INNER PRODUCTS AND THE CANONICAL BASIS OF QUANTUM AFFINE $\mathfrak{sl}_n$

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ABSTRACT. We give a geometric interpretation of the inner product on the modified quantum group for affine $\mathfrak{sl}_n$. We give some applications, including a positivity result for the inner product.

1. INTRODUCTION

Let $\mathbf{U}$ be a quantum group. The positive part $\mathbf{U}^+$ of $\mathbf{U}$ is well known to possess a canonical basis [K] [L91]. In contrast, there is no particularly natural basis for the $\mathbf{U}$ itself. Seeking to rectify this, Lusztig [L92] defined a variant of the quantized enveloping algebra known as the modified quantum group. This algebra has essentially the same representation theory, and can be given a canonical basis $\dot{\mathbf{B}}$ which packages together natural bases of the tensor product of a highest and lowest weight $\mathbf{U}$-module, in the same way that the canonical basis $\mathbf{B}$ of $\mathbf{U}^+$ packages together natural bases of highest weight representations. Just as for $\mathbf{B}$, (c.f. [GL], [K]) it is possible to characterize this basis, up to sign, in terms of an involution and an inner product.

In [BLM] the quantized enveloping algebra of $\mathfrak{gl}_n$ was constructed geometrically as a limit of certain convolution algebras. Subsequently Lusztig [L99], and independently Ginzburg and Vasserot [GV], observed that this construction could be extended to the case of quantum affine $\mathfrak{sl}_n$. More precisely, it can be shown that one can define a sequence of algebra $\mathfrak{A}_D$, and maps $\phi_D \colon \mathfrak{A}_D \to \mathfrak{A}_{D-n}$, and compatible maps ψ_D from the quantum group. In this paper we show that the inner product on the modified quantum group $\dot{\mathbf{U}}$ of affine $\mathfrak{sl}_n$ may be obtained geometrically in this context, and establish a positivity property for this case which is conjectured to hold in general. We also give a sheaf-theoretic description of the transfer maps of [L99a].

2. BACKGROUND

We begin by recalling the setup of [L99]. Fix a positive integer n. Let D be a positive integer, ϵ an indeterminate, $\mathbf{k}$ a finite field with q elements and v a square root of q. Given V a free $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module of rank D, a *lattice* in V is a free $\mathbf{k}[\epsilon]$-submodule of V, of rank D. Let $\mathcal{F}^n$ denote the set of *n-step periodic lattices* in V, that is, $\mathcal{F}^n$ consists of sequences of lattices $\mathbf{L} = (L_i)_{i\in\mathbb{Z}}$ where $L_{i-1} \subset L_i$, and $L_{i-n} = \epsilon L_i$ for all $i \in \mathbb{Z}$. We will also write $\mathcal{F}^n_D$ when we wish to emphasize the rank of V.

The group G of automorphisms of V acts on $\mathcal{F}^n$ in the natural way. We shall be interested in functions supported on $\mathcal{F}^n$ and its square which are invariant with

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respect to the action of G (where G acts diagonally on $\mathcal{F}^n \times \mathcal{F}^n$). Thus we first describe the orbits of G on these spaces. Let $\mathfrak{S}_{D,n}$ be the finite set of all $\mathbf{a} = (a_i)_{i\in\mathbb{Z}}$ such that

- $a_i \in \mathbb{N}$;
- $a_i = a_{i+n}$ for all $i \in \mathbb{Z}$;
- for all $i \in \mathbb{Z}$, $a_i + a_{i+1} + \cdots + a_{i+n-1} = D$.

For $\mathbf{L} \in \mathcal{F}^n$, let $|\mathbf{L}| \in \mathfrak{S}_{D,n}$ be given by $|\mathbf{L}|_i = \dim(L_i/L_{i-1})$. The G-orbits on $\mathcal{F}^n$ are indexed by this graded dimension: for $\mathbf{a} \in \mathfrak{S}_{D,n}$ set $\mathcal{F}_{\mathbf{a}} = \{\mathbf{L} \in \mathcal{F}^n \colon |\mathbf{L}| = \mathbf{a}\}$; then the $\mathcal{F}_{\mathbf{a}}$ are precisely the G-orbits on $\mathcal{F}^n$. The G orbits on $\mathcal{F}^n \times \mathcal{F}^n$ are indexed, slightly more elaborately, by the set of matrices $\mathfrak{S}_{D,n,n}$, where $A = (a_{i,j})_{i,j\in\mathbb{Z}}$, is in $\mathfrak{S}_{D,n,n}$ if

- $a_{i,j} \in \mathbb{N}$;
- $a_{i,j} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$;
- for any $i \in \mathbb{Z}$, $a_{i,*} + a_{i+1,*} + \cdots + a_{i+n-1,*} = D$;
- for any $j \in \mathbb{Z}$, $a_{*,j} + a_{*,j+1} + \cdots + a_{*,j+n-1} = D$.

Here

$$a_{i,*} = \sum_{j\in\mathbb{Z}} a_{i,j}; \qquad a_{*,j} = \sum_{i\in\mathbb{Z}} a_{i,j}.$$

For $A \in \mathfrak{S}_{D,n,n}$ set

$$r(A) = (a_{i,*})_{i\in\mathbb{Z}} \in \mathfrak{S}_{D,n} \qquad c(A) = (a_{*,j})_{j\in\mathbb{Z}} \in \mathfrak{S}_{D,n}.$$

For $A \in \mathfrak{S}_{D,n,n}$ the corresponding G-orbit $\mathcal{O}_A$ consists of pairs $(\mathbf{L}, \mathbf{L}')$ such that

$$a_{i,j} = \dim\left(\frac{L_i \cap L'_j}{(L_{i-1} \cap L'_j) + (L_i \cap L'_{j-1})}\right),$$

so $\mathbf{L} \in \mathcal{F}_{r(A)}$ and $\mathbf{L}' \in \mathcal{F}_{c(A)}$.

Let $\mathfrak{A}_{D;q}$ be the space of integer-valued G-invariant functions on $\mathcal{F}^n \times \mathcal{F}^n$ supported on a finite number of orbits. If e_A denotes the characteristic function of an orbit $\mathcal{O}_A$, the set $\{e_A \colon A \in \mathfrak{S}_{D,n,n}\}$ is a basis of $\mathfrak{A}_{D;q}$. The space $\mathfrak{A}_{D;q}$ has a natural convolution product which gives it the structure of an associative algebra. With respect to the basis of characteristic functions the structure constants are given as follows. For $A, B, C \in \mathfrak{S}_{D,n,n}$, let $\nu_{A,B,C}$ be the coefficient of e_C in the product $e_A e_B$. Then $\nu_{A,B,C}$ is zero unless $c(A) = r(B)$, $r(A) = r(C)$ and $c(B) = c(C)$. Now suppose these conditions are satisfied and fix $(\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_C$. Then $\nu_{A,B,C}$ is the number of points in the set

$$\{\mathbf{L}' \in \mathcal{F}_{c(A))} \colon (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A, (\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_B\}.$$

Clearly this is independent of the choice of $(\mathbf{L}, \mathbf{L}'')$, and moreover it can be shown that these structure constants are polynomial in q, allowing us to construct an algebra $\mathfrak{A}_D$ over $\mathbb{Q}(v)$ (we will, by deliberate misuse, treat v as both an indeterminate and a square root of q, depending on the context). This algebra is sometimes known as the affine q-Schur algebra. It is more convenient to use a rescaled version of the basis $\{e_A\}$ of $\mathfrak{A}_D$, with elements $[A] = v^{-d_A} e_A$ where

$$d_A = \sum_{i\geq k, j<l, 1\leq i\leq n} a_{ij} a_{kl}.$$

Note that if we define $\Psi([A]) = [A^t]$ then it is easy to check that Ψ is an algebra anti-automorphism, which we will sometimes call the transpose anti-automorphism.

Next we introduce quantum groups. In order to do this we recall the notion of a root datum from [L93].

Definition 2.1. A *Cartan datum* is a pair $(I, \cdot)$ consisting of a finite set I and a $\mathbb{Z}$-valued symmetric bilinear pairing on the free Abelian group $\mathbb{Z}[I]$, such that

- $i \cdot i \in \{2, 4, 6, \ldots\}$
- $2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \ldots\}$, for $i \neq j$.

A *root datum* of type $(I, \cdot)$ is a pair Y, X of finitely-generated free Abelian groups and a perfect pairing $\langle , \rangle \colon Y \times X \to \mathbb{Z}$, together with imbeddings $I \subset X$, $(i \mapsto i)$ and $I \subset Y$, $(i \mapsto i')$ such that $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$.

Given a root datum, we may define an associated quantum group $\mathbf{U}$. Since it is the only case we need, we will assume that our datum is symmetric and simply laced so that $i \cdot i = 2$ for each $i \in I$, and $i \cdot j \in \{0, -1\}$ if $i \neq j$. In this case, $\mathbf{U}$ is generated as an algebra over $\mathbb{Q}(v)$ by symbols E_i, F_i, K_μ, $i \in I$, $\mu \in Y$, subject to the following relations.

- $K_0 = 1$, $K_{\mu_1} K_{\mu_2} = K_{\mu_1 + \mu_2}$ for $\mu_1, \mu_2 \in Y$;
- $K_\mu E_i K_\mu^{-1} = v^{\langle \mu, i' \rangle} E_i, \quad K_\mu F_i K_\mu^{-1} = v^{-\langle \mu, i' \rangle} F_i$ for all $i \in I$, $\mu \in Y$;
- $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}$;
- $E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i$, for $i, j \in I$ with $i \cdot j = 0$;
- $E_i^2 E_j + (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$;
- $F_i^2 F_j + (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$.

Thus $\mathbf{U}$ is naturally X–graded, $\mathbf{U} = \bigoplus_{\nu \in X} \mathbf{U}_\nu$.

We also need to consider the modified quantum group $\dot{\mathbf{U}}$. This is defined by

$$\dot{\mathbf{U}} = \bigoplus_{\lambda \in X} \mathbf{U} 1_\lambda; \qquad \mathbf{U} 1_\lambda = \mathbf{U} / \sum_{\mu \in Y} \mathbf{U}(K_\mu - v^{\langle \mu, \lambda \rangle}).$$

Here the multiplicative structure is given in the natural way:

$$1_\lambda x = x 1_{\lambda - \nu}, \quad x \in \mathbf{U}_\nu; \qquad 1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda.$$

To see the connection between our convolution algebra and quantum groups, we will need the following notation. For $\mathbf{a} \in \mathfrak{S}_{D,n}$ let $\mathbf{i_a} \in \mathfrak{S}_{D,n,n}$ be the diagonal matrix with $(\mathbf{i_a})_{i,j} = \delta_{i,j} a_i$. Let $E^{i,j} \in \mathfrak{S}_{1,n,n}$ be the matrix with $(E^{i,j})_{k,l} = 1$ if $k = i + sn$, $l = j + sn$, some $s \in \mathbb{Z}$, and 0 otherwise. Let $\mathfrak{S}^n$ be the set of all $\mathbf{b} = (b_i)_{i \in \mathbf{Z}}$ such that $b_i = b_{i+n}$ for all $i \in \mathbb{Z}$. Let $\mathfrak{S}^{n,n}$ denote the set of all matrices $A = (a_{i,j})$, $i, j \in \mathbb{Z}$, with entries in $\mathbb{Z}$ such that

- $a_{i,j} \geq 0$ for all $i \neq j$;
- $a_{i,j} = a_{i+n,j+n}$, for all $i, j \in \mathbb{Z}$;
- For any $i \in \mathbb{Z}$ the set $\{j \in \mathbb{Z} \colon a_{i,j} \neq 0\}$ is finite;
- For any $j \in \mathbb{Z}$ the set $\{i \in \mathbb{Z} \colon a_{i,j} \neq 0\}$ is finite.

Thus we have $\mathfrak{S}_{D,n,n} \subset \mathfrak{S}^{n,n}$ for all D. For $i \in \mathbb{Z}/n\mathbb{Z}$ let $\mathbf{i} \in \mathfrak{S}^n$ be given by $\mathbf{i}_k = 1$ if $k = i \bmod n$, $\mathbf{i}_k = -1$ if $k = i + 1 \bmod n$, and $\mathbf{i}_k = 0$ otherwise. We write $\mathbf{a} \cup_i \mathbf{a}'$ if $\mathbf{a} = \mathbf{a}' + \mathbf{i}$. For such $\mathbf{a}, \mathbf{a}'$ set ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'} \in \mathfrak{S}^{n,n}$ to be $\mathbf{i_a} - E^{i,i} + E^{i,i+1}$, and ${}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} \in \mathfrak{S}^{n,n}$ to be $\mathbf{i}_{\mathbf{a}'} - E^{i+1,i+1} + E^{i+1,i}$. Note if $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}_{D,n}$ then ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}, {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} \in \mathfrak{S}_{D,n,n}$. For $i \in \mathbb{Z}/n\mathbb{Z}$ set

$$E_i(D) = \sum [{}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}], \qquad F_i(D) = \sum [{}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}],$$

where the sum is taken over all $\mathbf{a}, \mathbf{a}'$ in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$. For $\mathbf{a} \in \mathfrak{S}^n$ set

$$K_{\mathbf{a}}(D) = \sum_{\mathbf{b} \in \mathfrak{S}_{D,n}} v^{\mathbf{a}\cdot\mathbf{b}}[\mathbf{i_b}]$$

where, for any $\mathbf{a}, \mathbf{b} \in \mathfrak{S}^n$, $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^n a_i b_i \in \mathbb{Z}$. If we let $X = Y = \mathfrak{S}^n$, and $I = \mathbb{Z}/n\mathbb{Z}$, with the embedding of $I \subset X = Y$ and pairing as given above, we obtain a symmetric simply-laced root datum. We call the quantum group associated to it $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. It can be shown [L99] that the elements $E_i(D), F_i(D), K_{\mathbf{a}}(D)$, generate a subalgebra $\mathbf{U}_D$ which is a quotient of the quantum group $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, via map the notation suggests. Note that this gives the algebra $\mathfrak{A}_D$ the structure of a $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$-module.

3. Inner product on $\mathbf{U}_D$

Definition 3.1. We define a bilinear form

$$(,)_D \colon \mathfrak{A}_{D;q} \times \mathfrak{A}_{D;q} \to \bar{\mathbb{Q}}_l$$

by

$$(f, \tilde{f})_D = \sum_{\mathbf{L},\mathbf{L}'} v^{\sum |\mathbf{L}|_i^2 - \sum |\mathbf{L}'|_i^2} f(\mathbf{L}, \mathbf{L}')\tilde{f}(\mathbf{L}, \mathbf{L}'),$$

for f and $\tilde{f}$ in $\mathfrak{A}_{D,q}$, where $\mathbf{L}$ runs over $\mathcal{F}^n$ and $\mathbf{L}'$ runs over a set of representatives for the G-orbits on $\mathcal{F}^n$.

Let $\mathcal{O}_A$ be a G-orbit on $\mathcal{F}^n \times \mathcal{F}^n$, and let

$$X_A^{\mathbf{L}} = \{\mathbf{L}' \in \mathcal{F}^n \colon (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A\}.$$

It is easy to check that

$$2d_A - 2d_{A^t} = \sum_{i=1}^n a_{i,*}^2 - \sum_{j=1}^n a_{*,j}^2. \tag{3.1}$$

Thus if A, A' are in $\mathfrak{S}_{D,n,n}$ we find that

$$(e_A, e_{A'})_D = \delta_{A,A'} q^{d_A - d_{A^t}} \#|X_{A^t}^{\mathbf{L}'}|,$$

where $\mathbf{L}'$ is any lattice in $\mathcal{F}_{c(A)}$. Note that this makes it clear that the bilinear form is symmetric, which is not immediate from the initial definition. If $\{\eta_{A,B;q}^C\}$ are the structure constants of $\mathfrak{A}_{D;q}$ with respect to the basis $\{[A] \colon A \in \mathfrak{S}_{D,n,n}\}$, then we have

$$([A], [A'])_D = \delta_{A,A'} v^{d_A - d_{A^t}} \eta_{A^t,A;q}^{\mathbf{i}_{c(A)}}. \tag{3.2}$$

We therefore obtain an inner product on $\mathfrak{A}_D$ taking values in $\mathbb{Q}(v)$ by defining

$$([A], [A'])_D = \delta_{A,A'} v^{d_A - d_{A'}} \eta_{A^t,A,\mathbf{i}_{c(A)}} \in \mathbb{Z}[v, v^{-1}] \tag{3.3}$$

We now give some basic properties of this inner product:

Proposition 3.2. *Let $A \in \mathfrak{S}_{D,n}$, and let $f, \tilde{f} \in \mathfrak{A}_D$. Then we have*

$$([A]f, \tilde{f})_D = v^{d_A - d_{A^t}} (f, [A^t]\tilde{f})$$

Proof. Clearly it suffices to establish this equation in the algebra $\mathfrak{A}_{D;q}$. Since the characteristic functions of G-orbits form a basis of $\mathfrak{A}_{D;q}$, we may assume that $f = e_B$ and $\tilde{f} = e_C$, moreover we may assume that

$$r(A) = r(C), \quad c(A) = r(B), \quad c(B) = c(C). \tag{3.4}$$

as both sides are zero otherwise. It follows immediately that

$$[A] \cdot e_B = v^{-d_A} e_A \cdot e_B, \qquad v^{d_A - d_{A^t}} [A^t] \cdot e_C = v^{d_A - 2d_{A^t}} e_{A^t} \cdot e_C.$$

Hence if $(\tilde{\mathbf{L}}, \mathbf{L}') \in \mathcal{O}_C$ is fixed,

(3.5)

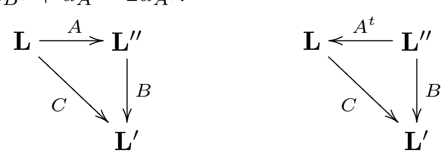
$$\begin{aligned} ([A] \cdot e_B, e_C)_D &= q^{d_C - d_{C^t}} \#|X_{C^t}^{\mathbf{L}'}| \cdot v^{-d_A} \#\{\mathbf{L}'' \colon (\tilde{\mathbf{L}}, \mathbf{L}'') \in \mathcal{O}_A, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B\} \\ &= v^{\alpha} \#\{\mathbf{L}, \mathbf{L}'' \colon (\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_A, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B, (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_C\}, \end{aligned}$$

where $\alpha = 2d_C - 2d_{C^t} - d_A$. Similarly, if $(\tilde{\mathbf{L}''}, \mathbf{L}') \in \mathcal{O}_A$ is fixed

(3.6)

$$\begin{aligned} v^{d_A - d_{A^t}} (e_B, [A^t] \cdot e_C)_D &= q^{d_B - d_{B^t}} \#|X_{B^t}^{\mathbf{L}'}| \cdot v^{d_A - 2d_{A^t}} \#\{\mathbf{L} \colon (\tilde{\mathbf{L}''}, \mathbf{L}) \in \mathcal{O}_F, (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_B\} \\ &= v^{\beta} \#\{\mathbf{L}, \mathbf{L}'' \colon (\mathbf{L}'', \mathbf{L}) \in \mathcal{O}_{A^t}, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B, (\mathbf{L}, \mathbf{L}' \in \mathcal{O}_C)\}, \end{aligned}$$

where $\beta = 2d_B - 2d_{B^t} + d_A - 2d_{A^t}$.



As the diagram clearly shows, the last line of equation (3.5) is the same as the last line of equation (3.6) if $\alpha = \beta$, that is, if

$$2d_C - 2d_{C^t} - d_A = 2d_B - 2d_{B^t} + d_A - 2d_{A^t} \tag{3.7}$$

But this follows directly from equation (3.1) and equation (3.4). □

We have the following easy consequence:

Corollary 3.3. *Let $i \in \mathbb{Z}$, and let $f, \tilde{f} \in \mathfrak{A}_D$ and $\mathbf{c} \in \mathfrak{S}^n$. Then we have*

1. $(E_i(f), \tilde{f})_D = (f, vK_{\mathbf{i}} F_i(\tilde{f}))_D$
2. $(F_i(f), \tilde{f})_D = (f, vK_{-\mathbf{i}} E_i(\tilde{f}))_D$
3. $(K_{\mathbf{c}}(f), \tilde{f})_D = (f, K_{\mathbf{c}}(\tilde{f}))_D$

Proof. We may assume that $f = e_A$ and $\tilde{f} = e_B$. The third equation can then be checked immediately from the formulas above. The second equation follows from the other two, so it only remains to prove the first. We may assume that $r(A) = r(B) - \mathbf{i}$ and $c(A) = c(B)$, as both sides are zero otherwise. Set $\mathbf{a} = r(A)$, $\mathbf{b} = r(B)$ (see section 2).

Then from the definitions we have

$$E_i(e_A) = [{}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}] \cdot e_A, \qquad vK_{\mathbf{i}} F_i(e_B) = v^{1 + \mathbf{i} \cdot \mathbf{a}} [{}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}] \cdot e_B.$$

Since ${}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}} = {}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}^t$, and $d_{{}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}} - d_{{}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}} = 1 + \mathbf{i} \cdot \mathbf{a}$ the result now follows immediately from the previous proposition. □

Remark 3.4. There is a unique algebra anti-automorphism $\rho\colon \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right) \to \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right)$ such that

$$\rho(E_i) = vK_{\mathbf{i}}F_i, \quad \rho(F_i) = vK_{-\mathbf{i}}E_i \quad \rho(K_i) = K_i$$

With this we may state the result of the previous corollary in the form

$$(u(f), \tilde{f})_D = (f, \rho(u)\tilde{f})_D, \qquad u \in \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right), \quad f, \tilde{f} \in \mathfrak{A}_D.$$

Lemma 3.5. (1) *For* $A \in \mathfrak{S}_{D,n,n}$, $([A],[A])_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$
(2) *For* $A, A' \in \mathfrak{S}_{D,n,n}$ *and* $A \neq A'$, $([A],[A'])_D = 0$

Proof. The second part of the statement is obvious. For the first, note that $X^{\mathbf{L}'}_{A^t}$ is an irreducible variety of dimension d_{A^t}, (see [L99, 4.3]). Since we have

$$([A],[A'])_D = \delta_{A,A'}q^{-d_{A^t}}\#|X^{\mathbf{L}'}_{A^t}|,$$

the Lang-Weil estimates [LW] then show that $([A],[A])_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$, as required. □

Remark 3.6. The results of this section are almost identical to the results of [L99, section 7]; however, as our inner product is not quite the same as that of [L99, 7.1], the proofs seem somewhat simpler.

4. INNER PRODUCT ON $\dot{\mathbf{U}}$

Notice that if $\mathbf{a} \in \mathfrak{S}^n$ then the sum $a_{i_0} + \cdots + a_{i_0+n-1}$ is independent of $i_0 \in \mathbb{Z}$; denote it by $\nabla_{\mathbf{a}}$. Let $Y = \{\mathbf{a} \in \mathfrak{S}^n\colon \nabla_{\mathbf{a}} = 0\}$. Let X be the quotient of $\mathfrak{S}^n$ by the subgroup generated by $\mathbf{b}_0$, the element with all entries equal to 1. Clearly the pairing on $\mathfrak{S}^n$ given in section 1 induces a non-singular pairing $Y \times X \to \mathbb{Z}$.

Let $I = \mathbb{Z}/n\mathbb{Z}$, and define maps $I \to X$, $I \to Y$ sending i to $\mathbf{i} \in \mathfrak{S}^n$ (see the end of section 2), taking the appropriate coset in X. This is the root datum of $\widehat{\mathfrak{sl}}_n$. Let $\mathbf{U}$ be the quantized enveloping algebra associated to this datum, and let $\dot{\mathbf{U}}$ be the modified algebra corresponding to $\mathbf{U}$. We wish to obtain an inner product on $\dot{\mathbf{U}}$ using those on $\mathbf{U}_D$.

We begin with some technical lemmas. Given $A \in \mathfrak{S}^{n,n}$ let $a_{i,\geq s} = \sum_{j \geq s} a_{i,j}$, and $a_{i,>s}$, $a_{i,\leq s}$, etc. similarly.

Lemma 4.1. *a) Let* $A \in \mathfrak{S}_{D,n,n}$ *and* $\mathbf{a}' = r(A)$. *If there is an* $\mathbf{a} \in \mathfrak{S}_{D,n}$ *such that* $\mathbf{a} \cup_i \mathbf{a}'$ *(i.e. if* $a'_{i+1} > 0$*) then we have*

$$[_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}][A] = \sum_{s \in \mathbb{Z}, a_{i+1,s} \geq 1} v^{a_{i,\geq s} - a_{i+1,>s}} \left(\frac{1 - v^{-2(a_{i,s}+1)}}{1 - v^{-2}}\right)[A + E^{i,s} - E^{i+1,s}], \tag{4.1}$$

where $A = (a_{i,j})$.

b) Let $A' \in \mathfrak{S}_{D,n,n}$ *and* $\mathbf{a} = r(A')$. *If there is an* $\mathbf{a}' \in \mathfrak{S}_{D,n}$ *such that* $\mathbf{a} \cup_i \mathbf{a}'$ *(i.e. if* $a_i > 0$*) then we have*

$$[_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}][A'] = \sum_{s \in \mathbb{Z}, a_{i,s} \geq 1} v^{a'_{i+1,\leq s} - a'_{i,<s}} \left(\frac{1 - v^{-2(a'_{i+1,s}+1)}}{1 - v^{-2}}\right)[A' - E^{i,s} + E^{i+1,s}], \tag{4.2}$$

where $A' = (a'_{i,j})$.

Proof. This follows by rescaling the statement of Proposition 3.5 in [L99]. □

Let $\mathcal{R}$ be the subring of $\mathbb{Q}(v)[u]$ generated by $\{v^j : j \in \mathbb{Z}\}$, and

$$\prod_{i=1}^{t}(v^{-2(a-i)}u^2 - 1)/(v^{-2i} - 1); \qquad a \in \mathbb{Z}, t \geq 1.$$

For $A \in \mathfrak{S}^{n,n}$ let ${}_pA$ be the matrix with $({}_pA)_{i,j} = a_{i,j} + p\delta_{i,j}$. We have the following partial analogue of [BLM, 4.2].

Lemma 4.2. *Let $A_1, A_2, \ldots, A_k$ be matrices of the form ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}$ or ${}_{\mathbf{a}}\mathbf{f}_{\mathbf{a}'}$, for $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}^n$, and A any element of $\mathfrak{S}^{n,n}$. Then there exist matrices $Z_1, Z_2, \ldots, Z_m \in \mathfrak{S}^{n,n}$ and $p_0 \in \mathbb{Z}$ such that*

$$[{}_pA_1][{}_pA_2]\ldots[{}_pA_k][{}_pA] = \sum_{i=1}^{m} G_i(v, v^{-p})[{}_pZ_i], \qquad G_i \in \mathcal{R} \tag{4.3}$$

for all $p \geq p_0$.

Proof. Use induction on k. When $k = 1$ the result follows from the previous lemma, once we note that both $a_{i,\geq s} - a_{i+1,>s}$ and $a_{i+1,\leq s} - a_{i,<s}$ are unchanged when A is replaced with $A + pI$. □

There is a surjective homomorphism $\phi_D : \dot{\mathbf{U}} \to \mathbf{U}_D$ [L99, Lemma 2.8] which, for $\lambda \in X$, sends $E_i 1_\lambda \mapsto E_i(D)[\mathbf{i_a}]$ and $F_i 1_\lambda \mapsto F_i(D)[\mathbf{i_a}]$ if there is an $\mathbf{a}$ in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} = \lambda \bmod \mathbb{Z}\mathbf{b}_0$, otherwise both $E_i 1_\lambda$, $F_i 1_\lambda$ are sent to zero.

Let $\mathbf{f}$ be the algebra attached to the root datum described above (see [L92, chapter 3]). Pick a monomial basis of $\mathbf{f}$, $\{\zeta_i : i \in J\}$ say. Then the triangular decomposition for $\dot{\mathbf{U}}$ [L92, 23.2.1] shows that $\mathfrak{B} = \{\zeta_i^+ \zeta_j^- 1_\lambda : i, j \in J, \lambda \in X\}$ is a basis of $\dot{\mathbf{U}}$, where $+ : \mathbf{f} \to \mathbf{U}^+$, and $- : \mathbf{f} \to \mathbf{U}^-$ are the standard maps given in [L92, 3.1.1]. Define a bilinear pairing $\langle , \rangle_D$ on $\dot{\mathbf{U}}$ via ϕ_D as follows:

$$\langle x, y \rangle_D = (\phi_D(x), \phi_D(y))_D$$

Proposition 4.3. *Let $k \in \{0, 1, \ldots, n-1\}$, then if $x, y \in \dot{\mathbf{U}}$*

$$\langle x, y \rangle_{k+pn}$$

converges in $\mathbb{Q}((v^{-1}))$, as $p \to \infty$, to an element of $\mathbb{Q}(v)$.

Proof. We may assume that x, y are elements of $\mathfrak{B}$. Then we need to show that

$$\langle \zeta_{i_1}^+ \zeta_{j_1}^- 1_\lambda, \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn} \qquad i_1, i_2, j_1, j_2 \in J; \lambda, \mu \in X$$

converges as $p \to \infty$. Let $\iota : \mathbf{f} \to \mathbf{f}$ is the $\mathbb{Q}(v)$-algebra anti-automorphism fixing the generators $\theta_i, 1 \leq i \leq n$. Using Proposition 3.3, it is easy to see that this inner product differs from

$$\langle 1_\lambda, \iota(\zeta_{j_1})^+ \iota(\zeta_{i_1})^- \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn} \tag{4.4}$$

by a power of v which is independent of p. But then the definition of the inner product and the previous proposition show that (4.4) may be written as $G(v, v^{-p})$ for some $G \in \mathcal{R}$. The result then follows immediately from the definition of $\mathcal{R}$. □

Definition 4.4. We define

$$(,) : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Q}(v),$$

a symmetric bilinear form on $\dot{\mathbf{U}}$ given by

$$(x,y) = \sum_{k=0}^{n-1} \lim_{p\to\infty} \langle x,y\rangle_{k+pn}.$$

Remark 4.5. Note that the proof of the last proposition actually allows us to conclude that

$$(\phi_D(\zeta_i^+\zeta_j^- 1_\lambda), [_pA])_{k+pn}$$

converges to an element of $\mathbb{Q}(v)$, as $p\to\infty$, for any $A\in\mathfrak{S}^{n,n}$. We will need this in the next section.

5. Comparison of inner products

There is a natural definition of an inner product on $\dot{\mathbf{U}}$ in the algebraic setting.

Theorem 5.1. *There exists a unique $\mathbb{Q}(v)$ bilinear pairing $\langle,\rangle\colon \dot{\mathbf{U}}\times\dot{\mathbf{U}}\to\mathbb{Q}(v)$ such that*

(1) $\langle 1_{\lambda_1}x1_{\lambda_2}, 1_{\mu_1}y1_{\mu_2}\rangle = 0 \qquad \forall x,y\in\dot{\mathbf{U}}$ *unless* $\lambda_1=\mu_1, \lambda_2=\mu_2$;
(2) $\langle ux,y\rangle = \langle x,\rho(u)y\rangle \qquad \forall x,y\in\dot{\mathbf{U}}, u\in\mathbf{U}$; *and*
(3) $\langle x^-1_\lambda, y^-1_\lambda\rangle = (x,y), \qquad \forall x,y\in\mathbf{f},\lambda\in X.$

Here (x,y) is the standard inner product on $\mathbf{f}$, (see [L93, 1.2.5]*). The resulting inner product is automatically symmetric.*

Proof. See [L93, 26.1.2]. □

Theorem 5.2. *The inner products $(,)$ of section 4 and $\langle,\rangle$ of Theorem 5.1 coincide.*

The remainder of this section is devoted to the proof of this theorem. The first property listed in Theorem 5.1 clearly holds for $(,)$, as the representatives for elements of X in $\mathfrak{S}_{D,n}$ are distinct when they exist. The second follows from Proposition 3.3; thus it only remains to verify the third. Fix $\lambda\in X$.

The algebra $\mathbf{f}$ is naturally graded: $\mathbf{f}=\bigoplus_{\nu\in\mathbb{N}I}\mathbf{f}_\nu$. For $\nu\in\mathbb{Z}[I]$, with $\nu=\sum_{i\in I}\nu_i i$ let $tr(\nu)=\sum_{i\in I}\nu_i$. If z is homogeneous we set $|z|=\nu$, where $z\in\mathbf{f}_\nu$. Thus for the third property we may assume that $x,y\in\mathbf{f}$ are homogeneous, i.e. $x,y\in\mathbf{f}_\nu$ for some ν, and proceed by induction on $N=\mathrm{tr}(\nu)$. If $N=0$ then we are reduced to the equation

$$(1_\lambda,1_\lambda)=1,$$

which is trivial. Now suppose that $N>0$ and the result is known for $x,y\in\mathbf{f}_\nu$ when $\mathrm{tr}(\nu)<N$. If x,y are in $\mathbf{f}_\nu$, $\mathrm{tr}(\nu)=N$, then we may assume that they are monomials, and $y=\theta_i z$ for some $z\in\mathbf{f}_{\nu-\mathbf{i}}$. Then we have

$$\begin{aligned}(x^-1_\lambda, y^-1_\lambda) &= (x^-1_\lambda, F_iz^-1_\lambda)\\ &= (vK_{-\mathbf{i}}E_ix^-1_\lambda, z^-1_\lambda).\end{aligned}$$

Using standard commutation formulas (see [L93, 3.1.6]) this becomes

$$(vK_{-\mathbf{i}}x^-E_i1_\lambda, z^-1_\lambda)+\frac{1}{1-v^{-2}}\left(({}_ir(x)^- - vK_{-\mathbf{i}}r_i(x)^-K_{-\mathbf{i}})1_\lambda, z^-1_\lambda\right)$$

and tidying this up we get

$$\frac{1}{1-v^{-2}}({}_ir(x)^-1_\lambda, z^-1_\lambda)+\left(v^{\mathbf{i}\cdot|x|-\mathbf{i}\cdot\lambda-1}\left(x^-E_i-\frac{v^{-\mathbf{i}\cdot\lambda}}{v-v^{-1}}r_i(x)^-\right)1_\lambda, z^-1_\lambda\right)$$

The properties of $(,)$ on $\mathbf{f}$ show that $\frac{1}{1-v^{-2}}({}_ir(x), z) = (x, \theta_i z)$, thus we are done by induction if we can show that

$$\left(x^- E_i - \frac{v^{-\mathbf{i}\cdot\lambda}}{v - v^{-1}} r_i(x)^-\right) 1_\lambda$$

annihilates $\mathbf{U}^- 1_\lambda$. To see this we need an explicit result about multiplication in $\mathfrak{A}_D$.

Lemma 5.3. *Let $A \in \mathfrak{S}^{n,n}$ be such that $a_{r,s} = 0$ for $r < s$ unless $r = s-1$ and $r = i$ mod n, when $a_{r,r+1} \in \{0,1\}$; then the following hold for p sufficiently large.*

(1) *For $j \neq i$ we have*

$$F_j[{}_pA] = \sum_{k=1}^{m} g_k(v)[{}_pZ_k]$$

where $g_k(v) \in \mathbb{Z}[v, v^{-1}]$ are independent of $\{a_{r,s} \colon r \leq s\}$, and $Z_k \in \mathfrak{S}^{n,n}$ have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$.

(2)

$$F_i[{}_pA] = \sum_{k=1} g_k(v)[{}_pZ_k] + v^{1-\mathbf{i}\cdot r(A)}\left(\frac{1 - v^{-2(a_{i+1,i+1}+1+p)}}{1 - v^{-2}}\right)[{}_p(A + E^{i+1,i+1} - E^{i,i+1})]$$

where $g_k(v) \in \mathbb{Z}[v, v^{-1}]$ are independent of $\{a_{r,s} \colon r \leq s\}$, and $Z_k \in \mathfrak{S}^{n,n}$ have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$, and the final term occurs only if $a_{i,i+1} = 1$.

Proof. Both of these formulas are consequences of the following, which is valid for any A (see Lemma 4.1).

$$F_j[{}_pA] = \sum_{k \colon ({}_pA)_{j,k} \geq 1} v^{a_{j+1,\leq k} - a_{j,<k}} \left(\frac{1 - v^{-2(a_{j+1,k} + p\delta_{j+1,k} + 1)}}{1 - v^{-2}}\right)[{}_pA + E^{j+1,k} - E^{j,k}].$$

□

Let $\sum_{j=1}^n \lambda_j = k \bmod n$, where $k \in \{0, 1, \ldots, n-1\}$, and suppose that $D = k + pn$ for some p. Let $\mathfrak{A}_D^- = \mathrm{span}\{[A] \colon a_{r,s} = 0, \forall r < s\}$, and note that Lemma 5.3 shows that $\phi_D(x^- 1_\lambda) \in \mathfrak{A}_D^-$ for any $x \in \mathbf{f}$. In fact, it is also clear that

$$\phi_D(x^- E_i 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[{}_pB_k] + \sum_{k=1}^{m_2} g_k(v)[{}_pH_k]$$

where $(B_k)_{i,i+1} = 1$ and $(H_k)_{i,i+1} = 0$, and a_k, g_k are independent of λ and p. Moreover from the formula in the proof of the Lemma 5.3 it is easy to see that

$$\phi_D(x^- 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[{}_pB_k + E^{i+1,i+1} - E^{i,i+1}].$$

We are now ready to set up the key step in the proof of Theorem 5.2: Let $\pi_D \colon \mathfrak{A}_D \to \mathfrak{A}_D^-$ be the orthogonal projection. Define $s_D \colon \mathbf{f} \to \mathfrak{A}_D^-$ by setting

$$x \mapsto \pi_D(\phi_D(x^- E_i 1_\lambda))$$

and define $r_D\colon \mathbf{f} \to \mathfrak{A}_D^-$ by setting

$$x \mapsto \frac{v^{-\mathbf{i}\cdot\lambda}}{v-v^{-1}}\phi_D(r_i(x)^- 1_\lambda)$$

Proposition 5.4. *Let $x \in \mathbf{f}$.*

$$s_D(x) - r_D(x) = v^{-2p}\left(\sum_{k=1}^{m} c_k(v)[_pZ_k]\right)$$

for some $Z_k \in \mathfrak{S}^{n,n}$, independent of p.

Proof. We may assume that x is a monomial, and proceed by induction on $\mathrm{tr}(|x|)$. It is easy to check that $s_D(1) = r_D(1) = 0$, so we may assume that $x \in \mathbf{f}_\nu$, $\mathrm{tr}(\nu) > 0$, and that $x = \theta_j z$ where $z \in \mathbf{f}_{\nu-\mathbf{j}}$. Now as above we have

$$\phi_D(z^- E_i 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[_pB_k] + \sum_{k=1}^{m_2} g_k(v)[_pH_k] \qquad (B_k)_{i,i+1} = 1, (H_k)_{i,i+1} = 0,$$

and so $s_D(z) = \sum_{k=1}^{m_2} g_k(v)[_pH_k]$. Let $E = E^{i+1,i+1} - E^{i,i+1} \in \mathfrak{S}^{n,n}$. Using the lemma we see that since

$$\phi_D(x^- E_i 1_\lambda) = F_j \phi_D(z^- E_i 1_\lambda),$$

we have

$$\begin{aligned} s_D(x) = \delta_{i,j} &\sum_k a_k(v) v^{1-\mathbf{i}\cdot r(B_k)}\left(\frac{1-v^{-2((B_k)_{i+1,i+1}+p+1)}}{1-v^{-2}}\right)[_p(B_k+E)] \\ &+ F_j s_D(z). \end{aligned} \tag{5.1}$$

Now $r(B_k) = \lambda + \mathbf{i} - |x|$, hence $1 - \mathbf{i}\cdot r(B_k) = \mathbf{i}\cdot(|x|-\lambda) - 1$, so

$$v^{1-\mathbf{i}\cdot r(B_k)}\left(\frac{1-v^{-2((B_k)_{i+1,i+1}+p+1)}}{1-v^{-2}}\right) = \left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v-v^{-1}}\right)(1 - v^{-2p}v^{-2((B_k)_{i+1,i+1}+1)}).$$

The definition of r_i shows that

$$\begin{aligned} r_D(x) &= r_D(\theta_j z) \\ &= \delta_{i,j}\left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v-v^{-1}}\right)\phi_D(z^- 1_\lambda) + F_j r_D(x) \\ &= \delta_{i,j}\left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v-v^{-1}}\right)\sum_k a_k(v)[_p(B_k+E)] + F_j r_D(x), \end{aligned}$$

so we see that

$$\begin{aligned} s_D(x) - r_D(x) = &F_j(s_D(z) - r_D(z)) \\ &- \delta_{i,j} v^{-2p}\sum_k a_k(v)\left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v-v^{-1}}\right)v^{-2((B_k)_{i+1,i+1}+1)}[_p(B_k+E)] \end{aligned}$$

and so using induction and the lemma again, we are done. □

Corollary 5.5. *Let $x \in \mathbf{f}$, then*

$$u = \left(x^- E_i - \frac{v^{-\mathbf{i}\cdot\lambda}}{v-v^{-1}} r_i(x)^-\right)1_\lambda$$

is orthogonal to $\mathbf{U}^- 1_\lambda$.

Proof. Let $y \in \mathbf{f}$ be a monomial. Then we have

$$(u, y^{-}1_\lambda) = \lim_{p\to\infty} \langle u, y^{-}1_\lambda\rangle_{k+pn},$$

and by definition

$$\langle u, y^{-}1_\lambda\rangle_{k+pn} = (s_{k+pn}(x) - r_{k+pn}(x), \phi_{k+pn}(y^{-}1_\lambda))_{k+pn}. \tag{5.2}$$

By the previous proposition,

$$s_{k+pn}(x) - r_{k+pn}(x) = v^{-2p}\left(\sum_{j=1}^{m} c_j(v)[{}_pZ_j]\right), \qquad Z_j \in \mathfrak{S}^{n,n},$$

and by the remark at the end of section 2, we know that $([{}_pZ_j], \phi_{k+pn}(y^{-}1_\lambda))_{k+pn}$ converges in $\mathbb{Q}((v^{-1}))$ as $p \to \infty$. Thus the right-hand side of Equation 5.2 tends to zero as required. □

This completes the proof of Theorem 5.2.

6. Geometric Interpretation

Recall from [L99, section 4] that $\mathfrak{A}_D$ possesses a canonical basis $\mathfrak{B}_D$ consisting of elements $\{A\}$, $A \in \mathfrak{S}_{D,n,n}$. To define these elements we must assume $\mathbf{k}$ is algebraically closed (either the algebraic closure of $\mathbb{F}_q$, in which case we must use sheaves in the étale topology, or $\mathbb{C}$ in which case we use the analytic topology). Fix $A \in \mathfrak{S}_{D,n}$, and $\mathbf{L} \in \mathcal{F}_{r(A)}$. The space $\mathcal{F}_D$ can be given the structure of an ind-scheme such that the set $X_A^{\mathbf{L}}$ (see section 3) lies naturally in a projective algebraic variety. Thus it makes sense to consider its closure $\bar{X}_A^{\mathbf{L}}$. Let $\mathcal{A}_{\mathbf{L}}$ be the simple perverse sheaf on $\bar{X}_A^{\mathbf{L}}$ whose restriction to $X_A^{\mathbf{L}}$ is $\mathbb{C}[d_A]$. Let $\mathcal{H}^s(\mathcal{A}_{\mathbf{L}})$ to be the s-th cohomology sheaf of $\mathcal{A}_{\mathbf{L}}$. For $A_1 \in \mathfrak{S}_{D,n,n}$ such that $X_{A_1}^{\mathbf{L}} \subset \bar{X}_A^{\mathbf{L}}$ we write $A_1 \leq A$, and set

$$\Pi_{A_1,A} = \sum_{s\in\mathbb{Z}} \dim(\mathcal{H}_y^{s-d_{A_1}}(\mathcal{A}_{\mathbf{L}}))v^s \in \mathbb{Z}[v^{-1}],$$

where $\mathcal{H}_y^{s-d_{A_1}}(\mathcal{A}_{\mathbf{L}})$ is the stalk of $\mathcal{H}^{s-d_{A_1}}(\mathcal{A}_{\mathbf{L}})$ at a point $y \in \mathbf{X}_{A_1}^{\mathbf{L}}$ (since $\mathcal{A}^{\mathbf{L}}$ is constructible with respect to the stratification of $\bar{X}_A^{\mathbf{L}}$ given by $\{X_{A_1}^{\mathbf{L}} : A_1 < A\}$, this is independent of the choice of y). We have

$$\{A\} = \sum_{A_1; A_1 \leq A} \Pi_{A_1,A}[A_1].$$

Note that the following is an immediate consequence of the definitions and Lemma 3.5.

Lemma 6.1. *Let $A, A' \in \mathfrak{S}_{D,n,n}$, then,*

$$(\{A\}, \{A'\})_D \in \delta_{A,A'} + v^{-1}\mathbb{Z}[v^{-1}].$$

□

The algebra $\mathfrak{A}_D$ may be viewed as a convolution algebra of (equivariant) complexes on $\mathcal{F}^n$. We wish to give an interpretation of the inner product of section 3 in this context. Suppose that $A, B \in \mathfrak{S}_{D,n,n}$. We want to describe $(\{A\}, \{B\})$. We may assume that $r(A) = r(B) = \mathbf{a}$ and $c(A) = c(B) = \mathbf{b}$. Let $\mathbf{L}' \in \mathcal{F}_{\mathbf{b}}$. Let $\mathcal{A}^t_{\mathbf{L}'}$ and $\mathcal{B}^t_{\mathbf{L}'}$ denote the simple perverse sheaves on $\bar{X}^{\mathbf{L}'}_{A^t}$ and $\bar{\mathbf{X}}^{\mathbf{L}'}_{B^t}$ respectively. Then define

$$\langle \{A\}, \{B\} \rangle^D = \sum_{i \in \mathbb{Z}} \dim(H^i_c(\mathcal{F}_{\mathbf{a}}, \mathcal{A}^t_{\mathbf{L}'} \otimes \mathcal{B}^t_{\mathbf{L}'}))v^i. \tag{6.1}$$

$\langle , \rangle^D$ extends to an inner product on the whole of $\mathfrak{A}_D$ (viewed as an algebra of equivariant complexes on $\mathcal{F}^n$). We want to show that it is the same as the inner product $(,)_D$ of section 3 , at least on the subalgebra $\mathbf{U}_D$. We start by showing that $\langle , \rangle$ satisfies the properties of Proposition 3.3.

Lemma 6.2. *Let $A, B, C \in \mathfrak{S}_{D,n,n}$, and suppose that $\mathcal{O}_A$ is a closed orbit. Then*

$$\langle \{A\}\{B\}, \{C\} \rangle^D = v^{d_A - d_{A^t}} \langle \{B\}, \{A^t\}\{C\} \rangle^D.$$

Proof. Both sides are obviously zero unless $r(A) = r(C) = \mathbf{a}$, $c(A) = r(B) = \mathbf{b}$ and $c(B) = c(C) = \mathbf{c}$, thus we assume these equalities from now on. Pick $\mathbf{L}_0 \in \mathcal{F}_{\mathbf{c}}$. We need to recall the definition of the convolution product. Pick a subset Y of $\mathcal{F}_{\mathbf{b}}$ such that Y is a smooth projective variety containing $X^{\mathbf{L}_0}_{B^t}$. Let

$$Z_A = \{(\mathbf{L}, \mathbf{L}') \in O_A : \mathbf{L}' \in Y\},$$

where $r(A) = \mathbf{a}$, $c(A) = \mathbf{b}$. We have maps $p_1 : Z \to \mathcal{F}_{\mathbf{a}}$ and $p_2 : Z \to Y$, the first and second projections respectively. The map p_1 is clearly proper (as the fibre is $X^{\mathbf{L}}_A \cap Y$) and the map p_2 is smooth with fibre dimension d_{A^t}.

$$(\mathbf{A} * \mathbf{B})^t = (p_1)_! p_2^*(\mathbf{B^t})[d_{A^t}],$$

Now

$$\begin{aligned}(\mathbf{A} * \mathbf{B})^t \otimes \mathbf{C^t} &= (p_1)_! p_2^*(\mathbf{B^t})[d_{A^t}] \otimes \mathbf{C^t} \\ &= (p_1)_!(p_2^*(\mathbf{B^t}) \otimes p_1^*(\mathbf{C^t})[d_{A^t}]),\end{aligned} \tag{6.2}$$

where we use the projection formula in the second equality.

On the other hand, to compute the product $\{A^t\}\{C\}$ we pick a smooth projective variety $W \subset \mathcal{F}_{\mathbf{a}}$ which contains $X^{\mathbf{L}_0}_{C^t}$, and consider the variety

$$Z_{A^t} = \{(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A : \mathbf{L} \in W\}.$$

As above there are projection maps p_1,p_2, and the product is given by

$$(\{A^t\}\{C\})^t = (p_2)_! p_1^*(\mathcal{C}^t)[d_{A^t}].$$

and

$$\begin{aligned}\{B^t\} \otimes (\{A^t\}\{C\})^t &= \{B^t\} \otimes (p_2)_! p_1^*(\mathcal{C}^t)[d_{A^t}] \\ &= (p_2)_!(p_2^*(\{B^t\}) \otimes p_1^*(\mathcal{C}^t))[d_{A^t}].\end{aligned}$$

Now as tensor product is a local functor, we may restrict to $Z_A \cap Z_{A^t}$, and then it is clear that the compactly supported cohomologies will be equal up to shifts, with the difference in shifts being $d_A - d_{A^t}$ as required.

□

Lemma 6.3. *Let $A, B \in \mathfrak{S}_{D,n,n}$, and $\mathbf{c} \in \mathfrak{S}^n$. Then*

(1) $\langle E_i\{A\}, \{B\} \rangle^D = \langle \{A\}, vK_{\mathbf{i}}F_i\{B\} \rangle^D$.
(2) $\langle F_i\{A\}, \{B\} \rangle^D = \langle \{A\}, vK_{-\mathbf{i}}E_i\{B\} \rangle^D$
(3) $\langle K_{\mathbf{c}}\{A\}, \{B\} \rangle^D = \langle \{A\}, K_{\mathbf{c}}\{B\} \rangle^D$

Proof. This follows from the previous lemma exactly as in the proof of corollary 3.3, since the varieties $X^{\mathbf{L}}_{\mathbf{a}+i\mathbf{e_a}}$ are closed. □

The algebra $\mathbf{U}_D$ is spanned by elements of the form $T_1T_2\dots T_N[\mathbf{i_a}]$ where T_s is either E_i or F_i for some i. Thus the previous lemma shows we need only check that

$$\langle T_1T_2\dots T_N[\mathbf{i_a}],[\mathbf{i_a}]\rangle^D = (T_1T_2\dots T_N[\mathbf{i_a}],[\mathbf{i_a}])_D$$

But this will follow if we can show that

$$\langle\{A\},[\mathbf{i_a}]\rangle^D = (\{A\},[\mathbf{i_a}])_D$$

for all $A \in \mathfrak{S}_{D,n,n}$, as $\{\{A\} \colon A \in \mathfrak{S}_{D,n,n}\}$ is a basis of $\mathfrak{A}_D$. But as the simple perverse sheaf corresponding to $\{\mathbf{i_a}\} = [\mathbf{i_a}]$ is just the skyscraper sheaf at the point $\mathbf{L}'$, this last equality follow from directly from the definitions. We have therefore shown the following result.

Proposition 6.4. *On the algebra $\mathbf{U}_D$ the inner products $\langle,\rangle^D$ and $(,)_D$ coincide.* □

Remark 6.5. It can be shown that the algebra $\mathfrak{A}_D$ is generated by the elements $\{A\}$ for which $X^{\mathbf{L}}_A$ is closed, and so the above argument adapts to show that the inner products in fact agree on the whole of $\mathfrak{A}_D$. Henceforth we will use the notation $(,)_D$ when referring to the inner product on $\mathfrak{A}_D$ in either of its incarnations.

7. TRANSFER MAPS

In this section we give a sheaf-theoretic description of the "transfer maps" [L99a], using the "hyperbolic localization" of [Br]. In [L99a] Lusztig defines homomorphisms

$$\psi_D \colon \mathfrak{A}_D \to \mathfrak{A}_{D-n},$$

which are characterized , at least on $\mathbf{U}_D$, by the following,

- $\psi_D(E_i(D)) = E_i(D-n)$;
- $\psi_D(F_i(D)) = F_i(D-n)$;
- $\psi_D(K_{\mathbf{a}}(D)) = v^{\mathbf{a}\cdot\mathbf{b}_0}K_{\mathbf{a}}(D-n)$.

The definition of [L99a], which works with functions on the varieties defined over $\mathbf{F}_q$, has two parts: the first a kind of coproduct, and second a combination of a diagonal operator with an extension of the sign representation. We first give a sheaf theoretic construction of the "coproduct". Grojnowski has informed the author that this was already done in the unpublished preprint [G92]. Indeed the idea that coproducts should be viewed as localizations was also first pointed out to the author by Grojnowski, see for example [G95]. Fix integers D_1 and D_2 with $D_1 + D_2 = D$. Fix a decomposition $V = V_1 \oplus V_2$, into a direct sum of two free modules of rank D_1 and D_2 respectively. Consider the $\mathbb{C}^*$ action on V where $z \in \mathbb{C}^*$ acts on V_1 by z and on V_2 by z^{-1}. The induced action on $\mathcal{F}^n$ has a fixed point set $\dot{\mathcal{F}}^n$ isomorphic to $\mathcal{F}^n_{D_1} \times \mathcal{F}^n_{D_2}$. Indeed, somewhat more precisely, if we restrict the action to a component $\mathcal{F}_{\mathbf{a}}$, then the fixed point set $\dot{\mathcal{F}}^n_{\mathbf{a}}$ is isomorphic to

$$\bigcup_{\mathbf{a}_1+\mathbf{a}_2=\mathbf{a}} \mathcal{F}_{\mathbf{a}_1} \times \mathcal{F}_{\mathbf{a}_2},$$

where $\mathbf{a}_1 \in \mathfrak{S}_{D_1,n}$, and $\mathbf{a}_2 \in \mathfrak{S}_{D_2,n}$.

Fix $A \in \mathfrak{S}_{D,n,n}$ and pick $\mathbf{L} \in \dot{\mathcal{F}}_{r(A)}$. Consider the intersection cohomology sheaf $\mathbf{A}$ associated to the $G_{\mathbf{L}}$–orbit $X^{\mathbf{L}}_A$. Let $h \colon \dot{\mathcal{F}}^n \to \mathcal{F}^n$ be the obvious inclusion.

Since the inclusion f is the fixed point set of a $\mathbb{C}^*$–action, we have a natural map in the opposite direction, $\pi\colon \mathcal{F}^n \to \dot{\mathcal{F}}^n$ which sends $\mathbf{L} \mapsto \lim_{z\to 0} z \cdot \mathbf{L}$. Given a connected component of the fixed point set $\dot{\mathcal{F}}_{\mathbf{a}}$, its preimage under the map π is a locally closed subvariety of $\mathcal{F}_{\mathbf{a}}$. Let $Y_{\mathbf{a}}$ be the disjoint union of these pieces, let g be the obvious map from $Y_{\mathbf{a}}$ to $\mathcal{F}_{\mathbf{a}}$, and let f be the inclusion of the fixed point set $\dot{\mathcal{F}}_{\mathbf{a}}$ into $Y_{\mathbf{a}}$. Then, following [Br], we define the hyperbolic localization of $\mathbf{A}$ to be $\mathbf{A}_{\mathbb{C}^*} = f^! g^*(\mathbf{A})$. By the main result of that paper, this complex is semisimple. The simple perverse sheaves which occur must be invariant under the action of the group $\mathrm{Aut}(V_1) \times \mathrm{Aut}(V_2)$, and hence must be the tensor product of intersection cohomology complexes on $\mathcal{F}^n_{D_1}$ and $\mathcal{F}^n_{D_2}$. It follows that by associating to each simple perverse sheaf its corresponding canonical basis element we obtain a map $\Delta\colon \mathfrak{A}_D \to \mathfrak{A}_{D_1} \otimes \mathfrak{A}_{D_2}$. We first show that Δ is an algebra map, for which we require the following lemmas, the first of which is well-known, see for example [KSc, 3.7.5].

Lemma 7.1. *Let E be a vector bundle over B, and let $\pi\colon E \to B$ be the bundle map, $i\colon B \to E$ be the inclusion of B as the zero section of E. Then for a conic sheaf F we have isomorphisms*

$$\pi_*(F) \simeq i^*(F); \qquad \pi_!(F) \simeq i^!(F).$$

It follows immediately from this lemma that if $\mathbf{A}$ be a semisimple complex on a proper variety X, and suppose X is equipped with an action of $\mathbb{C}^*$. Let X^+ be the disjoint union of the pieces in the Bialynicki-Birula decomposition of X, and let $f\colon X_{\mathbb{C}^*} \to X^+$ and $g\colon X^+ \to X$ be the obvious maps, Let $\pi : X^+ \to X_{\mathbb{C}^*}$ be the map given by $x \mapsto \lim_{z\to 0} z \cdot x$. Then we have:

$$\mathbf{A}_{\mathbb{C}^*} = f^! g^*(\mathbf{A}) \simeq \pi_! g^*(\mathbf{A}). \tag{7.1}$$

In other words, hyperbolic localization may be replaced by projection to the fixed point sets.

Proposition 7.2. *Hyperbolic localization is an algebra homomorphism with respect to convolution of sheaves.*

Proof. This is just the sheaf-theoretic version of the corresponding result in [L99a], we simply have to be slight more careful about the geometry. We use a similar setup to Lemma 6.2. Let $A, E \in \mathfrak{S}_{D,n,n}$ such that $c(A) = r(E)$. Set $r(E) = \mathbf{a}$, and $c(E) = \mathbf{b}$. Pick $\mathbf{L}_0 \in \dot{\mathcal{F}}^n_{r(A)}$. Assume $\mathcal{O}_E$ is a closed orbit, and let X be a subset of $\mathcal{F}_{\mathbf{a}}$ which is a smooth projective variety containing $X_A^{\mathbf{L}_0}$ as a locally closed subvariety (see [L99, section 4] for more details). Let

$$Z = \{(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_E \colon \mathbf{L} \in X\}.$$

Pick a smooth projective variety $Y \subset \mathcal{F}_{\mathbf{b}}$ such that Z is a subset of $X \times Y$. The maps $p\colon Z \to X$ and $q\colon Z \to \mathcal{Y}$ are the first and second projections. The map p is smooth and proper with fibre dimensions d_E, and the map q is proper. Let $\mathbf{A}$ and E be the intersection cohomology complexes on $\bar{X}_A^{\mathbf{L}_0}$ and $X_E^{\mathbf{L}}$ respectively. Then we have

$$(\mathbf{A} * \mathbf{E}) = q_! p^*(\mathbf{A})[d_E].$$

Consider the diagrams:

$$\begin{array}{ccccc} X^+ & \xleftarrow{p^+} & Z^+ & \xrightarrow{q^+} & Y^+ \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Z} & & \downarrow{\scriptstyle g_Y} \\ X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \end{array}$$

and

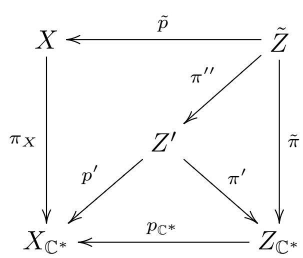
$$\begin{array}{ccccc} X^+ & \xleftarrow{p^+} & Z^+ & \xrightarrow{q^+} & Y^+ \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ X_{\mathbb{C}^*} & \xleftarrow{p_{\mathbb{C}^*}} & Z_{\mathbb{C}^*} & \xrightarrow{q_{\mathbb{C}^*}} & Y_{\mathbb{C}^*} \end{array}$$

where the maps $p_{\mathbb{C}^*}$ and $q_{\mathbb{C}^*}$ are the restrictions of p and q respectively, and the maps π_X, π, and π_Y are all obtained from the $\mathbb{C}^*$-action in the manner described above. By Equation 7.1, and the commutativity of the diagram we see that

$$\begin{aligned} (\mathbf{A} * \mathbf{E})_{\mathbb{C}^*} &\simeq (\pi_Y)_!(g_Y)^* q_! p^*(\mathbf{A})[d_E] \\ &\simeq (\pi_Y)_!(q^+)_!(g_Z)^* p^*(\mathbf{A})[d_E] \\ &\simeq (q_{\mathbb{C}^*})_! \pi_! (p^+)^*(g_X)^*(\mathbf{A})[d_E]. \end{aligned} \tag{7.2}$$

where the second equality holds because the map q is compatible with the $\mathbb{C}^*$ action and is smooth, so that the components of the fixed point sets $Z_{\mathbb{C}^*}$ and $Y_{\mathbb{C}^*}$ correspond and thus the right square in the first diagram is Cartesian.

We now want to consider the convolution $\mathbf{A}_{\mathbb{C}^*} * \mathbf{E}_{\mathbb{C}^*}$. For this we use the diagram:



where

$$Z' = \{(\mathbf{L}, \mathbf{L}') \in Z^+ : \mathbf{L} \in X_{\mathbb{C}^*}\},$$

and

$$\tilde{Z} = \{(\mathbf{L}, \mathbf{L}') \in X^+ \times Y^+ : (\pi_X(\mathbf{L}), \mathbf{L}') \in Z^+\},$$

and the maps are the obvious ones. Note that $X, X_{\mathbb{C}^*}, \tilde{Z}, Z'$ form a Cartesian square. Since π' is an affine bundle, $(\pi')^*(\pi')_!$ is simply a shift by twice the dimension of the fibre. As this dimension is exactly the difference between the fibre dimensions for the convolutions on Z and on the fixed points $\mathbf{Z}_{\mathbb{C}^*}$, we see that

$$\mathbf{A}_{\mathbb{C}^*} * \mathbf{E}_{\mathbb{C}^*} = (q_{\mathbb{C}^*})_!(\pi')_!(\pi')^* p^*_{\mathbb{C}^*}(\pi_X)_!(g_X)^*(\mathbf{A})[d_E],$$

and so we have by functoriality and the projection formula that

$$\begin{aligned} \mathbf{A}_{\mathbb{C}^*} * \mathbf{E}_{\mathbb{C}^*} &= (q_{\mathbb{C}^*})_!(\pi')_!(p')^*(\pi_X)_!(g^*_X)(\mathbf{A})[d_E] \\ &= (q_{\mathbb{C}^*})_!(\pi')_!(\pi'')_!(\tilde{p})^*(g_X)^*(\mathbf{A})[d_E] \\ &= (q_{\mathbb{C}^*})_!(\tilde{\pi})_!(\tilde{p})^*(g_X)^*(\mathbf{A})[d_E]. \end{aligned} \tag{7.3}$$

Thus in order to show that $(\mathbf{A} * \mathbf{E})_{\mathbb{C}^*} = \mathbf{A}_{\mathbb{C}^*} * \mathbf{E}_{\mathbb{C}^*}$, by Equations 7.2 and 7.3 it is enough to show that

$$\pi_!(p^+)^*(\mathbf{B}) = (\tilde{\pi}_!)(\tilde{p})^*(\mathbf{B}),$$

where $\mathbf{B} = g_X^*(\mathbf{A})[d_E]$. We use Equation 7.1 again — replacing projection to the fixed points of the $\mathbb{C}^*$ action in each case with hyperbolic localization. As the two smooth varieties Z and $\tilde{Z}$ have the same fixed-point set $Z_{\mathbb{C}^*}$, the components of the Bialynicki-Birula decomposition correspond. Let $\{Z_i \colon 1 \leq i \leq k\}$ be the pieces of the decomposition of Z, and let $\{\tilde{Z}_i \colon 1 \leq i \leq k\}$ be pieces of $\tilde{Z}$, ordered so that Z_i corresponds to $\tilde{Z}_i$, that is, Z_i and $\tilde{Z}_i$ are affine bundles over the same component F_i of $Z_{\mathbb{C}^*}$, their common fixed-point set.

Next note that if $X = \bigcup_{j \in [1,r]}$ is the Bialynicki-Birula decomposition of X, then Z_i and $\tilde{Z}_i$ map surjectively onto a single piece, $X_{j(i)}$ say, as the maps p and $\tilde{p}$ are $\mathbb{C}^*$-equivariant. We claim that $p^+_{|Z_i}$ and $\tilde{p}_{|\tilde{Z}_i}$ are smooth maps over $X_{j(i)}$ of the same fibre dimension. It is clear from the definitions that $\tilde{p}_{|\tilde{Z}_i}$ is smooth. To see that the same is true for $p^+_{|Z_i}$ we first let $\mathcal{U}$ be the subgroup of G whose elements act trivially on V_1 and V/V_1. The group $\mathcal{U}$ acts transitively on the fibres of the projection to the fixed points of the $\mathbb{C}^*$-action on $\mathcal{F}^n$ (see for example [L99a, 1.4]). Fix $\mathbf{L} \in X_{j(i)}$. Pick $u \in \mathcal{U}$ so that $\lim_{z\to 0} z \cdot \mathbf{L} = u \cdot \mathbf{L}$. Then the fibre of $p_{|Z_i}$ over $\mathbf{L}$ is

$$\{(\mathbf{L}, \mathbf{L}') \in Z : \lim_{z\to 0} z \cdot (\mathbf{L}, \mathbf{L}') \in F_i\}$$

but since Z is $\mathcal{U}$-invariant (as it is even G-invariant) we see that this is

$$\{(\mathbf{L}, \mathbf{L}') \colon (\lim_{z\to 0} z \cdot \mathbf{L}, u \cdot \mathbf{L}') \in Z, \lim_{z\to 0} z \cdot (\mathbf{L}, \mathbf{L}') \in F_i\},$$

which in turn is just

$$\{(\mathbf{L}, \mathbf{L}') \colon (\mathbf{L}, u \cdot \mathbf{L}') \in (\tilde{p}_{|\tilde{Z}_i})^{-1}(\mathbf{L})\}.$$

Thus we see immediately that $p_{|Z_i}$ is smooth, with fibres (noncanonically) isomorphic to the fibres of $\tilde{p}_{|\tilde{Z}_i}$. Moreover it is clear that the complex $\mathbf{B}$ is conic with respect to these bundles structures. Let d_i be there common fibre dimension, and let $f : F_i \to Z_i$ and $\tilde{f} F_i \to \tilde{Z}_i$ be the obvious inclusion maps.

By Equation 7.1 we then see that

$$\begin{aligned}
\pi_!(p^+)^*(\mathbf{B}) &= f^!(p^+)^*(\mathbf{B}) \\
&= f^!(p^+)^!(\mathbf{B})[2d_i] \\
&= \tilde{f}^!(\tilde{p})^!(\mathbf{B})[2d_i] \\
&= \tilde{\pi}_!\tilde{p}^*(\mathbf{B})
\end{aligned}$$

as required.

Hence we see that $\pi_! p^*(\mathbf{A}) = (\tilde{\pi})_!(\tilde{p})^*(\mathbf{A})$, and so $(\mathbf{A} * \mathbf{E})_{\mathbb{C}^*} = \mathbf{A}_{\mathbb{C}^*} * \mathbf{E}_{\mathbb{C}^*}$. Since $\mathfrak{A}_D$ is generated by elements corresponding to closed orbits, the proposition follows. □

We now need the other components of our transfer map. One of these is a "diagonal" operator $\xi : \mathfrak{A}_D \to \mathfrak{A}_D$ which acts as $v^{(\mathbf{a}:\mathbf{b})}$ on $[i_\mathbf{a}]\mathfrak{A}_D[i_\mathbf{b}]$, where

$$(\mathbf{a} : \mathbf{b}) = \sum_{i \in [1,n]} i(a_i - b_i); \qquad \mathbf{a}, \mathbf{b} \in \mathfrak{S}_{D,n}.$$

It is clear that ξ is an algebra homomorphism, and in the context of sheaf theory, it is just a shift operator, shifting by different amounts on different components of $\mathcal{F}^n \times \mathcal{F}^n$.

The last component of the transfer map is a character of the algebra $\mathfrak{A}_n$, which is an extension of the "sign character" of the Hecke algebra for affine GL_n. If $\mathbf{b}_0 = (\ldots, 1, 1, \ldots) \in \mathfrak{S}_{n,n}$, then this Hecke algebra is isomorphic to the subalgebra $[\mathbf{i}_{\mathbf{b}_0}]\mathfrak{A}_n[\mathbf{i}_{\mathbf{b}_0}]$, as we shall explain below. The map $\chi \colon \mathfrak{A}_n \to Z[v, v^{-1}]$ is just the sign character on this subalgebra, and zero everywhere else. It is shown to be an algebra map in [L99a]. The Weyl group W of affine GL_n is an extension by Ω, an infinite cyclic group, of a Coxeter group W'. This Coxeter part of W is generated by the involutions $I = \{s_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ subject the braid relation:

$$\begin{aligned} s_i s_j s_i = s_j s_i s_j, \qquad & |i - j| = 1; \\ s_i s_j = s_j s_i \qquad & |i - j| > 1. \end{aligned}$$

and Ω acts on W' by $\omega s_i \omega^{-1} = s_{i+1}$ where ω is a specified generator of Ω. We can write elements of W uniquely as a product γw where $w \in W'$ and $\gamma \in \Omega$. The standard length function extends to W by setting $l(\gamma) = 0$ for $\gamma \in \mathbb{Z}$, and the Bruhat order extends by setting $\gamma_1 w < \gamma_2 w'$ when $\gamma_1 = \gamma_2$ and $w < w'$. The Hecke algebra $\mathcal{H}$ is a $\mathbb{Z}[v, v^{-1}]$ algebra with a $\mathbb{Z}[v, v^{-1}]$-basis T_w $(w \in W)$, and multiplication given by $T_w T_{w'} = T_{ww'}$ if $l(w) + l(w') = l(ww')$, and the quadratic relation

$$(T_{s_i} - v)(T_{s_i} + v^{-1}) = 0, \qquad s_i \in I.$$

There is an involution $- : \mathcal{H} \to \mathcal{H}$ which sends $v \mapsto v^{-1}$ and $T_w \mapsto T_w^{-1}$, as can easily be checked from the defining relations. The Kazhdan-Lusztig basis of $\mathcal{H}$ is a subset $\{C_w \colon w \in W'\}$ where C_w is characterized by the properties that

$$C_w = T_w + \sum_{y<w} p_{y,w} T_y; \qquad p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}],$$

and $C_w = \bar{C}_w$. For convenience we set $p_{w,w} = 1$. The group W has a realization as a permutation group acting on $\mathbb{Z}$. Indeed it is precisely the set of all permutations of the integers w such that $w(i+n) = w(i)+n$ $(i \in \mathbb{Z})$, with ω being the permutation $i \mapsto i + 1$. It is then clear that $w \in W$ corresponds to an infinite matrix A_w with $(A_w)_{ij} = \delta_{i,w^{-1}(j)}$. This identification allows us to describe the isomorphism from $\mathcal{H}$ to the subalgebra $[\mathbf{i}_{\mathbf{b}_0}]\mathfrak{A}_n[\mathbf{i}_{\mathbf{b}_0}]$: the Kazhdan-Lusztig basis element C_w maps to $\{A_w\}$, or equivalently T_w maps to $[A_w]$.

The following lemma is presumably well-known.

Lemma 7.3. *If C_w is an element of the Kazhdan-Lusztig basis of the Hecke algebra, and χ is the sign character, then $\chi(C_w) = \delta_{0,l(w)}$. Hence $\chi(\{A\}) = 0$ for $A \in \mathfrak{S}_{n,n,n}$ unless $A = A_\gamma$ for some $\gamma \in \Omega$.*

Proof. The character $\chi \colon \mathcal{H} \to \mathbb{Z}[v, v^{-1}]$ is defined by $\chi(T_w) = (-v^{-1})^{l(w)}$. Hence we see immediately that if $-$ is the involution of $\mathbb{Z}[v, v^{-1}]$ sending $v \mapsto v^{-1}$, the character χ intertwines the two bar involutions. Thus we have

$$\chi(C_w) = \sum_{y \le w} (-1)^{l(w)} v^{-l(w)} p_{y,w},$$

and the right-hand side is in $v^{-1}\mathbb{Z}[v^{-1}]$ unless $l(w) = 0$, when it is clearly equal to 1. However, by our earlier remark, the right-hand side is also fixed by the bar involution, and if $l(w) > 0$ this can only happen if it vanishes. This proves the lemma. □

Remark 7.4. In order to interpret this map in terms of sheaves, we need to use the functor of vanishing cycles, (see for example [KSc] for its definition and basic properties). Pick $\mathbf{L} \in \mathcal{F}_{\mathbf{b}_0}$, and consider the set

$$\Xi = \{\mathbf{L}^k \colon L_i^k = L_{i+k}\}.$$

Thus Ξ is a countable set of discrete points in $\mathcal{F}_{\mathbf{b}_0}$. Given $A \in \mathfrak{S}_{n,n,n}$, let $\mathbf{A}$ be the intersection cohomology sheaf on $\bar{X}_A^{\mathbf{L}}$. For each point p of Ξ we pick a suitably generic function germ f_p, and consider the Poincare polynomial $\chi_p(\mathbf{A}) = \sum_{i\in\mathbb{Z}} \dim(\mathcal{H}_p^i(\phi_{f_p}(\mathbf{A})))v^i$, where ϕ_f is the vanishing cycles functor with respect to f_p. Since the support of $\mathbf{A}$ intersects Ξ in at most finitely many points, it makes sense to define $\chi(\mathbf{A}) = \sum_{p\in\Xi} \chi_p(\mathbf{A})$. It can then be shown that $\chi(\mathbf{A}) = \chi(\{A\})$.

The transfer map $\psi_D \colon \mathfrak{A}_D \to \mathfrak{A}_{D-n}$ is then defined to be the composite

$$\mathfrak{A}_D \xrightarrow{\Delta_{D-n,n}} \mathfrak{A}_{D-n} \otimes \mathfrak{A}_n \xrightarrow{\xi\otimes\chi} \mathfrak{A}_{D-n}$$

We claim that on the subalgebra $\mathbf{U}_D$ the map ψ_D is compatible with the canonical bases. More precisely, recall that Lusztig [L99] showed that the subalgebra $\mathbf{U}_D$ of $\mathfrak{A}_D$ is spanned by the elements $\{A\}$ of $\mathfrak{B}_D$ which are *aperiodic*, i.e. the matrix A has for any $p \neq 0$ some i such that $a_{i,i+p} = 0$.

8. Applications

In this section we give some applications of the results of the previous sections. Let $\hat{\mathbf{U}} = \varprojlim_D \mathfrak{A}_D$ where the limit is taken over the projective system given by the maps $(\psi_D)_{D\in\mathbb{N}}$ of the last section. Since the maps ϕ_D are compatible with this system, that is, $\psi_{D+n}\phi_{D+n} = \phi_D$, there is a unique map $\phi \colon \dot{\mathbf{U}} \to \hat{\mathbf{U}}$, which factors each of the maps ϕ_D through the canonical map $\hat{\mathbf{U}} \to \mathfrak{A}_D$. Theorem 5.2 allows us to give an alternative proof of an injectivity result due to Lusztig [L99a].

Proposition 8.1. *The homomorphism ϕ is injective.*

Proof. Let u be in the kernel of ϕ. Then for every D we have $\phi_D(u) = 0$, and hence by Theorem 5.2 we see that u is in the radical of the inner product on $\dot{\mathbf{U}}$. Since this inner product is nondegenerate it follows that $u = 0$. □

The modified quantum group $\dot{\mathbf{U}}$ is equipped with a canonical basis $\dot{\mathbf{B}}$ which generalizes the canonical basis of $\mathbf{U}^-$. We can use the compatibility of the inner products to show a kind of "asymptotic" compatibility of the canonical bases of $\dot{\mathbf{U}}$ and $\mathfrak{A}_D$.

Proposition 8.2. *Let $b \in \dot{\mathbf{B}}$. Then there exists λ such that $b \in \dot{\mathbf{U}}1_\lambda$. Set k to be the residue of $\sum_{i=1}^n \lambda_i$* mod n. Then there is a $p_0 > 0$ such that for all $p > p_0$ we have $\phi_{k+pn}(b) \in \mathfrak{B}_D$.

Proof. The canonical basis of $\dot{\mathbf{U}}$ is characterized (up to sign) by the properties of being invariant under the bar involution, lying in the integral form $\dot{\mathbf{U}}_{\mathcal{A}}$, and having self inner product 1 modulo $v^{-1}\mathbb{Z}[[v^{-1}]]$ (for a precise statement see [L93, Theorem 26.3.1]). Each of these ingredients has a counterpart for $\mathfrak{A}_D$, and using Lemma 6.1 it is easy to see that $\mathfrak{B}_D$ is also characterized in this way. Since the inner product

$\dot{\mathbf{U}}$ is obtained as a limit from the inner products on $\mathfrak{A}_D$, it follows that for large p we have

$$(b,b)_{k+pn} = 1 \bmod v^{-1}\mathbb{Z}[v^{-1}].$$

The bar involutions on $\dot{\mathbf{U}}$ and $\mathfrak{A}_D$ are compatible, as can be easily checked on generators, and the maps ϕ_D are compatible with the integral forms. Therefore at least for large p we have $\phi_D(b)$ is, up to sign, an element of $\mathfrak{B}_D$, the canonical basis of $\mathfrak{A}_D$. The issue of sign can be resolved by using induction, and the fact that for an element of $\mathbf{U}^{\pm}1_\lambda$ the compatibility can be obtained directly from geometry (see [L99a, section 3]), so that in fact $\phi_D(b)$ is an element of $\mathfrak{B}_D$. □

It is a theorem of Schiffmann and Vasserot [SV] (conjectured by Lusztig [L99a]) that the maps ϕ_D are all compatible with the canonical basis, *i.e.* if $b \in \dot{\mathbf{B}}$ then $\phi_D(b) \in \mathfrak{B}_D \cup \{0\}$. The previous result shows that this theorem would also follow if we could show that the maps ϕ_D are compatible with the bases $\mathbf{B}_D$ and $\mathbf{B}_{D-n}$. The previous section shows that this can be made into a purely geometric question concerning perverse sheaves. Note that it is *not* true that the maps ψ_D send $\mathfrak{B}_D$ to $\mathfrak{B}_{D-n} \cup \{0\}$, as was pointed out already in [L99a, 1.12]. Unfortunately, at the moment this compatibility does not seem evident to the author, but we hope to return to this question in the future.

We can also combine Theorem 5.2 and Proposition 6.4 to prove a positivity result for the inner product of two elements of $\dot{\mathbf{B}}$. This result was conjectured by Lusztig,

Theorem 8.3. *Let* $b_1, b_2 \in \dot{\mathbf{B}}$.

$$(b_1, b_2) \in \mathbb{N}[[v^{-1}]] \cap \mathbb{Q}(v).$$

Proof. We may assume that there is a $\lambda \in X$ such that $b_1 1_\lambda = b_1$, and $b_2 1_\lambda = b_2$. Let $k \in \{0, 1, \dots, n-1\}$ be such that $\sum_{j=1}^{n} \lambda_j = k \bmod n$. Then

$$(b_1, b_2) = \lim_{p \to \infty} (\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn}$$

By Proposition 8.2 we know that for all large enough p, $\phi_D(b_1), \phi_D(b_2)$ are in $\mathfrak{B}_D$, hence it is clear from Equation (6.1) that

$$(\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn} \in \mathbb{N}[v, v^{-1}].$$

However, it follows also from Lemma 6.1 that the left-hand side is in fact in $\mathbb{N}[v^{-1}]$ (this can also be seen directly, using the definition of intersection cohomology sheaves). Hence (b_1, b_2) is the limit of elements of $\mathbb{N}[v^{-1}]$, and the statement follows. □

Remark 8.4. All the results of this paper have analogues for the nonaffine case, which can be proved in exactly the same way. The module V is replaced by a D-dimensional vector space over $\mathbf{k}$, and the space $\mathcal{F}^n$ of n-step periodic lattices should be replaced by the space of n-step flags in that vector space. In this case it is the algebra corresponding to $\mathbf{U}_D$ is actually equal to the algebra analogous to $\mathfrak{A}_D$, hence the results are in sometimes easier in this case.

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