MacPherson [4]. The existence of such a lifting would establish a precise connection between the action of the quantum Frobenius and the canonical basis, at least on $\mathbf{U}^+$.

**Quiver varieties**: There are natural analogues of generalized $q$-Schur algebras for affine quantum groups, and I can show that the quantum Frobenius is also compatible with these affine algebras. Nakajima has given a geometric construction of a version of these affine generalized $q$-Schur algebras in his work on finite dimensional representations of affine quantum groups. I hope that an analogue of my construction with the Hall algebra is possible in this geometry, giving an interpretation of the quantum Frobenius in this context. It is likely that this work will have some relation to Nakajima's "$t$-analogues of $q$-characters" [15].

## 3. The affine Grassmannian and modular representation theory

Recent work of Mirkovic and Vilonen [14] gives a remarkable construction (in characteristic zero), of the tensor category of representations of a reductive algebraic group, in any characteristic, in terms of perverse sheaves on the affine Grassmannian of the Langlands dual group. Thus one can hope to understand aspects of modular representation theory in this context. Indeed it is classical that the affine Weyl group plays a prominent role in this representation theory, so a context where the affine Weyl group is utterly intrinsic should be a great advantage. The most basic issue here is to obtain an intrinsic understanding of the action of Frobenius in this realization, with the Steinberg tensor product theorem mentioned above as the natural first result to interpret. Another reason for seeking a geometric lifting of the quantum Frobenius is that I hope it will provide a model for these questions. Indeed one can use the techniques of [12] for the affine Grassmanian of type $A$ to study Hall-Littlewood functions at a root of unity [13].

## 4. Character sheaves

I am interested in the theory of representations of finite reductive groups, and its connection to perverse sheaves. In the geometric study of these groups, there are two distinct strands. One is the study of the construction of the representations themselves, through the seminal work of Deligne and Lusztig [3]. The other, motivated by Springer's definition of Green functions, is Lusztig's theory of character sheaves, which seeks to calculate the characters of a finite reductive group $G$, by producing from the geometry of the group, a basis of the space of class functions on $G$. Although work of Lusztig shows that these two theories are deeply related – both produce, for example, from quite different starting points, the *almost characters* of the group, a basis of the class functions of $G$ which is closely related, but not precisely equal, to the irreducible characters of $G$ – however the nature of the connection remains somewhat mysterious.

Gurevich and Hadani [5] have recently given a geometric construction of the Weil representation as a perverse sheaf. Their sheaf is a geometric version of the algebra of operators on the representation, thus a "geometrization" of the representation, not just its character. I would like to use this sheaf to investigate the behaviour of character sheaves under a geometric version of Howe duality, and perhaps give examples of other "representation sheaves" (rather than character sheaves) for finite reductive groups. Such examples might shed some light on the relation between the representation theory and character theory mentioned above.

## 5. Quiver varieties and the nilpotent cone

The study of quiver varieties, introduced by Kronheimer and Nakajima and subsequently deeply studied by Nakajima, is often informed by an analogy between their geometry and that of the nilpotent cone with its resolution, the cotangent bundle of the flag variety. In collaboration with Anthony Henderson, I am studying some situations where this is more than an analogy and should in fact be an isomorphism.

Our motivation comes from an observation of Mark Reeder [16]. Recall that if $G$ is an algebraic group, and $V$ is a rational representation of $G$, then the zero weight space of $V$ (the subspace on which $T$ a maximal torus acts trivially) is naturally a representation of the Weyl group of $G$. Although this representation is usually very difficult to calculate, for certain "small" representations of groups of type $A$,$D$ or $E$, Reeder was able to perform this calculation, and discovered that the representations were exactly those realized by Springer in the top-dimensional cohomology of certain Springer fibers (corresponding to a class of large nilpotent orbits in $\mathrm{Lie}(G)$).

Since, on the level of representations of the Lie algebra, the small representations can be realized in the cohomology of quiver varieties, it is natural to conjecture that these representation-theoretic identifications reflect a geometric identity – namely that the quiver varieties and Springer fibers are isomorphic. Indeed it is natural to go somewhat further and conjecture an identification between certain symplectic resolutions arising from quiver varieties and slices throught nilpotent orbits respectively. These identifications turn out to be already known in type $A$ from work of Nakajima and Maffei [10], but not as yet for type $D$ and $E$.

Since other invariants of these varieties, such as their equivariant $K$-theory, are also known to have representation-theoretic structure, our conjectural identifications also suggest possible connections between degenerate affine algebras and finite $W$-algebras. Some evidence of this is already given by recent work of Brundan and Kleshchev [2] in type $A$.

## 6. Elliptic algebras

It is well known that a one-dimensional algebraic group (over an algebraically closed field) can be one of only three possibilities: the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, or an elliptic curve. An affine quantum group can naturally be viewed as an algebra over $\mathbb{G}_m$, and it possesses a "degeneration" known as a Yangian, which is an algebra over the additive group $\mathbb{G}_a$. It is therefore intriguing to ask if there are families of affine algebras which live in some sense over an elliptic curve (or most likely better, the moduli of elliptic curves). In joint work with Ian Grojonowski, we are currently working to construct certain families of quantum algebras related to elliptic curves via the representations of affine Lie algebras.

## References


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