ALGEBRAIC GROUPS.

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1. AFFINE VARIETIES

Let k be a field, which is assumed most of the time, including at the moment, to be algebraically closed. The term k-algebra will be reserved for a commutative, associative, unital, algebra over k.

Let n be a positive integer. The polynomial ring

$$A_n = \mathsf{k}[t_1, t_2, \dots, t_n]$$

can be thought of as an algebra of k-valued functions on the set k^n – given a point $x \in \mathsf{k}^n$ we may evaluate $f \in A_n$ at the coordinates of $x = (x_1, x_2, \dots, x_n)$. (This gives an injection from A_n into the algebra of functions on k^n since k is algebraically closed and hence infinite, thus we need not distinguish between abstract elements of A_n and polynomial functions on k^n). An *algebraic set* is a subset of k^n defined by the vanishing of a collection of polynomials in A_n. More precisely, given a collection of polynomials $S = \{f_i : i \in I\}$ the algebraic set defined by S is

$$V(S) = \{x \in \mathsf{k}^n : f_i(x) = 0 \quad \forall i \in S\}.$$

Similarly, if $F \subseteq \mathsf{k}^n$ we may define

$$I(F) = \{f \in \mathsf{k}^n : f(x) = 0 \quad \forall x \in F\}.$$

Note that if I is the ideal generated by S, the $V(I) = V(S)$. Since it is known that any ideal in A_n is finitely generated (Hilbert's basis theorem[1]), one can always assume that S is finite. Clearly we have $S \subset I(V(S))$ and $F \subset V(I(F))$, and moreover since I and V are order reversing (for the natural containment relation) we immediately see that $I(V(I(F))) = I(F)$ and $V(I(V(S))) = V(S)$. This establishes a bijection between algebraic sets and ideals of the form $I(F)$. It is then natural to ask for a more explicit description of these ideal, which is the content of the following theorem.

Theorem 1.1. *(Hilbert's Nullstellensatz.) Let $S = \{f_i : i \in I\} \subseteq A_n$. If $f \in A_n$ is such that it vanishes at every $x \in V(S)$, that is, if $f \in I(V(S))$, then some power f^n lies in the ideal generated by S.*

Proof. (*not really*:) This is a cunning corollary of the fact that, over an algebraically closed field, a proper ideal I of $\mathsf{k}[t_1, t_2, \dots, t_n]$ has $V(I) \neq \emptyset$. This in turn is a consequence of the fact that a field $K \supset \mathsf{k}$ which is finitely generated as an algebra over k is in fact algebraic over k, which one can, for example, easily deduce from Noether normalization. □

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[1]If you don't already know this, prove it by induction: $A_n = A_{n-1}[t_n]$, so given an ideal one can consider "leading coefficients" of a generating set.

The *radical* of an ideal I is the set of $f \in A_n$ such that some power of f lies in I. It is an ideal, and is denoted $\sqrt{I}$. The previous theorem says that if J is an ideal in A_n, then $I(V(J)) = \sqrt{J}$. Indeed the theorem also shows that the ideals $I(F)$, for $F \subseteq \mathsf{k}^n$ are precisely the radical ideals of A_n, that is, ideals I with $I = \sqrt{I}$, and hence there is a bijection between radical ideals of A_n and the algebraic sets in k^n. The following lemma is elementary.

Lemma 1.2. *The sets $V(I)$ form the closed sets of a topology on k^n.*

This topology is know as the *Zariski topology*. Thus an algebraic set F in k^n is naturally a topological space, equipped with a ring of functions which are the restriction of the polynomial functions on k^n. This ring is clearly the quotient ring $A_n/I(F)$. The formal definition of an affine variety is just an abstraction of this data.

Definition 1.3. An *affine* k*-variety* (often just *affine variety*) is a pair (X, A) consisting of a set X and a ring of k-valued functions A on X, such that

- A is a finitely generated k-algebra;
- the points of X are in bijection with the algebra maps $A \to \mathsf{k}$, where a point $x \in X$ gives a map $e_x \colon A \to \mathsf{k}$, with $e_x(f) = f(x)$.

We can make X naturally into a topological space in the same way we made algebraic sets into topological spaces: let the closed sets of be given by the vanishing of subsets of A. A *morphism*of algebraic varieties $f \colon (X, A) \to (Y, B)$ is a map of sets $f \colon X \to Y$ such that the map on functions f^* from functions on Y to functions on X induces a map $f^* \colon B \to A$.

Affine varieties over k thus naturally form a category, which in fact is the opposite category of the category of finitely generated k-algebras with zero radical[2] (that is, k-algebras for which (0) is a radical ideal). We call A the ring of *regular functions* on X, and it is often denoted $\mathsf{k}[X]$ or $\mathcal{O}[X]$. The following lemma shows that this definition really does give an abstract version of an algebraic set.

Lemma 1.4. *Any algebraic set $F \subset \mathsf{k}^n$ is naturally an affine variety, and conversely any affine variety is isomorphic to an algebraic set.*

Proof. For the first part, let $J = I(F)$, the radical ideal defined by F. Then the functions A_n restricted to F are just the algebra $B = A_n/J$. To see that (F, B) is an affine variety, we just have to check that there is a bijection between the points of F and homomorphisms $B \to \mathsf{k}$. Since we can certainly evaluate at a point $x \in F$, we have a map $e \colon F \to \mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, \mathsf{k})$, which we must show is a bijection. To see that it is injective, note that if $s_i \in B$ is the image of the coordinate functions $t_i \in A_n$, then

$$x = (e_x(s_1), \ldots, e_x(s_n)).$$

To see that it is surjective, suppose that $\epsilon \colon B \to \mathsf{k}$ is a homomorphism, then we may set $x = (\epsilon(s_1), \ldots, \epsilon(s_n))$, and it is then easy to see that $x \in F$, and $\lambda = e_x$ as required.

For the converse, if (X, A) is an affine variety, then A is finitely generated. Pick a generating set $\{a_1, \ldots, a_n\} \subset A$, and define a map

$$\varphi \colon X \to \mathsf{k}^n, \quad x \mapsto (a_1(x), \ldots, a_n(x)).$$

[2]The point of taking the opposite category here is that it has great geometric intuition. See the next section for more discussion of categories.

Then it is easy to see that $A = \mathsf{k}[t_1, \ldots, t_n]/\mathrm{ker}(\varphi^*)$, and that the image of φ is an algebraic subset of k^n isomorphic to (X, A). □

It is easy to show from the definition of tensor product that if (X, A) and (Y, B) are affine varieties, then $(X \times Y, A \otimes_{\mathsf{k}} B)$ is again an affine variety, so that finite products of affine varieties exist (note that the topology on the product is finer than the product topology). Also, a closed subset of an affine variety is again an affine variety, with the ring of functions given by restriction of the ambient ring of functions.

If (X, A) is an affine variety, and $g \in A$, then $U_g = \{x \in X : g(x) \neq 0\}$ is clearly an open set in X. However it is also an affine variety, as we may identify U_g with a closed subset of $X \times \mathsf{k}$ given by $\{(x, \lambda) : g(x)\lambda = 1\}$. Notice however that the regular functions on U_g are not just the restrictions of those from X – the function $1/g$ is now also regular. From this it is easy to see that X has a basis of open sets which are themselves affine varieties. This is crucial both in the local study of affine varieties and the definition of a general algebraic variety[3], since it means that in the Zariski topology, affine varieties still look locally like affine varieties.

Example 1.5. Let $\mathbb{A}^1 = \mathsf{k}$, the affine line. Although the set $U = \mathsf{k} - \{0\}$ is not an algebraic subset of $\mathbb{A}^1$, it is an affine variety with ring of functions $\mathsf{k}[t, t^{-1}]$. On the other hand, the subset $V = \mathbb{A}^2 - \{(0, 0)\}$ is an open subset of V which is *not* an affine variety[4]. To see this, note that if $V_1 = \{(t_1, t_2) : t_1 \neq 0\}$ and $V_2 = \{(t_1, t_2) : t_2 \neq 0\}$, then $V = V_1 \cup V_2$. Each of V_1 and V_2 is an affine variety (both are isomorphic to the product $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$, and we know from the above that $\mathbb{A}^1 - \{0\}$ is affine). The ring of regular functions on V_1 is $\mathsf{k}[t_1, t_1^{-1}, t_2]$, and the ring of regular functions on V_2 is $\mathsf{k}[t_1, t_2, t_2^{-1}]$. But then the ring of regular functions on V should be the intersection of these two rings, which is $\mathsf{k}[t_1, t_2]$, and so the correspondence between points and algebra homomorphisms fails. (You should compare this with Hartog's lemma on removable singularities in several complex variables.)

2. ARBITRARY FIELDS

In this section we do not assume that our base field k is algebraically closed.

Remark 2.1. Notice that the Nullstellensatz shows that the points of an affine variety X correspond bijectively with the maximal ideals of $\mathsf{k}[X]$, thus we can recover the topological space X without even the k-algebra structure on $\mathsf{k}[X]$. However, we chose to attach points to homomorphisms $\mathsf{k}[X] \to \mathsf{k}$ rather than maximal ideals because this generalizes naturally to the case where the field k is not algebraically closed. Indeed notice that if $f \colon A_n \to \mathsf{k}$ is a homomorphism, it is completely determined by the numbers $a_i = f(t_i)$ $(1 \leq i \leq n)$. If I is an ideal in A_n, then a homomorphism $A_n/I \to \mathsf{k}$ is determined by an n-tuple (a_i) such that $f(a_1, a_2, \ldots, a_n) = 0$ for all $f \in I$. Thus the set of "zeros" of I in k^n can be identified with the set

$$\mathrm{Hom}_{\mathsf{k-alg}}(A/I, \mathsf{k}).$$

[3]Roughly, an algebraic variety is a space which looks locally like an affine variety. It's easiest to say this via sheaf theory however– see the appendix.

[4]There's a slight problem here, in that I'm being vague about what V actually *is*! As an open subset of an affine varieties, it's more than just a set, it is topological space with a sheaf of functions (such a thing is called a *ringed space*), which is not an affine variety – again, for more details, see the appendix.

This is utterly tautological, but useful linguistically in what follows. We will refer to these zeros as the *points of* A_n/I *over* k.

If k is not algebraically closed, and I is an ideal of $A_n = \mathsf{k}[t_1, t_2, \ldots, k_n]$, then we can certainly still consider the set of zeros $\{x \in \mathsf{k}^n : f(x) = 0, f \in I\}$. However, once we allow k nonalgebraically closed, this is a somewhat deficient tool with which to study ideals in $\mathsf{k}[t_1, t_2, \ldots, t_n]$: for example, the zeros in $\mathbb{R}^2$ of the ideal $(t_1^2 + t_2^2 + 1) \subset \mathbb{R}[t_1, t_2]$ don't allow you to distinguish it from the ideal (1)[5]. The situation is improved if we allow ourselves to consider also the set of zeros $\{x \in K^n : f(x) = 0, f \in I\}$ for field extensions $K \supset \mathsf{k}$ (*e.g.* the algebraic closure of k), or, even more generally, solutions $\{x \in R^n : f(x) = 0, f \in I\}$ for any k-algebra R. By analogy with the points of affine varieties, we will refer to these as the *R-points* of A_n/I.

Notice also that the zero radical assumption is no longer reasonable in general: in characteristic p, even if a k-algebra A has no nilpotents, extending to $A \otimes_{\mathsf{k}} \bar{\mathsf{k}}$ might introduce them (*can you give an example*?). Once again this emphasizes the deficiency of varieties, since at least in the naive sense, our finitely generated k-algebras can no longer be algebras of functions on a space.

One solution to this problem is to attach to a finitely generated algebra R rather than a space with an algebra of functions, something which captures R by considering its zeros over all k-algebras at the same time. The easiest language to express this in is that of *functors*[6]. The greater difficulty, on the other hand, is to think geometrically with them...

2.1. **Some Category Theory.** We begin by recalling the definition of a category.

Definition 2.2. A category $\mathcal{C}$ is a collection of *objects* $\mathrm{Ob}(\mathcal{C})$ such that for any two objects A and B we have a set of *morphisms*, $\mathrm{Mor}(A, B)$ and a composition law

$$\circ \colon \mathrm{Mor}(B, C) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C),$$

which is associative in the obvious sense. Moreover $\mathrm{Mor}(A, A)$ contains a distinguished element id_A such that if $f \in \mathrm{Mor}(A, B)$ then $f = f \circ \mathrm{id}_A = \mathrm{id}_B \circ f$.

Example 2.3. Many things in mathematics form categories:

(1) Sets with set maps as morphisms form a category, which we will denote $Sets$.
(2) Groups with group homomorphisms as morphisms form a category, which we will denote Grp. Similarly we get a category $Ring$ of rings etc.
(3) Topological spaces with continuous maps as morphisms form a category, which we denote as Top.
(4) Vector spaces over a field k with linear maps as morphisms form the category $Vect$, similarly for modules over a ring.
(5) Affine varieties over an algebraically closed field k form a category, as was mentioned before.

[5]The crucial difference is the failure of the Nullstellensatz

[6]There are many times in mathematics when you want to study an object which you know has some rich structure, and then find it isn't as nice as you would like it to be (it's not a manifold, orbifold, scheme....) It *is* however, almost surely a functor. One grabs on to this and the asks how nice a functor is it. It's a bit like wanting to find a unicorn and settling on first finding horned beasts, and then hoping some of them are pretty.

(6) Given a category $\mathcal{C}$ one can form a new category $\mathcal{C}^{\text{op}}$ with the same objects, but with the arrows of morphisms reversed, that is $\text{Mor}_{\mathcal{C}^{\text{op}}}(A,B) = \text{Mor}_{\mathcal{C}}(B,A)$.
(7) Let X be a topological space. Then we can attach a category Top_X to X by letting the objects be the open sets of X, and the morphisms to be inclusion maps. (In the same way you can attach a category to any poset).
(8) A group can be thought of as a category with one object, where all the morphisms are invertible. In the same way, a groupoid is just a category where every morphism is invertible.

Definition 2.4. Given two categories $\mathcal{C}, \mathcal{D}$, a functor $F\colon \mathcal{C} \to \mathcal{D}$ assigns for each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and for each morphism $f\colon A \to B$ in $\mathcal{A}$ a morphism $F(f)\colon F(A) \to F(B)$, compatibly with composition, and such that $F(\text{id}_A) = \text{id}_{F(A)}$. A functor $F\colon \mathcal{C}^{\text{op}} \to \mathcal{D}$ is usually called a *contravariant* functor[7] from $\mathcal{C}$ to $\mathcal{D}$.

Example 2.5. (1) There is always an identity functor from any category $\mathcal{C}$ to itself.
(2) Since a ring R is a group under addition, there is a "forgetful functor" F from $Ring$ to Grp sending a ring R to the additive group $(R,+)$.
(3) Forgetting even more, there is a functor G from Grp to Set which forgets the group structure of a group H, and just considers H as a set.
(4) More interestingly, the category $Vect$ has a contravariant functor * from $Vect^{\text{op}}$ to $Vect$ given by taking the dual vector space. Taking the dual twice gives a functor from $Vect$ to $Vect$.
(5) If you know any topology, there is a functor π_1 from Top^+ the category of pointed topological spaces to Grp taking (X,x) to $\pi_1(X,x)$, the group of homotopy classes of based paths $p\colon S^1 \to X$ with $p(1) = x$.
(6) If X is a topological space, then a functor from Top_X^{op} to, say, abelian groups, is called a presheaf (of abelian groups) on X. For example, for an open set U let $\mathcal{F}(U)$ be the continuous functions on U. Then $\mathcal{F}$ is a presheaf of rings on X – given a continuous function f on U, we can restrict it to a continuous function on V for any open set $V \subset U$.

There is an obvious notion of a map between functors:

Definition 2.6. Let F, G be functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\alpha\colon F \to G$ assigns to each object A of $\mathcal{C}$ a morphism $\alpha(A)\colon F(A) \to G(A)$, such that if $f\colon A \to B$ is any morphism in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha(A)}\downarrow & & \downarrow{\scriptstyle \alpha(B)} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

is commutative.

Example 2.7. Let $\mathbb{D}$ be the functor from $Vect$ to itself which takes a vector space V to its double dual V^{**} (the dual of the dual vector space $V^* = \text{Hom}(V, \mathsf{k})$). Then there is a natural transformation α from the identity functor I to $\mathbb{D}$: given a vector

[7] What I have called simply a functor here is then sometimes called a *covariant* functor, for clarity.

space V, let $\alpha(V)\colon V \to \mathbb{D}(V)$ be given by setting

$$\alpha(v)(\beta) = \beta(v), \qquad \beta \in V^*.$$

Clearly $\alpha(V)$ is injective for all V (at least assuming the axiom of choice), and it is an isomorphism on finite dimensional vector spaces.

2.2. k**-functors.** When working over an algebraically closed field, affine varieties allow us to think of a finitely generated k-algebra B as an algebra of functions on a space (and thus a "geometric" object). When working over a general field k, this picture breaks down: the set $\mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, \mathsf{k})$ is simply not rich enough. To remedy this, we consider instead the collection of all R-points of B for every finitely generated k-algebra R. That is, we consider all the sets

$$\mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, R),$$

as R runs over all finitely generated k-algebras. But notice now that we have rather more than a collection of sets: any time we have a map of k-algebras $\phi\colon R \to S$, there is an induced map from $\mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, R) \to \mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, S)$. Thus we have actually constructed a functor! We formalize this as follows:

Definition 2.8. A k-functor $\mathcal{X}$ is a functor from the category of k-algebras to the category of sets. Given a k-algebra B, there is an associated k-functor $\mathcal{X}_B$ given by

$$\mathcal{X}_B(R) = \mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, R).$$

$\mathcal{X}_B$ is called the *functor of points* of B.

We give a name to the k-functors which arise from k-algebras:

Definition 2.9. A k-functor $\mathcal{X}$ is *affine* if there is a finitely generated k-algebra B such that we may identify

$$\mathcal{X}(R) = \mathrm{Hom}_{\mathsf{k}\text{-alg}}(B, R).$$

for every k-algebra R (that is, there is an isomorphism of functors between $\mathcal{X}$ and $\mathcal{X}_B$). One also says that $\mathcal{X}$ is *representable*.

Example 2.10. Let $\mathbb{A}^n$ be the k-functor which sends a k-algebra A to the set A^n. Then it is easy to check that this functor is represented by the algebra $\mathsf{k}[t_1, t_2, \dots, t_n]$, and so is affine. As we saw before, for an ideal $I \subset \mathsf{k}[t_1, \dots t_n]$ the functor $\mathcal{X}_{A_n/I}$ gives all solutions to the equations $\{(a_1, a_2, \dots, a_n) \in A^n : f(a_1, a_2, \dots, a_n) = 0, \forall f \in I(F)\}$ in A^n.

Lemma 2.11. *(Yoneda Lemma) The map $A \to \mathcal{X}_A$ is a full (contravariant) embedding of the category of k-algebras into the category of k-functors.*

Proof. This is a deep triviality. One way to say it is to check that if F is any k-functor then

$$\mathrm{Mor}(\mathcal{X}_A, F) = F(A).$$

where Mor here is the natural transformations from $\mathcal{X}$ to F, and the identification claimed is natural in both F and A. To see this, suppose first we have α a natural transformation from $\mathcal{X}_A$ to F. Then $\mathrm{id}_A \in \mathcal{X}_A$, so applying α to it we get $\alpha(\mathrm{id}_A) \in F(A)$. On the other hand, if $x \in F(A)$, there is a natural transformation α_x from $\mathcal{X}_A$ to F given by taking, for $y \in \mathcal{X}_A(B)$,

$$\alpha_x(y) = F(y)(\mathrm{id}_A).$$

It is easy to check that these operations are natural and mutually inverse. $\square$

Thus we lose nothing if we replace a k-algebra A by its associated functor $\mathcal{X}_A$ (so at the very least we solved the problem of the k-points being insufficient, though you may not think we have done so very efficiently). On the other hand, one has to hope that you gain something by embedding k-algebras in k-functors and you do, in the sense that the world of k-functors is more flexible than that of k-algebras: one can develop all of scheme theory in terms of k-functors – what is missing is a (Grothendieck) topology to give a notion of locality and an appropriate notion of sheaves. We, however, will mostly make do with affine varieties and some basics of functors.

3. Affine Algebraic Groups

We begin with a simple example: consider the special linear group, $SL_n(\mathbb{C})$. This is the set of $n \times n$ matrices A with the single condition that $\det(A) = 1$. Since this is clearly a polynomial equation, $SL_n(\mathbb{C})$ is a hypersurface in $\mathbb{C}^{n^2}$, which is also a group. Better still, the operations of multiplication and inversion on $SL_n(\mathbb{C})$ are polynomial maps (*why is this the case for inversion?*) and so the group structure on $SL_n(\mathbb{C})$ is compatible with variety structure of $SL_n(\mathbb{C})$. This means that $SL_n(\mathbb{C})$ is in fact an affine group variety (*i.e.* a group and an affine variety compatibly[8]). There was nothing special about the field $\mathbb{C}$ here, and in fact given any algebraically closed field k we can attach to it a group variety $SL_n(\mathsf{k})$.

However there was no need to require k to be algebraically closed, indeed we might also consider, given any k-algebra A the group $SL_n(A)$ of n-by-n matrices with entries in A and determinant 1. In the manner of the Remark 2.1 this is just

$$\mathrm{Hom}_{\textsf{k-alg}}(S, A),$$

where $S = \mathsf{k}[X_{11}, \dots, X_{nn}]/(\det(X_{ij}) - 1)$. This leads us to our first definition:

Definition 3.1. A k-group functor G is a functor from the category of k-algebras to the category of groups.

Thus to each k-algebra R we have a group $G(R)$ and to each map of k-algebras $\alpha\colon R \to S$ we have a group homomorphism $G(\alpha)\colon G(R) \to G(S)$, which compose appropriately. Note that a k-group functor is also a k-functor, by composing it with the forgetful functor from $Groups$ to $Sets$.

A k-group functor G is said to be an *affine algebraic group* if there is a finitely generated k-algebra A such that

$$G(R) = \mathrm{Hom}_{\textsf{k-alg}}(A, R),$$

as k-functors (that is, G is affine as a k-functor).

The special linear group is of course an example of an affine algebraic group, where S is the representing k-algebra. We now give some more examples.

Example 3.2. (1) The functor $\mathbb{G}_a$ which attaches to the algebra R the group $(R, +)$ is an affine algebraic group, since

$$\mathbb{G}_a(R) = \mathrm{Hom}_{\textsf{k-alg}}(\mathsf{k}[t], R).$$

[8]This is *not* an algebraic group as we are about to define them, but it has one naturally attached to it.

(2) The functor $\mathbb{G}_m$ which attaches to each k-algebra R the group of units $(R^\times, \cdot)$ is an affine algebraic group, since

$$\mathbb{G}_m(R) = \mathrm{Hom}_{\mathsf{k}\text{-alg}}(\mathsf{k}[t, t^{-1}], R).$$

(3) The general linear group GL_n is the k-group functor which attaches to each k-algebra R the group

$$\mathrm{GL}_n(R) = \{x \in \mathrm{Mat}_{n,n}(R) : \det(x) \in R^\times\}$$

To see that it is an affine algebraic group, note that it is represented by the algebra

$$\mathsf{k}[X_{ij}, Y : 1 \leq i, j, \leq n]/(Y \det(X_{ij}) - 1)$$

(this is the same trick that produced the basic open sets in the Zariski topology).

From now on, we shall normally be sloppy and simply say algebraic group rather than affine algebraic group, as we will not be concerned with more general algebraic groups in this course. Given an affine algebraic group G, the k-algebra representing it is unique up to unique isomorphism, so we may refer to it as *the* coordinate ring of G, and denote it by $\mathsf{k}[G]$. When we write $x \in G$ without further comment, we will always mean $x \in G(\mathsf{k})$. Provided that k is algebraically closed and $\mathsf{k}[X]$ has no nilpotents, we can think of G as an affine variety, with regular functions $\mathsf{k}[G]$.

The functor $G \times G$ is represented by the algebra $\mathsf{k}[G] \otimes \mathsf{k}[G]$ and the trivial group is represented by k, thus the group operations of multiplication $m \colon G \times G \to G$, inverse $\iota \colon G \to G$, and the identity element $i \colon \{1\} \to G$, yield, by the Yoneda lemma, homomorphisms of k-algebras as follows, where we put $\mathsf{k}[G] = A$:

$$\Delta \colon A \to A \otimes_{\mathsf{k}} A, \quad \epsilon \colon A \to \mathsf{k}, \quad S \colon A \to A.$$

(alternatively, in the case of an algebraically closed field, on the variety $G(\mathsf{k})$ these are simply the pull-back operations on functions given by the group structure).

Example 3.3. For the additive group $\mathbb{G}_a$, we have $\mathsf{k}[\mathbb{G}_a] = \mathsf{k}[t]$, and the homomorphisms are $\Delta(f) = 1 \otimes f + f \otimes 1$, $\epsilon(t) = 0$, $S(t) = -t$.

A k-algebra with the additional structure of the homomorphisms Δ, ϵ, S satisfying axioms which correspond to the axioms for group multiplication etc. is called a commutative Hopf algebra[9]. For example the associativity axiom for group multiplication corresponds to the commutativity of the following diagram (which we call "coassociativity"):

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\mathrm{id} \otimes \Delta} & A \otimes A \\ \uparrow{\scriptstyle \Delta \otimes \mathrm{id}} & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array}$$

A *morphism* of k-group functors $\alpha \colon G \to H$, is a natural transformation – that is, a family of homomorphisms $\alpha(A) \colon G(A) \to H(A)$ such that if $f \colon A \to B$ is a morphism of k-algebras, then we have a commutative diagram

[9] In fact there is an equivalence of categories between affine algebraic groups and commutative Hopf algebras over k.

$$\begin{array}{ccc} G(A) & \xrightarrow{\alpha(A)} & H(A) \\ \downarrow & & \downarrow \\ G(B) & \xrightarrow{\alpha(B)} & H(B) \end{array}$$

By the Yoneda lemma, when G and H are algebraic groups, a morphism from G to H induces an algebra map $\alpha^*\colon \mathsf{k}[H] \to \mathsf{k}[G]$.

Given any k-group, one can define a sub-k-group to be a k-group functor H for which $H(A) < G(A)$ for each k-algebra A. If G and H are affine, there is then a corresponding map $i\colon \mathsf{k}[G] \to \mathsf{k}[H]$. We say that H is a closed (or algebraic) subgroup of G if this map is a surjection (and hence $\mathsf{k}[H]$ is a quotient of $\mathsf{k}[G]$.

Example 3.4.

(1) Recall the general linear group GL_n is an affine algebraic group. To each k-algebra A we attach the group

$$\mathrm{GL}_n(A) = \{x \in \mathrm{Mat}_{n,n}(A) : \det(x) \in A^\times\}$$

The group $T_n(A) \subset GL_n(A)$ of upper triangular matrices is an affine algebraic group which is an algebraic subgroup of GL_n.

(2) The group $U_n(A)$ of upper triangular matrices with 1s on the diagonal is an affine algebraic group.

(3) The subgroup of $GL_n(A)$ consisting of matrices in which each row and column contain exactly one nonzero entry is an affine algebraic group, known as the group of *monomial matrices*.

Remark 3.5. If $K \supset \mathsf{k}$, then given a k-group functor G, we obtain a K-group functor G_K, just by restricting the functor G – any K-algebra R is obviously also a k-algebra by the inclusion $\mathsf{k} \hookrightarrow K$. The universal property of tensor product immediately implies that if G is an algebraic group, so is G_K with coordinate algebra $K \otimes_{\mathsf{k}} \mathsf{k}[G]$, i.e. the group G_K corresponds to the base change to K of the coordinate algebra $\mathsf{k}[G]$.

We now wish to briefly to give examples of algebraic groups over nonalgebraically closed fields.

Example 3.6.

(1) Suppose that $\mathsf{k} = \mathbb{R}$. Then

$$\mathrm{SU}(2) = \{A = \begin{pmatrix} u & v \\ z & w \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}) : A.\bar{A}^t = I, \det(A) = 1\},$$

is a group. Moreover, it is not hard to see that it is the $\mathbb{R}$-points of an $\mathbb{R}$-algebraic group: the defining equations $A.\bar{A}^t = I, \det(A) = 1$, generate the ideal $(|u|^2 + |v|^2 - 1, v + \bar{z}, u - \bar{w})$ in $\mathbb{R}$-algebra $\mathbb{C}[u, v, w, z]$, and hence in fact we can consider the coordinate algebra to be

$$\mathbb{R}[x_1, x_2, y_1, y_2]/(x_1^2 + x_2^2 + y_1^2 + y_2^2 - 1)$$

where we identify $u = x_1 + ix_2$ and $v = y_1 + iy_2$.

(2) Similarly the group $\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{Mat}_2(\mathbb{R}) : \det(A) = 1\}$ is also the $\mathbb{R}$-points of an $\mathbb{R}$-algebraic group. This time it is more straightforward to check that the coordinate algebra can be indentified with

$$\mathbb{R}[x_1, x_2, y_1, y_2]/(x_1 y_2 - x_2 y_1 - 1),$$

by using coordinates $A = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$.

(3) Now notice that if we base change these two $\mathbb{R}$-algebraic groups to $\mathbb{C}$, we find in the second case that $\mathrm{SL}_2(\mathbb{R})_\mathbb{C}$ is obviously just the $\mathbb{C}$-algebraic group SL_2, however a moment's thought shows that in fact $\mathrm{SU}_2(\mathbb{R})_\mathbb{C}$ is also! (Over $\mathbb{C}$, it is easy to find a linear change of coordinates in x_1, x_2, y_1, y_2 which transforms the polynomial $x_1^2 + x_2^2 + y_1^2 + y_2^2$ to the polynomial $x_1y_2 - x_2y_1$, thus the two varieties are the same, and it's not hard then to check that the rest of the structure also matches).
(4) More generally, we could pick a nondegenerate Hermitian form on $\mathbb{C}^n$ of signature (p, q) say, and obtain an $\mathbb{R}$-algebraic group $\mathrm{U}(p, q)$.

Thus classifying groups over nonalgebraically closed fields is a more subtle issue than over an algebraically closed field, which, as we already see over $\mathbb{R}$, involves the arithmetic of the field.

Example 3.7. There is another reason to study algebraic groups as we have defined them (rather than the way they are defined in, say [H]). Suppose that $\mathsf{k} = \overline{\mathbb{F}}_p$ is the algebraic closure of the finite field $\mathbb{F}_p$. Consider the algebraic group GL_n over k. The map $Fr\colon \mathrm{GL}_n \to \mathrm{GL}_n$ given on the coordinate ring by $(X_{ij}) \mapsto (X_{ij}^p)$ is a morphism of algebraic groups. The kernel of this morphism is a closed algebraic subgroup of GL_n whose coordinate ring is finite dimensional as a k-vector space (of dimension p^{n^2}) and nilpotent. These finite algebraic groups are crucial in the study of the representation theory of algebraic groups in positive characteristic.

Definition 3.8. A representation of an algebraic group G on a k-vector space V is a natural transformation $\alpha\colon G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the algebraic group given by $A \mapsto \mathrm{GL}(V \otimes_\mathsf{k} A)$. By Yoneda's lemma this is equivalent to giving a map $m\colon V \to \mathsf{k}[G] \otimes_\mathsf{k} V$ satisfying the obvious compatibilities:

$$(\mathrm{id}_{\mathsf{k}[G]} \otimes m) \circ m = (\Delta \circ \mathrm{id}_V) \circ m,$$
$$(\epsilon \otimes \mathrm{id}_V) \circ m = \mathrm{id}_V.$$

(This is known as a *comodule* for the Hopf algebra $\mathsf{k}[G]$.) A subrepresentation is a k-subspace W of V for which $m(W) \subset \mathsf{k}[G] \otimes W$.

Note that a G-representation induces a representation of the group $G(\mathsf{k})$ on V. Indeed given $g \in G(\mathsf{k})$ we get an automorphism $\rho(g)$ of V by setting:

$$\rho(g)(v) = (g \otimes \mathrm{id}_V)(m(v)).$$

The definition of a comodule for $\mathsf{k}[G]$ then shows that this gives a representation of $G(\mathsf{k})$. In the algebraically closed case, this determines the G-representation, but in general this is not true.

We can make $\mathsf{k}[G]$ into a G representation using Δ itself. You should think of this as the analogue for an algebraic group of the "regular representation" of a finite group. Of course $\mathsf{k}[G]$ is in general an infinite dimensional representation. However it is not, in some sense, too infinite dimensional: more precisely, $(\mathsf{k}[G], \rho^*)$ is *locally finite* in the following sense.

Definition 3.9. A representation V of G is locally finite if any finite dimensional k-subspace of V is contained in a finite dimensional subrepresentation of V.

In fact, the property of being locally finite is not special to the representation on $\mathsf{k}[G]$. Indeed we have:

Lemma 3.10. *Let V be a G representation. Then V is locally finite.*

Proof. It is enough to show that, if $v \in V$, then the smallest G subrepresentation containing v is finite dimensional. Let $m\colon V \to \mathsf{k}[G] \otimes V$ be the comodule map. Then we have

$$m(v) = \sum_{i=1}^{n} \mu_i f_i \otimes w_i,$$

where $f_i \in \mathsf{k}[G]$ and $w_i \in V$. Now let

$$W = \text{span}\{w_i : 1 \leq i \leq n\}; \quad V' = \{u \in V : m(u) \in \mathsf{k}[G] \otimes W\}.$$

First note that $V' \subset W$, since if $u \in V'$ we may write $m(u) = \sum_{i=1}^{n} f_i \otimes w_i$, and hence $u = \sum_{i=1}^{n} \epsilon(f_i) w_i$ (note this also shows that $v \in V'$). We claim that V' is a finite dimensional subrepresentation of V, that is $m(V') \subset \mathsf{k}[G] \otimes V'$. But by definition $V' = m^{-1}(\mathsf{k}[G] \otimes W)$, and since $\mathsf{k}[G]$ is free over k (as k is a field!) it follows that

$$\mathsf{k}[G] \otimes V' = (\text{id}_{\mathsf{k}[G]} \otimes m)^{-1}(\mathsf{k}[G] \otimes \mathsf{k}[G] \otimes W),$$

and hence it is enough to show that

$$(\text{id}_{\mathsf{k}[G]} \otimes m) \circ m(V') \subset \mathsf{k}[G] \otimes \mathsf{k}[G] \otimes W. \tag{3.1}$$

But by the definition of a comodule, this is also $(\Delta \otimes \text{id}_V) \circ m(V')$, which is contained in the righthand side of (3.1) by definition. □

Proposition 3.11. *Let G be an affine algebraic group. Then there is an injective homomorphism $\phi\colon G \to GL(V)$ for some finite-dimensional vector space V, realizing G as an algebraic subgroup of $GL(V)$.*

Proof. Since $\mathsf{k}[G]$ is finitely generated, we may find a finite dimensional k-subspace W of $\mathsf{k}[G]$ which generates $\mathsf{k}[G]$ as a k-algebra. Using ρ^*, the coordinate algebra is a G-representation, and by the previous lemma, the subspace W lies in a finite dimensional subrepresentation V say, which *a fortiori* also generates $\mathsf{k}[G]$ as an algebra. Thus we have a map $\phi\colon G \to GL(V)$ given by the restriction of ρ^*. More explicitly, if $\{v_i : 1 \leq i \leq n\}$ is a basis of V, then we have $m(v_i) = \sum_{j=1}^{n} v_j \otimes f_{ij}$, so that ϕ is given by $x \mapsto (f_{ij}(x))$. To see that ϕ is an embedding we simply need to show that ϕ^* is surjective. But $\mathcal{O}(GL(V))$ is the algebra generated by the matrix coefficients, with the determinant function inverted. Since the f_{ij} are the pullbacks via ϕ of the matrix coefficients, and we have

$$v_i(x) = \sum_{j=1}^{n} v_j(e) f_{ij}(x)$$

the surjectivity follows from the fact that the v_i generate A. □

Remark 3.12. The previous proposition shows that any affine algebraic group is a closed subgroup of some $GL(V)$. This is why such groups are sometimes called "linear algebraic groups".

Finally we note that the regular representation contains every representation of G, in the following sense:

Lemma 3.13. *Every G-representation embeds into a direct sum of copies of the regular representation.*

Proof. Let V be a G-representation, and let V_0 be the underlying vector space. Then m gives a map

$$V \to \mathsf{k}[G] \otimes V_0,$$

where $\mathsf{k}[G] \otimes V_0$ is a G-representation via $\Delta \otimes \mathrm{id}_{V_0}$. That this is a map of G-representations follows from the fact that $(\mathrm{id}_{\mathsf{k}[G]} \otimes m) \circ m = (\mathrm{id} \otimes \Delta) \circ m$, and it is injective since $v = (\epsilon \otimes \mathrm{id})m(v)$ for all $v \in V$. □

3.1. **Jordan decomposition.** An endomorphism $\alpha \colon V \to V$ of a k-vector space is called *semisimple* if it is diagonalizable over the algebraic closure of k[10], and *unipotent* if $(\alpha - 1)$ is nilpotent, that is if $(\alpha - 1)^N = 0$ for N sufficiently large.

For the remainder of this section we assume that the field k is algebraically closed, though everything extends straightforwardly to the case of a perfect field.

Lemma 3.14. *Let V be a finite dimensional vector space, and $\alpha \colon V \to V$ an automorphism of V. Then there are unique automorphisms α_s, α_u such that α_s is semisimple, and α_u is unipotent, and*

$$\alpha = \alpha_s \alpha_u = \alpha_u \alpha_s.$$

Moreover α_s and α_u can be expressed as polynomials in α with zero constant term.

Proof. Let $p = \prod_{i=1}^k (T - \lambda_i)^{m_i}$ be the minimal polynomial of α, where $\lambda_i \in \mathsf{k} - \{0\}$ are distinct and $m_i \in \mathbb{N}$. Then $V = \bigoplus_{i=1}^k V_{\lambda_i}$ where

$$V_{\lambda_i} = \{v \in V : (\alpha - \lambda_i)^N v = 0, N \text{ sufficiently large}\},$$

is the generalized eigenspace of λ_i (so in fact $N = m_i$ suffices). By the Chinese Remainder Theorem, we may solve in $\mathsf{k}[T]$ the simultaneous congruences:

$$q_i(T) \cong 1 \bmod (T - \lambda_i)^{m_i}; \quad q_i(T) \cong 0 \bmod (T - \lambda_j)^{m_j}$$
$$q_i(T) \cong 0 \bmod T.$$

for each i, $1 \leq i \leq k$. Then $q_i(\alpha)$ is the projection to the subspace V_{λ_i} with kernel $\bigoplus_{j \neq i} V_{\lambda_j}$. Setting $\alpha_s = \sum_{i=1}^k \lambda_i q_i(\alpha)$, we see that α_s is semisimple and a polynomial in α with no constant term, hence α and α_s commute. Letting $\alpha_u = \alpha\alpha_s^{-1} = \alpha_s^{-1}\alpha$, we see α_u is unipotent.

To see that α_s, α_u are unique, note that if $\alpha = \beta_s\beta_u = \beta_u\beta_s$ is any decomposition of α into commuting semisimple and unipotent automorphisms, then β_s, β_u commute with α, and hence the distinct eigenspaces of β_s must be exactly the generalized eigenspaces of α: if V_λ is the λ eigenspace of β_s, and $v \in V_\lambda$, then

$$(\alpha - \lambda)^N(v) = (\beta_u\beta_s - \lambda)^N(v) = \lambda^N(\beta_u - 1)^N(v) = 0$$

for N sufficiently large. Thus β_s and hence β_u are uniquely determined by α. □

Corollary 3.15. *Let $\phi \colon V \to W$ be a linear map between finite dimensional vector spaces. Then if α and β are automorphisms of V and W respectively such that $\phi \circ \alpha = \beta \circ \phi$, we have $\phi \circ \alpha_s = \beta_s \circ \phi$ and $\phi \circ \alpha_u = \beta_u \circ \phi$. Thus in particular, the decomposition of the restriction of α to an invariant subspace, is given by the restrictions of α_s and α_u.*

Proof. It is enough to note that ϕ sends a generalized eigenspace of α into the corresponding generalized eigenspace of β. □

We now extend the notion of a Jordan decomposition to the case where V is infinite dimensional.

[10] *i.e.* $V \otimes_{\mathsf{k}} \bar{\mathsf{k}}$ has a basis of eigenvectors for α.

Definition 3.16. Let V be a k-vector space, and let $\alpha\colon V \to V$ be an endomorphism. We say that α is *unipotent* if for each $v \in V$ we have $(\alpha - 1)^N(v) = 0$ for sufficiently large N, and that α is semisimple if V has a basis of eigenvectors for α. We say the action of α is *locally finite* if for each vector $v \in V$ the span of $\{\alpha^n(v) : n \in \mathbb{Z}\}$ is finite dimensional. Note any semisimple or unipotent α acts locally finitely.

Example 3.17. Let $V = \mathsf{k}[t]$. Then the derivative $D = \frac{d}{dt}$ is a locally finite linear map on V (this is an "infinite Jordan block"). In fact it is locally nilpotent, *i.e.* $1 + D$ is unipotent.

Lemma 3.18. *If $g\colon V \to V$ is a locally finite automorphism then there are unique automorphisms g_s and g_u such that g_s is semisimple, g_u is unipotent, and $g = g_s g_u = g_u g_s$. Moreover any subspace of V stable under g is stable under g_s and g_u.*

Proof. This follows by reducing to the finite dimensional case, using Corollary 3.15. □

Definition 3.19. Now recall that $\mathsf{k}[G]$ is canonically a G-representation, so we have an action ρ of $G(\mathsf{k})$ on $\mathsf{k}[G]$. But Lemma 3.10 implies that for any $g \in G(\mathsf{k})$, the automophism $\rho(g)$ is locally finite. Hence we may define $g \in G$ to be *semisimple* or *unipotent* if $\rho(g)$ is. For $g \in G(\mathsf{k})$, if $g = g_s g_u = g_u g_s$ where $g_s \in G(\mathsf{k})$ is semisimple and $g_u \in G(\mathsf{k})$ is unipotent, then we say (g_s, g_u) is the *Jordan decomposition* of g. Note that if it exists, Lemma 3.18 shows that it is unique, since ρ is clearly a faithful $G(\mathsf{k})$-representation.

Lemma 3.20. *Suppose that $G = GL(V)$, and $g \in G$. Then if $g = g_s g_u$ is the Jordan decomposition of g as an automorphism of V, then g_s is semisimple and g_u is unipotent. In particular, every element of G has a Jordan decomposition.*

Proof. Note that the naive Jordan decomposition is compatible with the operations of tensor product, direct sums and duals of vector spaces (this is an easy exercise). Thus (g_s, g_u) is the Jordan decomposition the action of g on $\mathrm{Sym}(V \otimes V^*)$. Since the $\mathsf{k}[G]$ is just the localization of this algebra at the determinant function (an eigenvector for the action of g), it follows immediately that $\rho(g_s)$ and $\rho(g_u)$ are the Jordan decomposition of $\rho(g)$. □

Theorem 3.21. *There is a Jordan decomposition for any $g \in G(\mathsf{k})$.*

Proof. Pick an embedding $\phi\colon G \to \mathrm{GL}(V)$, and let $B = \mathsf{k}[\mathrm{GL}(V)]$ be the coordinate algebra of $GL(V)$ and $I = \ker(\phi^*)$, where $\phi^*\colon B \to \mathsf{k}[G]$. Let ρ be the representation of $GL(B)$ on B (the "regular representation" of $\mathrm{GL}(V)$). By definition, $g \in GL(V)(\mathsf{k})$ lies in $G(\mathsf{k})$ if and only if $g(I) = 0$.

Now any element of $g \in \mathrm{GL}(V)(\mathsf{k})$ has a Jordan decomposition $g = g_s g_u$, thus we need only show that $g \in G(\mathsf{k})$ implies that g_s, g_u are also in $G(\mathsf{k})$. By the above, this is true if we can show:

$$g(I) = 0 \implies g_s(I) = g_u(I) = 0.$$

From the definitions, for any $h \in \mathrm{GL}(V)(\mathsf{k})$, we have $h = \varepsilon \circ \rho(h)$. Hence as $\varepsilon(I) = 0$, it follows that if $\rho(g)(I) \subseteq I$, then $g \in G(\mathsf{k})$. But if $\rho(g)$ preserves I, then it will follow from the "moreover" part of Lemma 3.18 that $\rho(g_s)$ and $\rho(g_u)$ do also, and hence g_s and g_u will lie in $G(\mathsf{k})$ as required.

Thus it remains only to show that if $g \in G(\mathsf{k})$ then $\rho(g)$ preserves I. This follows immediately from the commutativity of the following diagram:

$$\begin{array}{ccccc}
B & \xrightarrow{\Delta_{\mathrm{GL}(V)}} & B \otimes_{\mathsf{k}} B & \xrightarrow{\mathrm{id}_B \otimes g} & B \otimes_{\mathsf{k}} \mathsf{k} \\
\downarrow & & \downarrow & & \downarrow \\
B/I & \xrightarrow{\Delta_G} & B/I \otimes_{\mathsf{k}} B/I & \xrightarrow{\mathrm{id}_{B/I} \otimes g} & B/I \otimes_{\mathsf{k}} \mathsf{k}
\end{array}$$

since the composition of the maps in the rows are the actions of g on the regular representation of $\mathrm{GL}(V)$ and G respectively. $\square$

One can show that the Jordan decomposition is functorial in the sense that if $\phi\colon G \to H$ is a homomorphism of algebraic groups, then

$$\phi(g)_s = \phi(g_s); \qquad \phi(g_u) = \phi(g)_u.$$

Exercise 3.22. Check that $g = \epsilon \circ \phi(g)$ from the axioms for a Hopf algebra. Check also the functoriality claimed above.

4. Smoothness and Tangent spaces

4.1. **Tangent Spaces.** If M is a smooth surface in $\mathbb{R}^3$, then at any point x there is a well-defined tangent plane – intuitively it is the space of possible velocities of a particle moving on the surface as it passes through the point x, and it is easy to use the derivative to turn this into a rigorous definition. Indeed let $p\colon (-\delta,\delta) \to M$ be a smooth curve with $p(0) = x$. Then we say $\frac{dp}{dt}(0)$ is a tangent vector to M at x, and the set of all such vectors T_xM is the tangent space[11] of M at x. Recall that if $M \subset \mathbb{R}^n$ is a smooth k-dimensional submanifold, and $x \in M$, then near x we can write M as the zeros of $n-k$ smooth functions, $f_1, f_2, \dots f_{n-k}$ (this follows from the implicit function theorem). Any curve $p\colon (-\delta,\delta) \to M$ with $p(0) = x$ will have $f_i \circ p(t) = 0$, and so $f_i(x + t\frac{dp}{dt}(0)) = 0$ to first order in t (by the definition of the derivative). In fact we have

$$T_xM = \{v \in \mathbb{R}^n : Df_i(x)(v) = 0, i = 1, 2, \dots, n-k\},$$

where $Df_i\colon \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ is the derivative of f_i, that is, T_xM is exactly the set of vectors for which $f_i(x + t\frac{dp}{dt}(0)) = 0$ to first order. This characterization is easy to make sense of in algebraic geometry.

Thus suppose that $X \subset \mathbb{A}^n$ is an affine variety, and let I be the radical ideal defining X. Then if $f \in I$ and v is a tangent vector v at $x \in X$ then by analogy with the above discussion $f(x + \epsilon v)$ should differ from $f(x) = 0$ by terms of order at least ϵ^2 – in other words we should have

$$f(x + \epsilon v) \equiv 0 \mod \epsilon^2 k[\epsilon], \quad \forall f \in I. \tag{4.1}$$

This gives an adequate definition for a tangent vector, but we want to rephrase it in terms of the coordinate algebra of X. Notice that the condition given by Equation (4.1) is equivalent to the statement that $g \mapsto g(x + \epsilon v) \mod \epsilon^2$ is an algebra homomorphism $\phi_v\colon \mathsf{k}[X] \to \mathbb{D}$, where $\mathbb{D} = \mathsf{k}[\epsilon]/(\epsilon^2)$ is the algebra of *dual numbers* and as usual $\mathsf{k}[X] = A_n/I$. Moreover it is not hard to see that we may recover v from the homomorphism ϕ_v. Motivated by this we make the following definition:

[11]The only problem with this definition is that it is not immediately clear that T_xM is a vector space, as you cannot add curves on M.

Definition 4.1. Let X be an affine variety with coordinate algebra $\mathsf{k}[X]$. The *tangent bundle* TX is the set

$$TX = \mathrm{Hom}_{\mathsf{k}\text{-alg}}(\mathsf{k}[X], D).$$

There is a natural map $\pi\colon TX \to X$ given by $\phi \mapsto z \circ \phi$ where $z\colon \mathsf{k}[\epsilon]/(\epsilon^2) \to \mathsf{k}$ is the canonical homomorphism (using the identification of X with $\mathrm{Hom}_{\mathsf{k}\text{-alg}}(\mathsf{k}[X], \mathsf{k})$).

Any such map ϕ can be written as $\phi(f) = e(f) + d(f)\epsilon$, where d, e are k-linear maps from $\mathsf{k}[X]$ to k. Now $e = \pi(\phi)$, so that $e \in \mathrm{Hom}_{\mathsf{k}\text{-alg}}(\mathsf{k}[X], \mathsf{k})$, while it is easy to check that $d\colon \mathsf{k}[X] \to \mathsf{k}$ is a k-linear map which satisfies:

$$d(fg) = d(f)e(g) + e(f)d(g).$$

If we write x for the point in X corresponding to e then the above equation becomes $d(fg) = g(x)d(f) + f(x)d(g)$, which is just the Leibniz rule for the derivative of a product, and therefore d is known as a derivation at the point x. We say that ϕ is a tangent vector at x, and denote the set of such tangent vectors as $T_xX = \pi^{-1}(x)$. A tangent vector at $x \in X$ is determined by the function d. The following lemma shows that the tangent vectors at $x \in X$ can be naturally identified with the dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal defined by x.

Lemma 4.2. *The set of derivations at $x \in X$ is isomorphic to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ as a k-vector space.*

Proof. Clearly the set of derivations at x forms a k-vector space. Given a derivation d, if $f, g \in \mathfrak{m}_x$ then

$$d(fg) = f(x).d(g) + g(x).d(f) = 0,$$

and so d annihilates $\mathfrak{m}_x^2$. Moreover any derivation vanishes on constants, as $d(1) = d(1.1) = 2d(1)$, hence since any $f \in \mathsf{k}[X]$ can be written as $f(x) + (f - f(x)) \in f(x) + \mathfrak{m}_x$, it follows that tangent vectors are at x are determined by the linear map $\delta\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathsf{k}$ that they induce. Conversely, it is easy to see that, starting with a linear map $\delta\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathsf{k}$, we obtain a derivation at x by setting

$$d(f) = \delta(\bar{f}), \quad (f \in \mathsf{k}[X]),$$

where $\bar{f}$ is the image of $f - f(x).1$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$. □

4.2. **Smooth varieties.** In differential geometry we deal almost exclusively with smooth objects. In contrast, in algebraic geometry many natural varieties are not smooth. Nevertheless smoothness is an important property, though at first sight an appropriate algebraic notion is a little harder to get your hands on. We start with a notion of dimension for an affine variety.

Definition 4.3. A topological space is said to be Noetherian if every stricly decreasing chain of closed subspaces

$$F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots,$$

is finite. If $(X, \mathsf{k}[X])$ is an affine variety, it is clear that X with the Zariski topology is a Noetherian space (just because $\mathsf{k}[X]$ is a Noetherian ring).

Exercise 4.4. Classify all Hausdorff topological spaces which are Noetherian.

An irreducible space is one which cannot be expressed as the union of two nonempty closed subspaces. It is easy to see that a Noetherian space is canonically the union of finitely many irreducible subspaces, which are known as it's irreducible components.

Definition 4.5. If X is irreducible, we may define the *dimension* of X to be the supremum of the lengths of chains of strictly decreasing irreducible closed subspaces in X. If X is not irreducible, we define it's dimension to be the maximum of the dimension of its irreducible components.

It is not difficult to show that an affine variety is irreducible precisely when $\mathsf{k}[X]$ is an integral domain. Moreover, for an irreducible variety, it's dimension is the supremum of the lengths of chains of distinct prime ideals in $\mathsf{k}[X]$. This clearly extends to give a notion of dimension for any k-algebra, which is known as the *Krull dimension* of the algebra.

Theorem 4.6. *Let X be an irreducible variety, and let $\mathsf{k}(X)$ be the function field of X, that is the field of rational functions on X. Then the transcendence degree of $\mathsf{k}(X)$ over k is equal to the dimension of X.*

Proof. This follows from Noether normalization and the fact that affine space $\mathbb{A}^n$ has dimension n. □

Given a point $x \in X$, we may form the *local ring* at x, by taking the localization $\mathcal{O}_{X,x}$ of $\mathsf{k}[X]$ at the maximal ideal $\mathfrak{m}_x$ (geometrically this is taking the germs of functions which are regular at x). Since the prime ideals in $\mathcal{O}_{X,x}$ are in bijection with the prime ideals in $\mathsf{k}[X]$ which lie in $\mathfrak{m}_x$ the ring $\mathcal{O}_{X,x}$ is local (that is, it has a unique maximal ideal)[12], with maximal ideal $\mathfrak{n}_x = \mathfrak{m}_x\mathcal{O}_{X,x}$.

Definition 4.7. Let X be an n-dimensional irreducible variety, and let $x \in X$ with associated maximal ideal $\mathfrak{m}_x$, and $\mathfrak{n}_x = \mathfrak{m}_x\mathcal{O}_{X,x}$ as above. The following conditions are equivalent:

- $\dim_{\mathsf{k}}(\mathfrak{n}_x/\mathfrak{n}_x^2) = n$;
- $\mathfrak{n}_x$ is generated by n elements;
- $\dim_{\mathsf{k}}(\mathfrak{m}_x/\mathfrak{m}_x^2) = n$;
- $\mathfrak{m}_x$ is generated by n elements.

Their equivalence follows easily from the definition of localization and Nakayama's lemma. A variety is *smooth* at x if any of these conditions hold. It is *smooth* if it is smooth at every point. If x is not a smooth point, we say it is a *singular* point.

Remark 4.8. A Noetherian local ring R with maximal ideal M has a Krull dimension, $\dim(R) < \infty$. One can show that a minimal generating set for M has at least $\dim(R)$ elements. When there is a generating set with $\dim(R)$ elements, the local ring is said to be *regular*. A rephrasing of the above definition is that a point x on a variety X is smooth if and only if its local ring $\mathcal{O}_{X,x}$ is regular. There is a rich theory of regular local rings – they are automatically integral domains, and even in fact UFDs.

Corollary 4.9. *A point $x \in X$ is smooth if and only if $\dim_{\mathsf{k}}(T_xX) = \dim(X)$.*

Proof. We saw that $T_xX \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ as vector spaces. The result follows immediately. □

In fact, when X is irreducible, one can show that one always has $\dim_{\mathsf{k}}(T_xX) \geq \dim(X)$. Using this, one shows the following result, which guarantees that any variety is in fact smooth at most of its points.

[12] If you don't know what these terms mean, see for example Ash's notes on commutative algebra at www.math.uiuc.edu/˜r-ash/ComAlg.html

Proposition 4.10. *The set of smooth points of an affine variety is a nonempty open set.*

Proof. One first shows that the set of points where $\dim(T_x X) \geq k$ is a closed set, for any $k \in \mathbb{N}$, and so the set of singular points of X is closed. Then one reduces to the case of a hypersurface in $\mathbb{A}^n$ to check that the set of smooth points is nonempty. □

4.3. **Smoothness of affine algebraic groups.**

Definition 4.11. Let A be a k-algebra. We say that A is *reduced* if $\sqrt{(0)} = 0$, that is, if A has no nonzero nilpotents.

If the characteristic of k is nonzero, then even if A is reduced, it is not necessarily the case that $\bar{\mathsf{k}} \otimes_{\mathsf{k}} A$ is also. However, a finitely generated k-algebra is reduced if and only if the intersection of all maximal ideals of A is trivial (this is an easy consequence of Nullstellensatz, since the map $A \to \bar{\mathsf{k}} \otimes_{\mathsf{k}} A$ is injective). The following lemma is an easy consequence of this.

Lemma 4.12. *Let A be a finitely generated k-algebra over a perfect field k. Then if A is reduced, so is $K \otimes_{\mathsf{k}} A$ for any field $K \supset \mathsf{k}$.*

Definition 4.13. Suppose that G is an affine algebraic group. Then we say G is *reduced* if $\mathsf{k}[G]$ is reduced, and that G is *smooth* if $\bar{\mathsf{k}} \otimes_{\mathsf{k}} \mathsf{k}[G]$ is reduced.

Any affine algebraic group G has an associated reduced algebraic group $\bar{G}$ where $\mathsf{k}[\bar{G}] = \mathsf{k}[G]/R$ where $R = \sqrt{(0)}$ is the radical of $\mathsf{k}[G]$, so that there is a morphism $G \to \bar{G}$. Notice that we have given a notion of dimension for an affine variety, so since $\bar{G}$ has an associated affine variety $G(\mathsf{k})$ we may set $\dim(G)$ to be the dimension of $G(\mathsf{k})$. Alternatively, one can use the algebraic version of Krull dimension, letting $\dim(G)$ be the supremum of lengths of chains of prime ideals in $\mathsf{k}[G]$. Since the nilradical lies in every prime ideal, the two definitions are equivalent.

Lemma 4.14. *Let k be algebraically closed. Then an affine algebraic group G is smooth if and only if $\mathcal{O}_{G,x}$ is regular for each local ring $\mathcal{O}_{G,x}$.*

Proof. If $\mathcal{O}_{G,x}$ is regular, then it is reduced (in fact it is an integral domain). If this is true for all maximal ideals of $\mathsf{k}[G]$ (which we are assuming since k is algebraically closed) then this implies that $\mathsf{k}[G]$ is reduced.

On the other hand, if $\mathsf{k}[G]$ is reduced then $\mathsf{k}[G]$ is just the coordinate algebra of the affine variety $G(\mathsf{k})$. The set of smooth points of $G(\mathsf{k})$ is a nonempty open set S say. Thus suppose that $g \in S$, so that $\dim(T_g G) = \dim(G(\mathsf{k}))$. The group $G(\mathsf{k})$ acts on itself by automorphisms $A_x\colon G(\mathsf{k}) \to G(\mathsf{k})$, where $g \mapsto gx$ (these are the maps on $G(\mathsf{k})$ induced by the automorphisms $\rho(g)\colon \mathsf{k}[G] \to \mathsf{k}[G]$, so the action is a right action). For any $h \in G(\mathsf{k})$, it follows that $A_{g^{-1}h}$ is an automorphism taking g to h, and hence $T_g(G)$ to $T_h(G)$. Thus certainly

$$\dim(T_g(G(\mathsf{k}))) = \dim(T_h(G(\mathsf{k}))), \quad \forall h \in G(\mathsf{k}),$$

and so since $g \in G(\mathsf{k})$ is a smooth point, all points of $G(\mathsf{k})$ are smooth, that is, $G(\mathsf{k})$ is a smooth variety.

□

Remark 4.15. In characteristic zero, it can be shown that all affine algebraic groups are smooth [O].

4.4. **The Lie algebra.**

Definition 4.16. A *Lie algebra* over a field k is a k-vector space $\mathfrak{g}$ equipped with a skew-linear pairing:

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},$$

which satisfies the *Jacobi identity*: for all $x, y, z \in \mathfrak{g}$ we have

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Thus $\mathfrak{g}$ is a nonassociative algebra without unit[13].

Example 4.17. Let A be any associative algebra over k. Then we may give A the structure of a Lie algebra by setting the bracket to be the commutator:

$$[a,b]=ab-ba.$$

The Jacobi identity is then the equality

$$[a(bc-cb)-(bc-cb)a]+[b(ca-ac)-(ca-ac)b]+[c(ab-ba)-(ab-ba)c]=0,$$

(which is true!) The Poincare-Birkhoff-Witt theorem (which we wont prove at the moment) implies that every Lie algebra is a (Lie) subalgebra of this kind of Lie algebra, and hence Lie algebras are in some sense no worse than associative algebras.

Example 4.18. To make the previous example more explicit, consider $\text{Mat}_n(\mathsf{k})$ the algebra of $n\times n$ matrices over k [14]. Then taking $[\cdot,\cdot]$ as the commutator bracket for this algebra we get a Lie algebra denoted $\mathfrak{gl}_n$. A little more interestingly, consider the subspace of traceless matrices:

$$\mathfrak{sl}_n=\{X\in\mathfrak{gl}_n : \text{tr}(X)=0\}.$$

Then since $\text{tr}(XY)=\text{tr}(YX)$, this subspace is a Lie subalgebra of $\mathfrak{gl}_n$, even though it is of course not a subalgebra of $\text{Mat}_n(\mathsf{k})$.

One of the basic achievements of Lie theory is to realize that much of the study of Lie groups (smooth manifolds which are compatibly groups, or in more elaborate language, group objects in the category of smooth manifolds) can be done via a linear approximation, namely the Lie algebra, in imitation the study of differentiable functions via their derivative. An algebraic version of this theory also exists, though in characteristic p the Lie algebra is much less closely bound to it's algebraic group than in characteristic zero.

Definition 4.19. Let A be a k-algebra and M an A-module. A *derivation* (or k-derivation) $d\colon A\to M$ is a k-linear map such that

$$d(fg)=fd(g)+gd(f).$$

We denote the space of k-derivations by $\text{Der}_{\mathsf{k}}(A,M)$.

Example 4.20. If $x\in \mathsf{G}(k)=\text{Hom}_{\mathsf{k}\text{-alg}}(\mathsf{k}[G],\mathsf{k})$, then x gives k the structure of a $\mathsf{k}[G]$-module, which we denote by k_x. A derivation at x in the sense of the previous section is simply an element of $\text{Der}_{\mathsf{k}}(\mathsf{k}[G],\mathsf{k}_x)$.

---

[13]but don't think of it that way, since that makes it sound horrible.

[14]a noncommutative algebra, hence I do note refer to it as a "k-algebra".

Lemma 4.21. *Let A be a* k*-algebra, and consider the space of derivations $D = Der_{\mathsf{k}}(A, A)$. If d_1, d_2 are derivations, then their commutator $[d_1, d_2]$ is also a derivation, hence D is naturally a Lie algebra.*

Proof. This is an easy calculation: indeed

$$\begin{aligned}[d_1, d_2](fg) &= (d_1 d_2 - d_2 d_1)(fg) \\ &= d_1(fd_2(g) + gd_2(f)) - d_2(fd_1(g) + gd_1(f)) \\ &= d_1(f)d_2(g) + fd_1d_2(g) + d_1(g)d_2(f) + gd_1d_2(f) \\ &\quad - d_2(f)d_1(g) - fd_2d_1(g) - d_2(g)d_1(f) - gd_2d_1(f) \\ &= g[d_1, d_2](f) + f[d_1, d_2](g).\end{aligned}$$

so $[d_1, d_2]$ is a derivation as required. □

Now if G is an algebraic group, we could certainly attach to it the Lie algebra $\mathrm{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G])$. This, however is in general rather large, and moreover, it does not take into account the group structure of G at all (we could make the same definition for the coordinate algebra of any affine variety for example).

4.5. **Group varieties and Lie algebras.** Suppose for the moment that k is algebraically closed and $\mathsf{k}[G]$ is reduced. Then $G(\mathsf{k})$ is an affine variety.

If $D \in \mathrm{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G])$, and $x \in G(\mathsf{k})$ (thought of as a homomorphism from $\mathsf{k}[G]$ to k) then $x \circ D$ is a derivation at x, *i.e.* $x \circ D \in \mathrm{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}_x)$, so that $x \circ D$ gives a tangent vector in $T_x G(\mathsf{k})$. In this way D gives a map $s_D \colon G(\mathsf{k}) \to TG(\mathsf{k})$ such that $\pi \circ s_D = \mathrm{id}_{G(k)}$. We call such a map a *section* of TX, or a *vector field*. Moreover D is completely determined by the corresponding vector field s_D. Now since $G(\mathsf{k})$ is a group, given a tangent vector at the identity, we can translate it by the group action to get a tangent vector at every point of the group (one can use left or right translation) and thus a vector field. Such a vector field is said to be (left or right) invariant. More precisely, let $T_g \colon G(\mathsf{k}) \to G(\mathsf{k})$ denote the (left) translation $x \mapsto gx$, and T_g^* the corresponding map on $\mathsf{k}[G]$. Then we require that

$$s_D(g) = dT_g(s_D(e))$$

where $dT_g(\delta) = \delta \circ T_g^*$ is the induced map on the tangent spaces $T_x G(\mathsf{k}) \to T_{gx} G(\mathsf{k})$. More explicitly, recalling the definition of s_D this is the same as: $g \circ D = \epsilon \circ D \circ T_g$, and since $g = \varepsilon \circ T_g^*$, this becomes

$$\epsilon \circ T_g^* \circ D = \epsilon \circ D \circ T_g^*, \quad \forall g \in G(\mathsf{k}).$$

Actually it follows from this, the fact that $T_{xy}^* = T_y^* T_x^*$, and that evaluation at x is just $\epsilon \circ T_x$, that this condition is in fact equivalent to $T_x^* \circ D = D \circ T_x^*$ for all $x \in G(\mathsf{k})$. Hence what we require is simply that D commute with all the translation operations T_x^*. This condition is clearly compatible with the Lie bracket on derivations, so that we get a Lie algebra of invariant vector fields on $G(\mathsf{k})$, which we declare to be *the* Lie algebra of G.

4.6. **Lie algebras and algebraic groups.** We now extend this to arbitrary (affine) algebraic groups. This is done by noticing that T_g is given on $\mathsf{k}[G]$ by the operation $(g \otimes 1)\Delta$, so that the condition $T_g^* \circ D = D \circ T_g^*$ for a vector field to be invariant becomes $(g \otimes 1)\Delta \circ D = (g \otimes D) \circ \Delta$ for every $g \in G(\mathsf{k})$. In the case of a group variety this is equivalent to insisting that $\Delta \circ D = (1 \otimes D) \circ \Delta$. This leads to the following definition:

Definition 4.22. Let $\text{Lie}(G)$, the Lie algebra of G, be the space of left invariant derivations, that is:

$$\text{Lie}(G) = \{D \in \text{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G]) : \Delta \circ D = (1 \otimes D) \circ \Delta\}.$$

Since this subspace of $\text{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G])$ is closed under the commutator product, it is a Lie algebra.

We now show that $\text{Lie}(G)$ is finite dimensional. Note that even if G is not reduced, and hence does not come from an affine variety, our definition of the tangent bundle as $G(D)$ still makes sense, and hence also T_xG for any k-point of G.

Lemma 4.23. *The map $L\colon \text{Lie}(G) \to T_eG(\mathsf{k})$ given by $D \mapsto \epsilon \circ D$ is a isomorphism of vector spaces.*

Proof. We construct an inverse to the map L as follows: given $v \in T_eG(\mathsf{k})$, we may associate to v a derivation $d \in \text{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}_\epsilon)$. Define $D \in \text{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G])$ by setting

$$D = (1 \otimes d) \circ \Delta.$$

Then using $(1 \otimes \epsilon) \circ \Delta = \text{id}_G$, one checks that D is a derivation, and it follows from coassociativity (*i.e.* $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$) that $D \in \text{Lie}(G)$. It is easy to see that this is an inverse for L as required. ☐

Since it is often easier to work with ϵ-derivations (*i.e.* tangent vectors at the identity element), it is useful to spell out the Lie bracket operation in these terms. From the above, given $d_1, d_2 \in T_eG$ we have derivations $D_i = (d_i \otimes 1) \circ \Delta$. Now using invariance we have

$$\begin{aligned}
D_1 D_2(f) &= (1 \otimes d_1)\Delta(D_2(f)) \\
&= (1 \otimes d_1)(1 \otimes D_2) \circ \Delta(f) \\
&= (1 \otimes d_1)(1 \otimes ((1 \otimes d_2) \circ \Delta))) \circ \Delta(f) \\
&= (1 \otimes d_1)(1 \otimes 1 \otimes d_2)(1 \otimes \Delta)\Delta(f) \\
&= (1 \otimes (d_1 \otimes d_2))(1 \otimes \Delta)\Delta(f) \\
&= (1 \otimes (d_1 \otimes d_2) \circ \Delta)\Delta(f).
\end{aligned}$$

It follows that $[D_1, D_2] \in \text{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G])$ corresponds to the ϵ-derivation $[d_1, d_2]$ given by

$$[d_1, d_2] = (d_1 \otimes d_2 - d_2 \otimes d_1) \circ \Delta. \tag{4.2}$$

To summarize, we have shown the following:

Proposition 4.24. *The Lie algebra of G is isomorphic to the space of ϵ-derivations equipped with the Lie bracket given by Equation (4.2). Hence if G is a smooth algebraic group, $\text{Lie}(G)$ has dimension $\dim(G(\mathsf{k}))$ as a vector space over k.*

We can use this to calculate the Lie algebra of some algebraic groups.

Example 4.25. Let $G = \text{GL}_n$. We can think of T_eG as the points of D-points of G mapping to the identity in $G(\mathsf{k})$, where $D = \mathsf{k}[\varepsilon]/(\varepsilon^2)$ is the algebra of dual numbers, it follows

$$T_eG = \{I + \varepsilon A \in \text{Mat}_n(D) : I + \varepsilon A \text{ is invertible}\}.$$

But $(I+\varepsilon A)(I-\varepsilon A)=I$, so T_eG is exactly $\{I+\varepsilon A : A \in \mathrm{Mat}_n(\mathsf{k})\}$. To calculate the Lie bracket, notice that the derivation (say denoted d_A), associated to the matrix $A \in \mathrm{Mat}_n(\mathsf{k})$ is given by $d_A(X_{ij}) = a_{ij}$, hence since

$$[d_A, d_B] = (d_A \otimes d_B - d_B \otimes d_A) \circ \Delta$$

we have

$$\begin{aligned}[d_A, d_B](X_{ij}) &= (d_A \otimes d_B - d_B \otimes d_A)(\sum_{k=1}^{n} X_{ik} \otimes X_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} - b_{ik}a_{kj}\end{aligned}$$

which is the derivation corresponding to $[A,B] = AB - BA$, the commutator of A and B. Thus we see that

$$\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n.$$

It is similarly straightforward to check that the Lie algebra of SL_n is just $\mathfrak{sl}_n$, since $\det(I+\varepsilon A) = I + \varepsilon \mathrm{tr}(A)$

Example 4.26. The Lie algebra of $\mathbb{G}_a$ is one dimensional as $\mathbb{G}_a$ is smooth, and it's Lie bracket then has to be zero.

Example 4.27. Strange things can happen in positive characteristic: Let k be an algebraically closed field of characteristic $p > 0$. Let μ_p the algebraic group of p-th roots of unity. This is the k-group functor which attaches to a k-algebra A the group $\{x \in A : x^p = 1\}$. This is clearly an affine algebraic group with coordinate algebra $\mathsf{k}[x]/(x^p-1)$, thus it is a finite algebraic group. Clearly it therefore has zero Krull dimension, however, it's Lie algebra is one-dimensional, as $T_e\mu_p = \{1+\varepsilon b : (1+b\varepsilon)^p = 1\}$, and since $(1+\varepsilon b)^p = 1 + \varepsilon pb = 1$, we see that b is arbitrary.

Lemma 4.28. *A morphism of algebraic groups $\phi\colon G \to H$ induces a Lie algebra map $d\phi\colon \mathit{Lie}(G) \to \mathit{Lie}(H)$, which is injective if ϕ is a closed embedding.*

Proof. If $\phi^*\colon \mathsf{k}[H] \to \mathsf{k}[G]$ is the corresponding map on coordinate algebras, then $d\phi$ is simply the induced action of ϕ on derivations

$$d\phi\colon \mathrm{Der}_{\mathsf{k}}(\mathsf{k}[G], \mathsf{k}[G]) \to \mathrm{Der}_{\mathsf{k}}(\mathsf{k}[H], \mathsf{k}[H]),$$

given by $D \mapsto D \circ \phi^*$. Since ϕ is a map of algebraic group, $\Delta_G \circ \phi^* = \phi^* \circ \Delta_H$, and hence $d\phi$ restricts to a map $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$. If ϕ is a closed embedding then ϕ^* is surjective and hence $d\phi$ must be injective on $T_e(G)$. □

Lemma 4.29. *Let G be an algebraic group. Then G is smooth if and only if $\dim(G) = \dim_{\mathsf{k}}(\mathit{Lie}(G))$.*

Proof. Here $\dim(G)$ is the Krull dimension of $\mathsf{k}[G]$, *i.e.* the supremum of the length of ascending chains of distinct prime ideals in $\mathsf{k}[G]$ (the algebraic analogue of dimension for Noetherian topological spaces which we gave before). We may assume that k is algebraically closed (since G is smooth by definition if $G_{\bar{\mathsf{k}}}$ is). If the equality holds, then $\mathcal{O}_{G,e}$ is a regular local ring, and then it follows that all the local rings $\mathcal{O}_{G,x}$ are regular by homogeneity, which implies that G is smooth. Conversely, if G is smooth, then all the local rings $\mathcal{O}_{G,x}$ are regular, and so

$$\dim_{\mathsf{k}}(\mathrm{Lie}(G)) = \dim_{\mathsf{k}}(T_eG(\mathsf{k})) = \dim(G).$$

□

5. Group actions

Definition 5.1. Let G be an algebraic group and X an affine variety (or if you want k-functor). Then an action of G on X is a morphism $a\colon G \times X \to X$ such that $e \in G$ acts by id_X and the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}_X} & G \times X \\ \downarrow{\scriptstyle \mathrm{id}_G \times a} & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow{a} & G \end{array}$$

commutes, where $m\colon G \times G \to G$ is the multiplication map. If X has a G-action we will say that X is a G-space.

If X is an affine variety, then we may rephrase this in terms of the coordinate algebras: we get a map $a^*\colon \mathsf{k}[X] \to \mathsf{k}[G] \otimes \mathsf{k}[X]$, which satisfies a diagram dual to one above, or in other words such that $(\mathrm{id}_G \otimes a^*) \circ a^* = (\Delta \otimes \mathrm{id}_{\mathsf{k}[X]}) \circ a^*$. Note that this is the same as saying that $\mathsf{k}[X]$ is a G-representation.

Any action of an algebraic group on an affine variety comes from a linear action in the sense that the following proposition makes precise.

Proposition 5.2. *Let X be an affine G-space. Then there is a finite dimensional G-representation V and an embedding $i\colon X \to V$ which respects the G-actions.*

Proof. This follows in exactly the same way we proved that an affine algebraic group was isomorphic to a closed subgroup of GL_n. Simply choose a finite dimensional subspace W of $\mathsf{k}[X]$ which generates $\mathsf{k}[X]$ as an algebra. Since $\mathsf{k}[X]$ is a G-representation, the action is locally finite, and hence we may enlarge W to a finite-dimensional subspace W' which is a G-representation. The representation W' gives the required embedding. □

6. Topology of smooth algebraic groups

. In this section we will assume that G is a smooth algebraic group over an algebraically closed field. With this assumption, $\mathsf{G}(\mathsf{k})$ is an affine variety, and $\mathsf{k}[G]$ is its ring of functions, thus this class of algebraic groups are equivalent to *group varieties* – affine varieties which are compatibly groups, hence we may be sloppy and identify G with its k-points.

Lemma 6.1. *Let G be a group variety, then G has finitely many smooth connected components. The component containing e, denoted G^0, is a normal subgroup, so that the set of components has a group structure.*

Proof. We may write G as a union of irreducible components $G = G^0 \cup G^1 \cup \ldots \cup G^r$, with say $e \in G^0$. Clearly there is a point $x \in G$ lying exactly one component of G. However, G acts transitively on itself by automorphisms, thus every point lies in exactly one component of G, and hence the irreducible components are actually the connected components of G. To see that G^0 is a normal subgroup, note that conjugation by $g \in G$ gives a map $Ad_g\colon G^0 \to G$. Since G^0 is connected, its image must be also, and since $Ad_g(e) = e \in G^0$ we see that $Ad_g(G^0) \subset G^0$, as required. □

Let $\phi\colon X \to Y$ be a morphism of varieties. It is a somewhat difficult question to define an "image" in the category of varieties (here is a situation where moving from affine varieties to varieties, or even schemes doesn't resolve the issue). There is however, always a set-theoretic image (*i.e.* whatever else it is or isn't $\phi(X)$ is certainly a subset of Y). Chevalley showed that the image of a morphism is not too wild a subset of Y.

Definition 6.2. If X is a topological space, then a subset Y is *locally closed* if it is open in its closure, or equivalently if it is the intersection of an open and a closed subset of X. A *constructible set* is a finite union of locally closed subsets.

Theorem 6.3. *If $\phi\colon X \to Y$ is a morphism of varieties, then the image of ϕ is a constructible subset of Y.*

Proof. The key step in the proof is to show that if $f(X)$ contains an open subset of its closure $\overline{f(X)}$. For a complete proof see, for example, [M]. □

Let X be a G-variety. Given a point $x \in X$ the *orbit* of x is the image of G under the map $g \mapsto g \cdot x$.

Proposition 6.4. *Let X be a G-space. Then the orbits of G are locally closed subsets of X. Moreover $\overline{\mathcal{O}}$ union of G-orbits, and X contains a closed G-orbit.*

Proof. Let $x \in X$ and let $\mathcal{O}$ be the orbit of G through x. Then $\mathcal{O}$ is the image of the morphism $G \to X$ given by the orbit map $g \mapsto g \cdot x$, hence by Chevalley's theorem we know $\mathcal{O}$ is constructible. Thus we may find an subset U which is open in $\overline{\mathcal{O}}$ the closure of $\mathcal{O}$. But clearly we have $\mathcal{O} = \bigcup_{g \in G} g(U)$, and hence $\mathcal{O}$ is open in $\overline{\mathcal{O}}$ as required.

Since G acts continuously it is clear that G preserves $\overline{\mathcal{O}}$, hence $\overline{\mathcal{O}}$ is a union of G-orbits. To see that X has a closed G-orbit, simply choose an orbit of minimal dimension, as $\dim(\overline{\mathcal{O}}\backslash\mathcal{O}) < \dim(\overline{\mathcal{O}})$. □

Corollary 6.5. *Let $\phi\colon G \to H$ be a morphism of group varieties. Then $\phi(G)$ is a closed subgroup of H.*

Proof. The morphism ϕ makes H a G-space. Clearly the G-orbits on H are just the coset of $\phi(G)$ in H, and hence they are all isomorphic subvarieties of H. But we know that G has a closed orbit in H, and hence all of the G-orbits must be closed, so that $\phi(G)$ (the orbit of the identity in H) is a closed subgroup of H as required. □

Remark 6.6. If you are familiar with Lie groups, then it is worth noting that this Corollary is in stark contrast to what happens with Lie groups, as the example of a dense winding in a torus shows.

Lemma 6.7. *Let G be a connected affine algebraic group. Suppose that A is a dense constructible subset of G.Then $G = A.A$.*

Proof. Let $\iota\colon G \to G$ denote the inversion map. Then note that if $g \in G$, then $g\iota(A)$ is also a dense constructible subset of A, and so $A \cap g\iota(A) \neq \emptyset$. Thus $G = A.A$. □

7. Quotients of algebraic groups

In this section k is algebraically closed. Given a closed subgroup H of an algebraic group G, we would like to have a good notion of a quotient G/H. In general this can be quite a subtle issue (even in the context of a group action on topological spaces, quotients can be a delicate issue – if you take the action of $\mathbb{C}^*$ on $\mathbb{C}$ then the quotient space has two points, but one is in the closure of the other, so it is not a Hausdorff topological space, which is perhaps unpleasant). We will shameless avoid the more complicated situations, and only work with nice quotients.

We assume that G and H are smooth algebraic groups over an algebraically closed field, with H a closed subgroup of G. Then we may equivalently work with the group varieties given by their k-points (and will be sloppy about identification, as we were before). Now G/H is certainly a collection of subsets of G, however we would like it to be a more geometric object. The issue is what kind of object we can ask it to be.

Example 7.1. Let $G = \mathrm{GL}_2(\mathsf{k})$, and

$$B = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : ac \neq 0\}.$$

Then it is easy to check (*exercise?*) that any polynomial function on G is constant on B-orbits if and only if it is constant. Now suppose that the set G/B had the structure of an affine variety, and that the map $\pi\colon G \to G/B$ was a morphism of affine varieties. Then for any regular function f on G/B we would obtain a regular function $\pi^*(f)$ which would be constant on B-orbits. But then it follows that f is constant, and so the only regular functions on G/B would be the constants, that is, if it were an affine variety, G/B would have to be a point.

7.1. **Quasi-projective varieties.** One way to deal with this problem is to enlarge the class of geometric objects we are working with. The appendix gives a definition of an *algebraic variety*, which is somewhat similar to the definition of a smooth manifold – a smooth manifold must locally look like $\mathbb{R}^n$, and satisfy some Hausdorff condition, while for algebraic varieties we insist that locally they look like affine varieties, and that they satisfy some analogue of the Hausdorff condition. In fact, much of the time one can work with a more restrictive, but more explicit class of geometric objects, call *quasi-projective* varieties.

We begin with the definition of a projective variety. Just as affine varieties are all isomorphic to closed subsets of affine n-space $\mathbb{A}^n$ for some n, projective varieties are closed subspaces of some projective space. The *projective space* $\mathbb{P}^n(\mathsf{k})$ over k is, as a set, the collection of lines through the origin in affine $(n+1)$ space, that is, the collection of one-dimensional subspaces of k^{n+1}. Clearly this is the same as the set of orbits $(\mathsf{k}^{n+1}\backslash\{0\})/\mathsf{k}^\times$ of $\mathsf{k}^\times$ on $\mathsf{k}^{n+1}\backslash\{0\}$, and so we can give *homogeneous coordinates* to $\mathbb{P}^n(\mathsf{k})$ – a point in $\mathbb{P}^n$ corresponds to an equivalence class (*i.e.* $\mathsf{k}^\times$-orbit) $[x_0, x_1, \ldots, x_n]$ of points in k^{n+1}. Notice that while a polynomial on k^{n+1} which is invariant under the action of $\mathsf{k}^\times$ must be constant, the equation $f(x_0, x_1, \ldots, x_n) = 0$ makes sense on homogeneous coordinates provided only f is *homogeneous* – that is if

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_0, x_1, \ldots, x_n).$$

Thus we can study $\mathbb{P}^n$ algebraically by considering zero sets

$$V(S) = \{[x_0, x_1, \ldots, x_n] \in \mathbb{P}^n : f(x_0, x_1, \ldots, x_n) = 0, \forall f \in S\}$$

where S consists of homogeneous polynomials. This is the same as studying zero sets of homogeneous ideals in $\mathsf{k}[t_0, t_1, \ldots, t_n]$ – ideals which are generated by homogeneous polynomials, or equivalently ideals which are preserved by the action of the multiplicative group $\mathsf{k}^\times$.

It is easy to check, as one does for affine varieties, that the sets $V(S)$ form the closed sets for a topology on $\mathbb{P}^n$. Moreover, each open set

$$U_i = \{[x_0, x_1, \ldots, x_n] : x_i \neq 0\}$$

is clearly in bijection with k^n where

$$[x_0, x_1, \ldots, x_n] \mapsto (x_0/x_i, x_1/x_i, \ldots, x_n/x_i)$$

(we omit the i-th entry, since $x_i/x_i = 1$). Thus $\mathbb{P}^n$ has a covering by $n+1$ open subsets each isomorphic to affine n-space. Thus we can equip each U_i with a structure sheaf $\mathcal{O}_{U_i}$ in the obvious way, and since these assignments are compatible on overlaps $\mathbb{P}^n$ acquires the structure of a variety (one can check that the diagonal is closed in $\mathbb{P}^n \times \mathbb{P}^n$).

Definition 7.2. A projective variety is a closed subvariety of $\mathbb{P}^n$ for some n. A quasi-projective variety is a locally closed subset of a projective space, that is, an open subvariety of a projective variety.

Remark 7.3. Projective varieties have products which are again projective varieties: to see this it is enough to check that this is true of projective spaces themselves. The classical way to do this is the *Segre embedding* which embeds the product into $\mathbb{P}^{nm+n+m}$ and is given in homogeneous coordinates as:

$$([z_0, z_1, \ldots, z_n], [w_0, w_1, \ldots, w_m]) \mapsto [z_0w_0, z_0w_1, \ldots, z_0w_m, z_1w_0, \ldots, z_nw_m].$$

To see that the image is closed, one shows that the equations defining the image are $x_{ij}x_{kl} = x_{kj}x_{il}$, where $\mathbb{P}^{nm+n+m}$ has homogeneous coordinates $[x_{ij}]$, $(0 \leq i \leq n, 0 \leq j \leq m\})$.

Example 7.4. We give some examples of projective varieties that arise naturally for us. Given a vector space V over k, let $\mathrm{Gr}_d(V)$ be the set of d-dimensional subspaces of V. By using the d-th exterior power operation, such a subspace corresponds to a line in $\wedge^d(V)$, and so we can realize $\mathrm{Gr}_d(V)$ as a subset of $\mathbb{P}(\wedge^d(V))$. It can be shown that this subset is closed, and so $\mathrm{Gr}_d(V)$ is naturally a projective variety. For example, the Grassmannian $\mathrm{Gr}_2(\mathsf{k}^4)$ of 2-planes in k^4 is embedded in $\mathbb{P}(\wedge^2(\mathsf{k}^4)) = \mathbb{P}^5$, as the quadric hypersurface:

$$\mathrm{Gr}(2,4) = \{[x_0, x_1, \ldots, x_5] : x_0x_1 - x_2x_3 + x_4x_5 = 0\}.$$

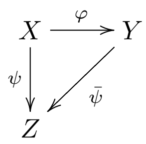
Example 7.5. Slightly more elaborately, let $\mathcal{F}(V)$ be the *flag variety* of V. As a set this is the collection of sequences of subspaces $(V_i)_{1 \leq i \leq n}$ where $V_i \subset V_{i+1}$ and $\dim(V_i) = i$. It is clear that we can realize $\mathcal{F}(V)$ as a subset of

$$\mathrm{Gr}_1(V) \times \mathrm{Gr}_2(V) \times \ldots \times \mathrm{Gr}_n(V),$$

and moreover, it is not hard to see that the containment conditions are closed, so that $\mathcal{F}(V)$ is again a projective variety.

With our larger class of varieties, we can give a definition of a quotient G/H. In fact, we give a slightly more general definition. Notice that the inclusion of H in G gives G the structure of an H-variety. The following is a definition of a quotient.

Definition 7.6. Let H be a smooth algebraic group and X a H-variety. The quotient X/H of X by H is a variety Y with a H-invariant map $\varphi \colon X \to Y$, such that any a H-invariant map $\psi \colon X \to Z$ to a variety Z factors uniquely through X, that is there is a unique morphism $\bar{\psi}$ making the following diagram commute:



Clearly if such a variety exists it is unique.

Applying this to the H-variety G, we get a definition of G/H, though we have yet to show it exists.

Remark 7.7. This definition is "categorical" in that it is crucial which kind of spaces with G-action you are considering. I have stated it for varieties with G-action. It would make equal sense to apply it to sets with a G-action, when the quotient is clearly just the set of cosets G/B. Considering Example 7.1 again, what it actually shows is that if we gave a definition of quotients in the category of affine varieties, then the point would actually be the "quotient" of G by B, in the sense it would satisfy the characterizing property among affine G-spaces. The "problem" here is then that the forgetful functor to sets does not commute with quotients. By working with varieties, this problem does not arise, that is, we can equip the set G/B with a variety structure making it the quotient of G by B in the category of G-varieties.

Example 7.8. We give a more interesting example of a quotient in the category of affine varieties. Let $G = \mathrm{SL}_2(\mathsf{k})$ act on itself by conjugation. Then you can show that the algebra of invariant function $\mathsf{k}[\mathrm{SL}_2]^{\mathrm{SL}_2}$ is $\mathsf{k}[T]$ where T is the trace function. The affine line (thought of as the variety attached to $\mathsf{k}[T]$) is then the quotient of SL_2 in the category of affine varieties. It's points do not quite correspond to the orbits – each of the points $\{\pm 2\}$ corresponds to the union of two orbits: the orbits of $[I]$ and $[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}]$ for 2, and the orbits of $[-I]$ and $[\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}]$ for -2 .

Remark 7.9. In this section we are only constructing quotients G/H and thus quotients for varieties with a *free* H-action, that is, the stabilizers of points are all trivial. In general if a group action is not free, the naive quotient is usually not the "correct" quotient. The above example of the conjugation action of SL_2 on itself is an example where the action is not free (indeed different points have non-conjugate stabilizers). There are thus many questions related to the above example which we will not deal with. The interested reader should take a look at a book on geometric invariant theory.

In order to show that there is a categorical quotient in the category of G-varieties, we want a criterion for when a homogeneous space for G is a quotient. We begin with some general properties of morphisms. Recall that a morphism $\phi\colon X \to Y$ is *dominant* if it has dense image.

Lemma 7.10. *(Technical facts.) We have the following properties of morphisms of varieties. Let $\phi\colon X \to Y$ be a dominant morphism, and let $r = \dim(X) - \dim(Y)$.*

(1) *There is an open set $U \subset Y$ such that $U \subset \phi(X)$, and if S is a closed subset such that $S \cap U \neq \emptyset$, then the components of $f^{-1}(S)$ dominating S have dimension at least $\dim(S) + r$, and those which intersect $f^{-1}(U)$ have dimension equal $\dim(S) + r$. In particular $r \geq 0$.*

(2) *Suppose that for every irreducible closed subset $S \subset Y$ the components of $\phi^{-1}(S)$ are all of dimension $\dim(S) + r$. Then ψ is an open mapping, i.e. ψ sends open sets to open sets.*
(3) *Suppose that ϕ is a bijective map. If $d\phi\colon T_xX \to T_{\phi(x)}Y$ is an isomorphism at all smooth points $x, \phi(x)$ of X and Y respectively, then ϕ restricts to an isomorphism of the smooth part X^0 of X onto its image.*
(4) *(a form of Zariski's Main Theorem) If Y is a normal variety (in particular if Y is smooth), and ϕ is birational and bijective, then ϕ is an isomorphism.*

Proof. All of these are fairly standard facts, found in Milne's notes [M] or Mumford's Red Book [Mu]. Statement (4) is the deepest, but we use it below only for smooth varieties, where it is an easier result. □

Proposition 7.11. *Let $\pi\colon G \to Q$ be a morphism with every fiber a coset gH of H, such that $d\pi$ is surjective at every point. Then Q is a quotient of G by (right translation by) H.*

Proof. Suppose that $\psi\colon G \to R$ is a H-invariant morphism. Then we may factor ψ through the map $\tilde{\psi} = \pi \times \psi\colon G \to Q \times R$. Now the nonempty fibers of $\tilde{\psi}$ have dimension at least $d = \dim(H)$ since $\tilde{\psi}$ is H-invariant, but also at most d, since this is true of the fibers of π. But then it follows that the dimension of all nonempty fibers $\phi^{-1}(y)$, $(y \in Y)$ are the same, and so by part (2) of Lemma 7.10 the image of $Z = \tilde{\psi}(G)$ is open in its closure. Thus Z is a variety. (Strictly speaking here we need to assume G is connected here, so that Z is irreducible, but this is a harmless assumption).

Now the map $d\pi$ vanishes on tangent vectors to H-cosets, so that the dimension of its image is at most $\dim(G) - \dim(H)$, which is just $\dim(Q)$. But this implies the dimension of the tangent spaces of Q are all equal to $\dim(Q)$, and so Q is smooth. The map $\rho\colon Z \to Q$ given by projection is clearly a bijection, and moreover it is surjective on tangent spaces, since π is. It follows that the restriction of this map to the smooth part of Z is an isomorphism and so ρ is birational. But then by Zariski's Main Theorem, ρ must be an isomorphism, and we see that ψ factors through Q via the identification of Z with Q as required. Since the points of Q are in bijection with the H-cosets, it is easy to see that the factorization is unique. □

Having established a criterion for the construction of homogeneous spaces, we now construct them as quasi-projective varieties.

Given a representation V of G, if W is a subspace of V, then the stabilizer of W is the subgroup H given functorially by

$$H(R) = \{g \in G(R) : g(W \otimes R) \subseteq W \otimes R\}.$$

Pick a basis $\{v_i\}$ of V such that $v_1, v_2, \dots v_r$ is a basis of W. To see that H is a closed subgroup, note that if $m\colon V \to V \otimes \mathsf{k}[G]$ is the comodule map, and $m(v_i) = \sum_j v_j \otimes a_{ij}$., then the defining ideal of H is generated by the functions $\{a_{ij}\}$, where $i \leq r < j$. The next Proposition, due to Chevalley, shows that in fact all closed subgroups arise in this way.

Proposition 7.12. *Let H be a closed subgroup of G. Then there is a finite dimensional representation $\pi\colon G \to GL(V)$ of G and a subspace $W \subset V$ such that*

$$H = \{g \in G : g(W) = W\}, \quad \mathfrak{h} = \{x \in \mathfrak{g} : d\pi(x)(W) \subseteq W\},$$

where $\mathfrak{g} = Lie(G), \mathfrak{h} = Lie(H)$.

Proof. Let I be the ideal defining H. Take a G-stable subrepresentation, V say, of $\mathsf{k}[G]$, which contains generators for I as an ideal of $\mathsf{k}[G]$, so that $\Delta(V) \subset V \otimes \mathsf{k}[G]$. Take $W = V \cap I$. If $\{v_i\}$ is a basis for V with $v_1, v_2, \dots, v_r$ a basis of W, then we have

$$\Delta(v_i) = \sum_j v_j \otimes a_{ij},$$

then the $\{a_{ij}\}_{i \leq r < j}$ generate the ideal for the stabilizer of W. But since $\Delta(I) \subseteq \mathsf{k}[G] \otimes I + I \otimes \mathsf{k}[G]$ it follows that all the a_{ij} with $i \leq r < j$ lie in I. Finally, since $\epsilon(I) = 0$, and

$$v_i = (\epsilon \otimes \mathrm{id}) \circ \Delta(v_i) = \sum_{j > r} \epsilon(v_j) a_{ij},$$

it follows that the ideal generated by the $\{a_{ij}\}_{i \leq r < j}$ must be exactly I as required. Similar considerations establish the Lie algebra version. □

By taking an exterior power of this representation, we can assume in fact that the subspace W in the above proposition is a line, since W is a subrepresentation if and only if $\wedge^r(W)$ is a subrepresentation of $\wedge^r(V)$.

Proposition 7.13. *Let G be a smooth algebraic group and H a smooth closed subgroup. Pick V a finite dimensional representation of G with a line L such that H is the stabilizer of L, and Lie(H) is the stabilizer of L in Lie(G). Then the orbit map $\phi\colon G \to \mathbb{P}(V)$, where $g \mapsto g(L)$ has full rank at each point, and gives a bijection between G/H and a quasi-projective subvariety of $\mathbb{P}(V)$. Hence the image is the quotient of G by H in the category of varieties.*

Proof. The image of ϕ is by definition a G-orbit, hence it must be locally closed, and thus a quasi-projective variety. Moreover, by construction it is clear that the fibers of the map ϕ are exactly the cosets of H in G, and hence $\dim(\phi(G)) = \dim(G) - \dim(H)$.

To see the condition on rank, it is enough, using the G-action, to check at the identity. Pick a nonzero vector $v \in L$. Then we have an orbit map $\tilde{\phi}\colon G \to V - \{0\}$ given by $g \mapsto g(v)$. It is easy to see that this orbit map is a composition of the map $\rho\colon G \to \mathrm{GL}(V)$ with the map $\psi\colon \mathrm{GL}(V) \to V - \{0\}$, where $\psi(\alpha) = \alpha(v)$. Now the derivative $d\psi_e\colon \mathfrak{gl}(V) \to T_v V$ is given similarly by $d\psi_e(X)(w) = X(w)$ for $X \in \mathfrak{gl}(V)$, $w \in T_v(V) \cong V$, hence the differential of $\tilde{\phi}$ is given by

$$X \mapsto d(\psi \circ \rho)_e(X) = d\psi_e \circ d\rho_e(X) = d\rho_e(X)(v).$$

Finally, if π is the projection map $\pi\colon V - \{0\} \to \mathbb{P}(V)$, then the kernel of $d\pi_v$ is exactly the line spanned by v. It follows that the kernel of $d\phi$ is exactly Lie(H), and since $\dim(\phi(G)) = \dim(G) - \dim(H)$, this finishes the proof. □

In summary, we have shown the following:

Theorem 7.14. *Let G be a smooth algebraic group and H a smooth closed subgroup. Then there is a quotient G/H which is a quasi-projective variety whose points are in bijection with the cosets of H in G.*

Example 7.15. Running through the proof of this theorem for the case $G = \mathrm{SL}_2$ and $H = B$ the upper triangular matrices as in Example 7.1, one can take $V = \mathsf{k}^2$ the standard representation of SL_2, since B is the stabilizer of the line spanned by e_1, the first standard basis vector. Then clearly G acts transitively on $V - \{0\}$, so that the orbit of $[e_1]$ is all of $\mathbb{P}^1$, and we see that $G/B = \mathbb{P}^1$. It is easy to see that $\mathsf{k}[\mathbb{P}^1] \cong \mathsf{k}$, and so the only regular functions we get via pull-back from the map $G \to G/B$ are the constant functions, which is consistent with our previous calculation.

We have now shown that, at least for smooth algebraic groups, there is a sensible quotient G/H in the category of varieties. However, if H is normal in G, then the set of cosets forms a group, which is clearly algebraic – that is, G/H is a group variety. Since we are only working with G, H affine, it is natural to ask if these quotient groups must be affine group varieties. The fundamental case here is for GL_n and its centre Z which is just $\mathbb{G}_m$ embedded as the scalar matrices. The centre of a smooth algebraic group can be defined as the centre of $G(\mathsf{k})$, since this is a closed subgroup of the group variety $G(\mathsf{k})$, and so has an associated algebraic group. In general on can give a functorial definition of the centre of an algebraic group, as shown in the exercises.

Definition 7.16. Let G be an algebraic group. Then G acts on itself by conjugation

$$Ad\colon G \to \mathrm{Aut}(G), \quad g \mapsto Ad_g = (x \mapsto gxg^{-1}), \quad g, x \in G(R).$$

Since taking the Lie algebra is a functor, it follows that G acts on $\mathrm{Lie}(G)$ by automorphisms, that is, we have a representation of G on $\mathrm{Lie}(G)$, known as the *adjoint representation*.

Explicitly, if $g \in G(\mathsf{k})$ then the canonical map $\mathsf{k} \to \mathbb{D}$ (the dual numbers) gives, for each $g \in G(\mathsf{k})$ an element $g' \in G(\mathbb{D})$, and conjugation by g' gives an automorphism of $\mathrm{Lie}(G) = \ker(G(\mathbb{D}) \to G(\mathsf{k}))$. Note that for $G = \mathrm{GL}_n$, this makes it clear that the adjoint representation is just the representation of GL_n on $\mathrm{End}(\mathsf{k}^n)$ given by conjugation.

Lemma 7.17. *Let V be a k-vector space, and let $G = GL(V)$ and let $Z = \mathbb{G}_m$ embedded in G as its centre. Then the quotient group G/Z is affine.*

Proof. Let $r\colon \mathrm{GL}(V) \to GL(\mathrm{End}(V))$ be the adjoint representation for $\mathrm{GL}(V)$, and let H be the image of $\mathrm{GL}(V)$. This is a closed subgroup of $\mathrm{GL}(\mathrm{End}(V))$, and the kernel of r is clearly Z, thus H is isomorphic to G/Z provided the derivative of r is surjective at each point. Because of the homogeneity it is enough to check this at a single point – the identity say. But we have already seen that the map $\mathrm{GL}(V) \to \mathrm{GL}(\mathrm{End}(V))$ is given by $g \mapsto (X \mapsto gXg^{-1})$, and it is straight-forward to check that the derivative of this map is:

$$dr(A) = (X \mapsto [A, X]), \quad X \in \mathrm{End}(V).$$

It follows that $dr(A) = 0$ if and only if $[A, B] = 0$ for all $B \in \mathrm{End}(V)$, that is, A is a scalar matrix. Since the scalar matrices are exactly $\mathrm{Lie}(Z)$, and H is smooth of dimension $\dim(\mathrm{GL}(V)) - \dim(Z)$ it follows that dr is surjective as required. □

Theorem 7.18. *Let G be a smooth algebraic group and N a smooth normal subgroup. Then the quotient G/N is affine.*

Proof. As usual we work with the associated group varieties. We may assume that G and N are connected, as G/N is a union of finitely many copies of $G^0/(G^0 \cap N)$. Let V be a G-representation and L a line in G whose stabilizer is N as in the proof of the existence of quotients, so that the Lie algebra stabilizer of L is $\mathrm{Lie}(N)$ also.

Now L determines a character of N, that is, a map $\chi\colon N \to \mathbb{G}_m$. Since N is normal, if $g \in G$ is any element of G, the line $g(L)$ is also N-stable, as

$$n.g(L) = g.(g^{-1}ng)(L) = g(L).$$

The lines $\{g(L) : g \in G\}$ clearly span a representation of G containing L, so we may assume (replacing V by this smaller space if necessary) that $V = \text{span}\{g(L)\}$. Then V is a sum of isotypic subspaces, V_ψ where $V_\psi = \{v \in V : n.v = \chi(n)v\}$, and $\psi\colon N \to \mathbb{G}_m$ is a character of N. Since the characters ψ are linearly independent (see the next section) this sum is direct, and so the collection of characters ψ for which $V_\psi \neq 0$ is finite. Now by assumption G acts transitively on this set, since we are assuming V is generated as a G-representation by L, but G is connected, so it acts trivially on any finite set, hence exactly one character of N can appear. Since N acts by χ on L it follows $V = V_\chi$.

Now consider the map $\rho\colon G \to \text{GL}(\text{End}(V))$ given by composing the representation V with the adjoint representation of $\text{GL}(V)$. Suppose that g is in the kernel of this map. Then g acts as a scalar on V, and so *a fortiori* it preserves L, which forces it to lie in N. Conversely any element of N lies in the kernel of this map by the previous paragraph, and so $N = \ker(\rho)$. The image of ρ is a closed subgroup of $\text{GL}(\text{End}(V))$, and hence affine. To check that this image is G/Z we just need to show the map is surjective on tangent spaces. By homogeneity, it is enough to do this at the identity, and then it follows by the assumption Lie algebra action on V and the calculations in the previous lemma for $\text{GL}(V)$. □

Remark 7.19. *The case of non-algebraically closed fields*: Let us mention briefly what goes on in the non-algebraically closed case. In this setting it is not enough to work with the group $G(\mathsf{k})$, and consider instead the full k-group functor.

In the setting of k-group functors, if G and H are k-group functors, the we certainly have a k-functor G/H which assigns to any k-algebra R the set $G(R)/H(R)$, and of course if H is normal in G, then this k-functor will again be a k-group functor. Now while this might at first seem good, the problem is that the k-functors G/H you get with this naive approach are rather wild, so they do not have nice "geometric" properties (they wont be schemes, for example). In order to get more geometric k-functors, however one takes the naive k-functor G/H and tries to "regularize" it to get a more geometric object. If we restrict to G and H smooth algebraic groups, then the the quotient G/H constructed above is the "correct" quotient (where we associate to the variety G/H a k-functor in a similar fashion to the way we produce a k-functor from an affine variety). It is still the case that for any k-algebra R we have a map $q\colon G(R) \to (G/H)(R)$ whose fibers are the $H(R)$-cosets, but it is *not* always the case that we have $(G/H)(R) = G(R)/H(R)$. In general the set $(G/H)(R)$ is larger than $G(R)/H(R)$. We will see an example of this below.

On the positive side, however, we can at least give an elementary definition of when a map $\phi\colon G \to H$ of affine algebraic groups is a quotient map even if k is not algebraically closed: we say ϕ is a quotient if $\phi^*\colon \mathsf{k}[H] \to \mathsf{k}[G]$ is injective. It is possible to show that this definition is equivalent to the categorical definition of a quotient in the category of affine algebraic groups.

Example 7.20. Let $G = \text{SL}_2$ and let Z be the center of G. Assume we are not in characteristic 2 so that Z is the smooth algebraic group $\mu_2(R) = \{r \in R : r^2 = 1\}$.

Then you can check that SL_2/μ_2 is PGL_2. Now suppose that k is not algebraically closed. The map

$$\mathrm{SL}_2(\mathsf{k})/\mu_2(\mathsf{k}) \to \mathrm{PGL}_2(\mathsf{k})$$

is always injective, however it is not necessarily surjective since a matrix cannot always be written as the product of a scalar matrix times a matrix of determinant one (indeed this fails to be true if $(\mathsf{k}^\times)^2 \neq \mathsf{k}^\times$, as is the case for $\mathbb{R}$ for example).

8. Diagonalizable Groups

We know begin the study of particular classes of algebraic groups. The idea is to begin with simple classes (for example, tori) and use them to study the more complicated classes of algebraic groups (*e.g.* reductive groups, as we will see later). In fact we work with a slightly larger class of groups than tori, known as diagonalizable groups.

Let G be an algebraic group. A *character* of G is a morphism $\phi\colon G \to \mathbb{G}_m$, the multiplicative group. On coordinate algebras it is given by a Hopf algebra map $\phi^*\colon \mathsf{k}[t, t^{-1}] \to \mathsf{k}[G]$. Such a map is determined by the element $a = \phi^*(t)$, and it follows immediately that $\Delta(a) = a \otimes a$, $S(a) = a^{-1}$ and $\epsilon(a) = 1$.

On the other hand we say an element $a \in \mathsf{k}[G]$ is *group-like* if $\Delta(a) = a \otimes a$. It follows from the Hopf algebra axioms[15] that this implies $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. Indeed we have

$$a = (\epsilon \otimes 1)\Delta(a) = \epsilon(a).a,$$

and hence $\epsilon(a) = 1$ as claimed. Moreover we then have

$$1 = \epsilon(a) = m(S \otimes 1)\Delta(a),$$

that is, $S(a)a = 1$, so that $S(a) = a^{-1}$. Thus any group-like element $a \in \mathsf{k}[G]$ determines a homomorphism $\psi\colon \mathsf{k}[t, t^{-1}] \to \mathsf{k}[G]$ by setting $\psi(t) = a$. Hence we see the set of characters of G is in bijection with the set of group-like elements of $\mathsf{k}[G]$. Moreover, since $\mathbb{G}_m$ is an abelian algebraic group the set $X(G)$ is naturally an abelian group. The set of group-like elements in $\mathsf{k}[G]$ inherits a group structure from the multiplication in $\mathsf{k}[G]$, since the product of two group-like elements is group-like (this follows from the fact that Δ is an algebra homomorphism). It is easy to check that the group structures are compatible with the identification of $X(G)$ with the group-like elements in $\mathsf{k}[G]$, so we shall feel free to identify the two groups.

Given any finitely-generated abelian group Γ, the group ring $\mathsf{k}[\Gamma]$ is a finitely generated k-algebra. If we set $\Delta(\gamma) = \gamma \otimes \gamma$, and $\epsilon(\gamma) = 1$, and $S(\gamma) = \gamma^{-1}$, for all $\gamma \in \Gamma$, and extend each linearly to $\mathsf{k}[\Gamma]$, then $\mathsf{k}[\Gamma]$ becomes a Hopf algebra.

Definition 8.1. Say that an algebraic group G is *diagonalizable* if its coordinate algebra is isomorphic to $\mathsf{k}[\Gamma]$ for some finitely generated abelian group Γ as above.

Example 8.2. Let $G = \mathbb{G}_m$ be the multiplicative group. Then we have $\mathsf{k}[G] = \mathsf{k}[t, t^{-1}]$. But it is easy to check that this is the Hopf algebra obtained from the group algebra of $\mathbb{Z}$, so $\mathbb{G}_m$ is diagonalizable. Similarly $(\mathbb{G}_m)^n$ is diagonalizable for any n. Similarly, if $G = \mu_n$ the group of n-th roots of unity, then it has coordinate

[15]More classically, you should check that any map $\phi\colon G \to H$ between groups G and H which is compatible with group multiplication is automatically compatible with the inversion, and sends the identity element of G to that of H.

algebra $\mathsf{k}[t]/(t^n - 1)$. It is easy to check that this is the Hopf algebra obtained from the group $\mathbb{Z}/n\mathbb{Z}$.

Lemma 8.3. *Let G be a diagonalizable group. Then G is a finite product of copies of $\mathbb{G}_m$ and various μ_n.*

Proof. Let us write $\mathsf{k}[G] = \mathsf{k}[\Gamma]$ where Γ is a finitely generated abelian group. Then from the structure theorem for such groups we know that

$$\Gamma \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}/p_i^{t_i}\mathbb{Z},$$

for various primes p_i. The lemma know follows from the previous example. □

The following lemma allows us to characterize diagonalizable groups amongst affine algebraic groups.

Lemma 8.4. *If A is a Hopf algebra over k the group-like elements in A are linearly independent over k.*

Proof. Suppose for the sake of contradiction that $b = \sum_{i=1}^{k} \lambda_i b_i$, where b and the b_i are group-like elements, and we assume that the b_i are linearly independent. Since Δ is an algebra morphism we have both $\Delta(b) = b \otimes b = \sum_{i,j} \lambda_i \lambda_j b_i \otimes b_j$, and $\Delta(b) = \sum_{i=1}^{k} \lambda_i b_i \otimes b_i$. Since the $\{b_i\}$ are linearly independent, we may compare the coefficients of these expressions to see that $\lambda_i \lambda_j = 0$ if $i \neq j$, and $\lambda_i^2 = \lambda_i$. From the second of these equations we see that each λ_i is 0 or 1, and then from the first that exactly one λ_i is nonzero. Hence the identity is the trivial dependence $b = b_i$, and we are done. □

Remark 8.5. Using the correspondence between group-like elements and characters, the previous lemma also shows that characters of a group are linearly independent, a fact we used in showing the quotient of a group variety by a closed normal subgroup was affine.

Theorem 8.6. *Let G be an affine algebraic group over a field k. Then G is diagonalizable if and only if its coordinate algebra is generated by group-like elements. There is an anti-equivalence between diagonalizable groups and finitely generated abelian groups, given by*

$$G \leftrightarrow X(G).$$

Proof. Let Γ be the set of group-like elements in $\mathsf{k}[G]$. We have already seen that Γ has a natural abelian group structure and choosing a coordinate t for $\mathbb{G}_m$ we obtain an isomorphism of abelian groups $X(G) \to \Gamma$. Suppose that $\mathsf{k}[G]$ is spanned by Γ. Then by the previous lemma, Γ must be a basis for $\mathsf{k}[G]$, and hence our isomorphism extends to an algebra isomorphism $\mathsf{k}[X(G)] \to \mathsf{k}[G]$. From the construction of the Hopf algebra structure on a group algebra, it is clear that this map is an isomorphism of Hopf algebras. The final part is clear, since the only choice is a coordinate for $\mathbb{G}_m$. □

Proposition 8.7. *Let G be a smooth closed subgroup of GL_n over an algebraically closed field. Then the elements of $G(\mathsf{k})$ can be simultaneously diagonalized if and only if the algebraic group G is diagonalizable. Thus $G(\mathsf{k})$ consists entirely of semisimple elements.*

Proof. Suppose the elements of $G(\mathsf{k})$ can be simultaneously diagonalized. Then by conjugating we may assume that they lie in T the subgroup of diagonal matrices in GL_n. But T is isomorphic to $(\mathbb{G}_m)^n$, a diagonalizable group, and hence $\mathsf{k}[T]$ is spanned by group-like elements. But then clearly $\mathsf{k}[G]$ is also, as it is a quotient of $\mathsf{k}[T]$.

Conversely, suppose that G is diagonalizable. Then let $\{f_i\}$ be a basis of $\mathsf{k}[G]$ consisting of group-like elements. Then the action of G on k^n gives a comodule map $\rho\colon \mathsf{k}^n \to \mathsf{k}^n \otimes \mathsf{k}[G]$. Given $v \in \mathsf{k}^n$ we may write $\rho(v) = \sum v_i \otimes f_i$. Then the comodule identity shows that

$$\sum \rho(v_i) \otimes f_i = \sum v_i \otimes f_i \otimes f_i,$$

and hence we see $\rho(v_i) = v_i \otimes b_i$, and each nonzero v_i is a simultaneous eigenvector for $G(\mathsf{k})$. Since $v = (1 \otimes \epsilon)\rho(v) = \sum v_i \epsilon(b_i)$, by picking a vector lying outside the span of the nonzero v_i, we may continue in this way to produce a basis of k^n constisting of eigenvectors for the $G(\mathsf{k})$-action as required. It follows immediately that $G(\mathsf{k})$ consists of semisimple elements. □

Definition 8.8. Let G be an algebraic group over k. Then we say G is a *torus* if $G_{\overline{\mathsf{k}}}$, the base-change of G to the algebraic closure of k, is isomorphic to a product of $\mathbb{G}_m$s, that is, it is a smooth connected diagonalizable group.

Corollary 8.9. *The automorphism group of an n-dimensional torus over an algebraically closed field is $GL_n(\mathbb{Z})$.*

Proof. By the above equivalence, the group $\mathrm{Aut}(\mathbb{G}_m^n)$ is isomorphic to $\mathrm{Aut}(\mathbb{Z}^n)$. □

Lemma 8.10. *Let k be algebraically closed and let T be a torus in a smooth algebraic group G. Then $Z_G(T)$ has finite index in $N_G(T)$, that is, $N_G(T)/T$ is a finite group.*

Proof. Pick an embedding $G \to \mathrm{GL}(V)$. Then V splits into a direct sum of subspaces V_χ given by the characters $\chi\colon T \to \mathbb{G}_m$ of T. Now $N_G(T)$ permutes these subspaces, and since only finitely many characters occur, this gives an action of $N_G(T)$ on a finite set. The kernel of this action is exactly the centralizer of T, since T acts via scalars on each V_χ and so commutes with every endomorphism which preserves the subspaces V_χ. Hence $Z_G(T)$ is finite in $N_G(T)$. □

Remark 8.11. In the previous lemma we used the obvious definitions of normalizer and centralizer for group varieties – taking the closed subgroup of $G(\mathsf{k})$ which centralizes/normalizes T. One can give a functorial definition of centralizers *etc.* which works for nonalgebraically closed fields. See the exercises for an example of this.

Finally, we show that diagonalizable groups have no deformations.

Proposition 8.12. *(Rigidity of tori.) Assume k is algebraically closed. Let G and G' be diagonalizable groups, and suppose V is a connected affine variety. Suppose that $\phi\colon V \times G \to G'$ is a morphism of affine varieties such that for any $v \in V$ the map $\phi(v, \cdot)$ is a homomorphism of groups. Then ϕ is independent of $v \in V$.*

Proof. Suppose that $\chi \in X(G')$. Then since G is diagonalizable we may write

$$\phi^*(\chi)(v, g) = \sum_{\eta \in X(G)} f_{\chi,\eta}(v)\eta(g),$$

then just as we showed that group-like elements are linearly independent, one can show that

$$f_{\chi,\eta_1} f_{\chi,\eta_2} = \delta_{\eta_1,\eta_2} f_{\chi,\eta_1}, \quad (\eta_1, \eta_2 \in X(G), \chi \in X(G')).$$

In particular it follows that the functions $f_{\chi,\eta}$ are idempotent in $\mathsf{k}[V]$. Since V is connected, it follows that the nonzero $f_{\chi,\eta}$ are identically one, and hence $\phi^*(\chi) = \eta$, and ϕ is independent of v as claimed. □

Corollary 8.13. *Suppose that a connected algebraic group G acts on a torus T via automorphisms. Then the action is trivial.*

Proof. Simply take $G = V$ in the above, and then since the identity must act trivially, the entire group G will. □

Remark 8.14. The proof of the rigidity theorem clearly did not use the algebraic closure of k (I assumed it only to speak of V as an affine variety). Thus the same argument will show that there are no non-trivial action of a connected algebraic group on a torus even if k is not algebraically closed.

9. SOLVABLE GROUPS

Definition 9.1. The *derived subgroup* of an algebraic group G, $\mathcal{D}(G)$, is the intersection of the closed normal algebraic subgroups N of G such that G/N is commutative.

Thus $\mathcal{D}(G)$ is the smallest closed subgroup of G such that $G/\mathcal{D}(G)$ is commutative. We now give a more explicit description of $\mathcal{D}(G)$. There are obvious maps of k functors:

$$\psi_n : G^{2n} \to G, \quad (g_1, h_1, g_2, h_2, \ldots, g_n, h_n) \mapsto \prod_{i=1}^{n} [g_i, h_i]$$

where $[g_i, h_i]$ is the commutator $g_i h_i g_i^{-1} h_i^{-1}$.

This gives a corresponding map on coordinate algebras $\psi_n^* : \mathsf{k}[G] \to \mathsf{k}[G]^{\otimes 2n}$, and moreover since the identity is a commutator, it is clear that if $I_n = \ker(\psi_n^*)$, then $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$. We set $I = \bigcap_{n \geq 1} I_n$, an ideal of $\mathsf{k}[G]$. Now the product of two elements of the image of ψ_n lies in the image of ψ_{2n} we see that if $f \in I_{2n}$ the $\Delta(f)$ is zero in $\mathsf{k}[G]/I_n \otimes \mathsf{k}[G]/I_n$. Hence we see that $\mathsf{k}[G]/I$ inherits a coproduct from $\mathsf{k}[G]$, and so I is the defining ideal of a closed subgroup of G. This subgroup is exactly $\mathcal{D}(G)$.

Notice that the same arguments show that given a pair of subgroups H_1, H_2 of an algebraic group G we may construct a closed subgroup $[H_1, H_2]$ "generated" by the commutators in the groups H_1, H_2.

Lemma 9.2. *If G is a smooth connected algebraic group over an algebraically closed field, then so is $\mathcal{D}(G)$.*

Proof. G is connected if and only if it is irreducible if and only if $\mathsf{k}[G]$ has no idempotents other than 0 and 1. Since $\mathsf{k}[G]/I_n \to \mathsf{k}[G^{2n}]$ is an injection, and G^{2n} is smooth and connected, it follows that $\mathsf{k}[G]/I_n$ has no idempotents except 0 and 1. It follows the same is true of $\mathsf{k}[G]/I$. It follows $\mathcal{D}(G)$ is connected. The same idea shows that $\mathcal{D}(G)$ is smooth. □

Definition 9.3. Let G be an algebraic group. Then let $\mathcal{D}^0(G) = G$, and for $n \geq 1$ let $\mathcal{D}^n(G) = \mathcal{D}(\mathcal{D}^{n-1}(G)$. The sequence of subgroups $\{\mathcal{D}^n(G)\}$ is *derived series* of G. Similarly, we set $G^{(0)} = G$, and $G^{(n)} = [G, G^{(n-1)}]$ for $n \geq 1$. The sequence of subgroups $G^{(n)}$ is called the *lower central series* of G. An algebraic group G is said to be *solvable* if $\mathcal{D}^n(G)$ is the trivial group for some n, and *nilpotent* if $G^{(n)}$ is trivial for some n. Clearly a nilpotent group is solvable, but the converse is false.

Remark 9.4. Note that we have defined our derived subgroup in the category of algebraic groups, not abstract groups. It follows from the construction that if k is algebraically closed, then $\mathcal{D}(G)(\mathsf{k})$ is the derived group of the abstract group $G(\mathsf{k})$. However, over a general field this might not necessarily be the case. For example, if $\mathsf{k} = \mathbb{F}_2$, then group $\mathrm{SL}_2(\mathbb{F}_2) \cong \mathrm{Sym}_3$ (*prove this*), the symmetric group on 3 letters, hence it has, as an abstract group, a derived subgroup of order 3. On the other hand, SL_2 is equal to its own derived group as an algebraic group.

10. Structure theory of solvable groups over algebraically closed fields

Assume in this section that k is algebraically closed, and that our groups are smooth, so that we may work with group varieties, and algebraic varieties.

Definition 10.1. A variety Z is *complete* if for every variety Y the projection map $\pi \colon Y \times Z \to Y$ is a closed map, *i.e.* takes closed sets to closed sets.

Remark 10.2. One can show that in the category of Hausdorff topological spaces, the above definition characterizes compact spaces.

Example 10.3. Consider the affine line $\mathbb{A}^1$. Then it is easy to see that it is *not* complete: let $Y = \mathbb{A}^1$ also, so that the product is $\mathbb{A}^2$. But it is easy to see that the projection $\mathbb{A}^2 \to \mathbb{A}^1$ is not a closed map: consider the closed set $Y = \{(x, y) \in \mathbb{A}^2 : xy = 1\}$: its image is obviously $\mathbb{A}^1 - \{0\}$, which is certainly not closed.

We list a few simple properties of complete varieties:

Proposition 10.4. *Let Z be a complete variety.*

1. *Any closed subvariety Y of Z is complete.*
2. *If Z is irreducible then $\mathsf{k}[Z] = \mathcal{O}(Z) = \mathsf{k}$.*
3. *If Z is affine then it is finite.*

Proof. The first property is obvious, as the intersection of closed sets is closed. If Z is affine, then take a regular function $g \in \mathsf{k}[Z]$, and consider the subset $\Gamma = \{(x, y)) \in Z \times \mathbb{A}^1 : g(x)y = 1\}$. Clearly this is a closed set subset of $Z \times \mathbb{A}^1$, hence its image under the second projection in $\mathbb{A}^1$ is closed. Thus it is a finite set, whose elements are the reciprocals of the nonzero values of g. Hence g takes finitely many values. Since Z is irreducible, it follows that g must be constant as required. The third property an obvious consequence of second. $\square$

Finally, there is a plentiful supply of complete varieties, as the next proposition shows.

Proposition 10.5. *Any projective variety is complete.*

Proof. By the previous proposition it is enough to show that each projective space is complete. This is done by showing the image of closed subset Y in $Z \times \mathbb{P}^n$ under the projection to Z is an intersection of "determinantal" subvarieties. $\square$

Recall in particular that we have the varieties $\mathrm{Gr}_d(V)$ the Grassmannian of d-planes in a vector space V, and the flag variety $\mathcal{F}(V)$ are projective varieties.

Theorem 10.6. *Let G be a smooth solvable algebraic group acting on a variety X. Then any G-orbit has no complete subvariety of positive dimension.*

Proof. We use induction on the dimension of G (the 0-dimensional case is trivial). We may assume that G is connected, as a G-orbit is a finite disjoint union of G^0-orbits. Let Y be a G-orbit in X, and let G_x be the stabilizer of a point $x \in Y$. The natural map $G/G_x \to Y$ is generically finite since both varieties have the same dimension, and hence by homogeneity it must be finite[16]. It follows that if Y contained a complete subvariety, then so would G/G_x, so that it is enough to consider the varieties G/G_x.

If G_x surjects onto $G/\mathcal{D}(G)$ then it follows that $\mathcal{D}(G)$ acts transitively on G/G_x, and so we may use induction to conclude that G/G_x contains no positive dimensional complete subvarieties. On the other hand, if G_x does not surject onto $G/\mathcal{D}(G)$ it follows G_x and $\mathcal{D}(G)$ generate a proper closed subgroup H of G, which is necessarily normal. But then G/G_x maps onto G/H with fibers H/G_x. Since G/H is affine, any complete subvariety is finite, and hence any complete subvariety of G/G_x lies in a finite union of copies of H/G_x. Hence we need only show H/G_x has no positive dimensional complete subvarieties. But since H is a proper closed subgroup, we must have $\dim(H) < \dim(G)$, and the result follows again by induction. □

Remark 10.7. I learnt the formulation of this theorem from a paper of Allcock [A].

One useful consequence of this is the following fixed point theorem.

Theorem 10.8. *(Borel fixed point theorem). If G is a connected solvable group variety acting on a complete variety X, then some point of X is fixed by G.*

Proof. A closed orbit Y for the action of G will be projective, and such orbits are guaranteed to exist (simply take one of minimal dimension. From the previous Theorem, it follows that Y must be zero-dimensional, and hence, since B is connected, it must be a point. □

Let us consider a more concrete corollary of this theorem.

Theorem 10.9. *(Lie-Kolchin): Let G be a solvable algebraic group, and let V be a representation of G. Then V has a basis with respect to which the algebraic group B is upper triangular.*

Proof. There is an induced action of G on $\mathcal{F}(V)$, and since $\mathcal{F}(V)$ is projective, and hence complete, it follows that G fixes a complete flag in V. Since G will be upper triangular with respect to any basis of V adapted to this flag we are done. □

Corollary 10.10. *Let G be a connected solvable algebraic group. Then the set G_u of unipotent elements of G is a closed nilpotent normal subgroup, and the quotient G/G_u is a torus.*

[16] Here we are using the fact that a *quasi-finite* morphism, *i.e.* one with finite fibers, is actually finite over an open subset of the target.

Proof. By the previous theorem, we may assume that G is a closed subgroup of the group U_n of upper-triangular matrices in GL_n. But we know that $g \in G$ is unipotent if and only if it is unipotent as an element of GL_n, and clearly this is true if and only if g has diagonal entries equal to one. The map from U_n to $\mathbb{G}_m^n$ given by taking the main diagonal entries of U_n is a homomorphsim, which restricts to a homomorphism $G \to \mathbb{G}_m^n$. It's kernel is precisely G_u, so this is a closed normal subgroup as required. Since the strictly upper triangular matrices are nilpotent, it follows G_u is also. Note since the codomain of homomorphism is abelian, G_u contains $\mathcal{D}(G)$. □

Theorem 10.11. *Let G be an connected solvable group. Then the subgroup G_u is also connected. If G is abelian, then the set of semisimple elements G_s is also a closed subgroup, and $G \cong G_s \times G_u$.*

Proof. Again embed G into $\mathrm{GL}(V)$ for some V and pick a basis making G upper triangular. If G is abelian, we may alter this basis so as to ensure that it simultaneously diagonalizes G_s, whence it is clear that G_s is a closed subgroup of G. By the uniqueness of the Jordan decomposition, the map $G_s \times G_u \to G$ given by multiplication is a bijection. To see that it is an isomorphism, it is enough to know that it is an isomorphism on the tangent spaces, and this can be check on Lie algebras, and it is clear that the images of the tangent spaces to the identity in G_s and G_u meet only in $\{0\}$. It follows immediately that G is connected if and only if G_s and G_u are.

How suppose that G is not necessarily abelian. Then since G_u is normal, we may quotient by G_u^0, and hence we may assume that G_u is finite. We claim that in this case G must be abelian. Since G_u is normal, and G/G_u is abelian, G-conjugation preserves the G_u cosets. If G_u is finite then G-conjugation must in fact be trivial each G_u-coset, and so G is abelian as required. □

To establish the structure of solvable group varieties, it is helpful to know the following basic theorem.

Theorem 10.12. *Let G be a connected one-dimensional group variety. Then G is isomorphic to either the additive group $\mathbb{G}_a$ or multiplicative group $\mathbb{G}_m$.*

Proof. (*Sketch.*) One way to prove this, if you know a little about curves, is to consider the curve C which is the smooth completion of G. First one shows that the action of G on itself extends to an action of G on C (which follows from the fact that the singularities of a birational map $X \to Y$ where X is smooth and Y is projective, are at most of codimension 2). From this it is easy to produce a vector field v on C vanishing only at $C \backslash G$. The genus of C is $g = \dim H^0(C, \Omega_C)$, the dimension of the space of 1-forms on X, but if $\omega \in H^0(C, \Omega_C)$ is nonzero, the pairing $\langle v, \omega \rangle$ is nonzero regular function on C. Since C is complete, $\mathsf{k}[C] = \mathsf{k}$, and hence $\langle v, \omega \rangle$ is a nonzero constant. But then ω vanishes nowhere, and hence trivializes Ω_X, so that $g = 1$, and moreover v vanishes nowhere so that $G = C$, and G is not affine. Hence we must have $H^0(C, \Omega_C) = 0$ and $g = 0$, that is, $C \cong \mathbb{P}^1$. The result then follows quickly, as then $G = \mathbb{P}^1 \backslash F$ where F is a finite set, and $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$. For details see, for example, [K, 10.7]. □

Definition 10.13. If G is a solvable group variety, any complement to G_u is called a *maximal torus*. They are maximal tori for G under containment.

Theorem 10.14. *(Structure theorem): Let G be a smooth connected solvable group variety. Then*

1. *The unipotent elements form a closed connected nilpotent normal subgroup G_u;*
2. *The group G/G_u is a torus.*
3. *The extension $1 \to G_u \to G \to G/G_u \to 1$ splits, that is, there are maximal tori in G. Moreover any two such splittings are conjugate.*
4. *Every semisimple element s lies in such a torus, and $Z_G(s)$ is connected.*
5. *There is a sequence of subgroups such that $1 = N_0 \subseteq N_1 \subseteq \dots N_d = G$ which are normal in G and such that N_i/N_{i-1} is isomorphic to the additive group $\mathbb{G}_a$.*

Proof. The first statement has already been established, except for the nilpotence, but this is a consequence of the fact that the group of strictly upper-triangular matrices N_n is nilpotent. For the second part, note that G/G_u is a connected subgroup of the diagonal matrices, and hence a torus.

To show the last part, one builds an explicit such chain of subgroups for the group N_n, and then takes their intersection with G_u – these are normal in G_u, as are their identity components. One may then take an appropriate subchain to finish the proof.

Finally, for the third and fourth part, assume to begin with that G_u is one-dimensional. By Theorem 10.12 we must have $G_u \cong \mathbb{G}_m$ or $G_u \cong \mathbb{G}_a$, but since the elements of G_u are unipotent it follows that $G_u \cong \mathbb{G}_a$. If G_u is central in G, then for any $g \in G$ we see that the semisimple part g_s of g lies in gG_u, and moreover it is the unique semisimple element in the coset gG_u, and so it is preserved by all conjugations in G. It follows that g_s is central, and hence g is central – that is, G is abelian, and we are done by Theorem 10.11.

Now suppose that G_u is not central. Then there is a semisimple element $s \in G$ which acts nontrivially on it. But the automorphisms of $\mathbb{G}_a$ are just $\mathbb{G}_m$, and hence it acts with nonzero derivative at the identity. Thus the map $Z_G(s) \to G/G_u$ is injective and moreover its derivative at the identity is an isomorphism. To see this, note that the kernel of the map $\mathrm{Lie}(G) \to \mathrm{Lie}(G/G_u)$ is exactly $\mathrm{Lie}(G_u)$, on which $Ad(s)$ acts by a nontrivial scalar, but on the other hand, we have

$$\mathrm{Lie}(Z_G(s)) \subseteq \{x \in \mathrm{Lie}(G) : Ad(s)(x) = x\},$$

hence $\mathrm{Lie}(Z_G(s))$ intersects $\mathrm{Lie}(G_u)$ trivially. It follows that $Z_G(s)$ projects isomorphically onto its image, and hence $Z_G(s)$ is connected, and a maximal torus for G.

Now the conjugacy class of g is

$$C(g) = \{xgx^{-1} : x \in G\} = \{g[g^{-1}, x] : x \in G\} \subseteq g\mathcal{D}(G) \subseteq gG_u.$$

Thus since $C(g)$ is not trivial, it must be equal to gG_u, and hence $\dim(Z_G(g)) = \dim(G/G_u)$, and so $Z_G(g)$ surjects onto G/G_u, and $Z_G(g)$ is a torus in G splitting the map $G \to G/G_u$. Now if T is any complementary torus, it contains a unique element of gG_u. Since $gG_u = C(g)$, we may conjugate g to this element, and then we must have $T = Z_G(g)$, hence T is unique up to conjugation. The same argument also shows that any semisimple element is contained in a maximal torus, from which it is clear that, if s is any semisimple element, then $Z_G(s)$ is either a maximal torus or all of G, and in each case $Z_G(s)$ is connected.

We now remove the assumption that $\dim(G_u) = 1$. We do this by induction: suppose that $\dim(G_u) > 1$. Then by part (4), we may find a subgroup $N \cong \mathbb{G}_a \subset$

G_u normal is G. By induction there is a complementary torus T' in G/N and any two such are conjugate in G/N. But then letting H be the preimage of T in G we have an exact sequence

$$1 \to N \to H \to T \to 1.$$

Applying the one-dimensional case, we see that this sequence splits uniquely up to conjugacy, hence there are complements to G_u in G, again unique up to conjugacy. An entirely similar argument shows that any semisimple element s lies in a maximal torus. To see that the groups $Z_G(s)$ are connected note that by induction we know $Z_{G/N}(\bar{s})$ is connected in G/N, where $\bar{s} = sN$. Let $G_1 = \{g \in G : sgs^{-1}g^{-1} \in N\}$, so that since N and $Z_{G/N}(\bar{s})$ are connected, G_1 is connected. Moreover, G_1 contains $Z_G(s)$ and N. If $G_1 \neq G$, then we are done by induction. If $G_1 = G$, then one checks, as for the case $\dim(G_u) = 1$, that the map $Z_G(s) \times N \to G$ is an isomorphism, and the connectedness of $Z_G(s)$ follows. □

Remark 10.15. A group variety G which consists entirely of unipotent elements is called a *unipotent group*. One can show that any such group variety is solvable and hence by the above structure theorem a unipotent group is nilpotent. In general we say an algebraic group is unipotent if every nonzero representation of G has a nonzero fixed vector (it is a theorem that for group varieties this is equivalent to the condition that every element is unipotent).

Proposition 10.16. *Let G be a connected solvable group variety and let H be a commutative subgroup consisting of semisimple elements. Then H lies in a maximal torus T of G, and all such maximal tori are conjugate in G by elements of $Z_G(H)$.*

Proof. If every element of H is central, then since central semisimple elements lie in every maximal torus we are done. Thus we may assume that there is an $h \in H$ which is not central. But then $Z_G(h)$ is a proper subgroup of G, and $H < Z_G(h)$, so we may use induction on $\dim(G)$. □

11. Classes of algebraic groups

We now want to attempt a rough classification of affine algebraic groups. We will assume that our field k is algebraically closed, though of course if it is not, one still gets some information by trying to classify first the groups $G_{\bar{\mathsf{k}}}$ obtained by base-changing to the algebraic closure of k. Similarly, we assume that G is smooth, since if $\mathfrak{N}$ is the radical of $\mathsf{k}[G]$, then $\mathsf{k}[G]/\mathfrak{N}$ yields a smooth algebraic group $\bar{G}$, which G maps to.

Definition 11.1. Given an smooth affine algebraic group G, we define the *radical* $R(G)$ of G to be the union of the smooth closed connected normal solvable subgroups of G. To see this exists, one must check that if H and N are normal smooth closed connected solvable subgroups, then HN is also, but this is a consequence of the isomorphism[17]

$$HN/N \cong H/H \cap N.$$

We set $R_u(G) = R(G)_u$, the *unipotent radical* of G. It may be characterized as the maximal normal closed connected unipotent subgroup of G.

[17]This is true in our category of algebraic groups, it isn't necessarily true in the world of group varieties – see Milne's notes on algebraic groups for a rant on this topic.

Clearly $R(G)$ is a solvable group, with unipotent radical $R_u(G)$ a nilpotent algebraic group. Thus we have a hierarchy:

$$
\begin{array}{ccc}
 & G & \\
\text{semisimple:} G/R(G) & | & \\
 & \mathrm{R}(G) & G/R_u(G)\text{: reductive} \\
\text{torus: } R(G)/R_u(G) & | & \\
 & R_u(G) & \text{unipotent} \\
 & | & \\
 & \{e\} &
\end{array}
$$

Definition 11.2. Let G be an affine algebraic group. Then we say G is *semisimple* if $R(G) = \{e\}$, and *reductive* if $R_u(G) = \{e\}$.

Lemma 11.3. *Let G be a smooth connected algebraic group over a perfect field* k*. Then $R(G) = \{e\}$ if and only if G has no nontrivial smooth connected commutative normal subgroup. Similarly $R_u(G) = \{e\}$ if and only if the only nontrivial smooth connected normal subgroups of G are tori.*

Proof. Clearly $R(G) = \{e\}$ implies that G has no nontrivial smooth connected commutative normal subgroups. To see the converse, one notes that $R(G)$ and $\mathcal{D}(G)$ are characteristic subgroups – that is, they are preserved by any automorphism of G. Considering the derived series of $R(G)$ we see that it's final nontrivial term will be an connected commutative normal subgroup of G. The structure theory of solvable groups immediately implies the claim about reductivity. □

Remark 11.4. Note that while all the adjectives "smooth connected commutative normal" sound cumbersome, none of them are redundant! Indeed simple considerations with SL_2 give the necessity of each.

Since we understand solvable groups reasonably well, the next natural class of algebraic groups to consider is the reductive and semisimple ones – that is, those with $R_u(G) = \{e\}$. At first it might seem that semisimple groups are the more natural objects, but it turns out reductive groups are a slightly better behaved class of algebraic groups (and many natural examples, such as GL_n, are reductive while not being semisimple)[18]. We now state some basic structural results.

Let G be an algebraic group. We say that G is *almost simple* if it is nonabelian and has no nontrivial closed normal subgroups of positive dimension. Thus, for example, SL_n is almost simple. We say that G is an *almost direct product* of subgroups $\{G_i : 1 \leq i \leq n\}$ if the natural map

$$G_1 \times \ldots \times G_n \to G,$$

is a quotient with finite kernel. Thus the subgroups G_i commute with each other and each G_i is normal.

Example 11.5. Assume char(k) $\neq 2$. Consider the form on k^4 given by $q(x) = x_0x_1 + x_2x_3$. Let

$$G(R) = \mathrm{SO}_4(R) = \{g \in \mathrm{GL}_4(R) : q(x) = q(g(x)), \det(g) = 1\}.$$

[18]In characteristic zero it is also the case that a connected algebraic group is reductive if and only if its category of representations is semisimple, that is, every representation is a direct sum of irreducible representations. In positive characteristic however, nothing like this is true.

Clearly $G(\mathsf{k})$ acts by automorphisms on $Q = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 : q(x) = 0\}$, a quadric surface in $\mathbb{P}^3$, and indeed is the subgroup of $\mathrm{SL}_4(\mathsf{k})$ which acts on Q. It is easy to see that this surface is just the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, hence we also have an action of $\mathrm{SL}_2(\mathsf{k}) \times \mathrm{SL}_2(\mathsf{k})$ on Q (since $\mathrm{SL}_2(\mathsf{k})$ obviously acts on $\mathbb{P}^1$). is the special orthogonal group on k^4. Thus we have a map $\mathrm{SL}_2(\mathsf{k}) \times \mathrm{SL}_2(\mathsf{k}) \to \mathrm{SO}_4(\mathsf{k})$. (In linear algebra terms we are just using the fact that $\mathsf{k}^2 \otimes \mathsf{k}^2 \cong \mathsf{k}^4$, and then noticing that the image of $\mathrm{SL}_2 \times \mathrm{SL}_2$ must preserve the cone of decomposable tensors.) It can be shown that this map realizes SO_4 as an almost direct product $\mathrm{SL}_2 \times \mathrm{SL}_2$, where the kernel is a central subgroup of order 2.

Proposition 11.6. *Every semisimple group G is an almost direct product of its minimal connected normal algebraic subgroups of positive dimension. In particular, there are only finitely many of these, and every connected normal subgroup is a product of a subset of these, and is centralized by the others.*

Proof. We sketch a proof assuming $\mathrm{char}(\mathsf{k}) = 0$: with this assumption the result is a consequence of Lie algebra structure theory. The corresponding result for Lie algebras is a relatively simple result – $\mathrm{Lie}(G)$ is the direct sum of its minimal Lie ideals $\{\mathfrak{g}_i : 1 \leq i \leq n\}$ say (an ideal in a Lie algebra $\mathfrak{g}$ is a sub-Lie-algebra $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$). The minimal connected normal subgroups G_i are then given as the identity components of the centralizers $G_i = Z_G(\bigoplus_{j \neq i} \mathfrak{g}_j)^0$ of these minimal ideals (recall that $\mathrm{Lie}(G)$ is a G representation via the adjoint action). The details involve establishing the precise connection between subgroups and subalgebras of $\mathrm{Lie}(G)$. These are closer in characteristic zero than in characteristic $p > 0$ because *all* algebraic groups in characteristic zero are smooth. □

Remark 11.7. The statement of the proposition remains true in positive characteristic, but it is less straightforward. Indeed it is a remarkable fact that the classification of (almost) simple algebraic groups over an algebraically closed field is independent of characteristic. The above proof fails in positive characteristic because the correspondence between closed subgroups and Lie subalgebras fails to be a bijection in general. For an example of the sort of thing that can go wrong, take $G = \mathrm{SL}_2$ and $B = T_2 \cap G$ in characteristic 2. Then you can check that $N_G(B) = B$. On the other hand, the Lie algebra normalizer of $\mathrm{Lie}(B)$ is

$$\{x \in \mathrm{Lie}(G) : [x, b] \in \mathrm{Lie}(B), \forall b \in \mathrm{Lie}(B)\}$$

which in this case is all of $\mathrm{Lie}(G)$. Thus $\mathrm{Lie}(B)$ is an ideal in $\mathrm{Lie}(G)$, even though B is not a normal subgroup of G.

Proposition 11.8. *Let G be a connected reductive algebraic group. Then we have*

$$R(G) = Z(G)^0, \quad G = R(G) \cdot \mathcal{D}(G),$$

and the subgroup $\mathcal{D}(G)$ is semisimple.

Proof. Since G is reductive, $R(G)$ must be a torus, and by definition $Z(G)^0$ is contained in $R(G)$. Now G acts by conjugation on $R(G)$, but it is known that a connected group acting on a torus via automorphisms must act trivially (this is known as the "rigidity of tori" – it essentially follows from the fact that the torsion points are dense in the torus, and a connected group must act trivially on this set). It follows that $R(G) \subset Z(G)$, so that $R(G) = Z(G)^0$.

For the second part, we begin by showing that $Z(G)^0 \cap \mathcal{D}(G)$ is finite. To see this, embed G into $\mathrm{GL}(V)$ and decompose V according to the eigenspaces of $Z(G)^0$:

$$V = \bigoplus_{\chi} V_\chi, \quad (\chi\colon Z(G)^0 \to \mathbb{G}_m),$$

(with $V_\chi = \{0\}$ for all but finitely many χ). Now the centralizer of $Z(G)^0$ in $\mathrm{GL}(V)$ consists of matrices which preserve the subspaces V_χ. Since $\mathcal{D}(G)$ consists of commutators, its image in $\mathrm{GL}(V_\chi)$ will lie in $\mathrm{SL}(V_\chi)$. Since the intersection of $\mathrm{SL}(V_\chi)$ with the scalar matrices is a finite group, the intersection $Z(G)^0 \cap \mathcal{D}(G)$ must be finite.

The subgroup $R(G) \cdot \mathcal{D}(G)$ is a normal in G, and the quotient is commutative and semisimple (since it is both a quotient of $G/R(G)$ and $G/\mathcal{D}(G)$), hence by the previous proposition we must have $G = R(G) \cdot \mathcal{D}(G)$. Finally the map $\mathcal{D}(G) \to G/R(G)$ is surjective with finite kernel, hence $\mathcal{D}(G)$ is semisimple. □

Thus reductive groups are very close to semisimple group.

Example 11.9. Let $G = \mathrm{GL}(V)$. Then the centre $Z(G)$ of G is $\mathbb{G}_m$ embedded as the diagonal matrices (it is thus connected). The derived group $\mathcal{D}(G)$ is $\mathrm{SL}(V)$, which intersects $Z(G)$ in the group μ_n of n-th roots of unity, which is finite. Clearly the map

$$\mathbb{G}_m \times \mathrm{SL}(V) \to \mathrm{GL}(V),$$

is surjective, and $\mathrm{SL}(V)$ is semisimple (and in fact almost simple).

12. Borel Subgroups

Definition 12.1. A *Borel* subgroup of G is a connected solvable subgroup of G which is maximal in the partial order on closed subgroups given by containment.

Proposition 12.2. *Let G be an algebraic group and B a Borel subgroup. The variety G/B is projective. Moreover, all Borel subgroups are conjugate in G.*

Proof. Let B_0 be a Borel subgroup of maximal dimension. As usual, pick a representation V of G for which there is a line L with stabilizer exactly B, and Lie algebra stabilizer $\mathrm{Lie}(B_0)$. Applying the Lie-Kolchin theorem to V/L we see that we may find a flag F of V whose line F_1 is L which is preserved by B. Let Y be the G-orbit of F in $\mathcal{F}(V)$. We claim this is isomorphic to G/B_0, and that it is closed. To see the latter we claim that it is of minimal dimensional among the G-orbits on $\mathcal{F}(V)$. Indeed note that if H stabilizes a flag $F' \in \mathcal{F}(V)$, then it is solvable since it's image in $\mathrm{GL}(V)$ is solvable as it is conjugate to a subgroup of the upper triangular matrices, and the kernel of the map $\rho\colon G \to \mathrm{GL}(V)$ is also solvable, since it lies in B_0. But then H^0 is solvable, and hence of dimension at most $\dim(B_0)$. It follows that $\dim(G \cdot F') = \dim(G) - \dim(H) > \dim(G) - \dim(B_0) = \dim(Y)$. Thus Y is closed as required.

To see that $Y \cong G/B$ note that $G/B = G \cdot L = X$ by assumption and there is a canonical map $\pi\colon X \to Y$ since Y is a homogeneous space for G with stabilizer B. On the other hand we have an obvious map $Y \to X$, projection a flag to its line, which clearly is the inverse to π.

To see the moreover part, suppose that B is any other Borel subgroup. Then B acts on the projective variety G/B_0, and so by the fixed point theorem there is a point gB_0 fixed by B. But then $g^{-1}Bg \subset B_0$, and since B is a Borel subgroup, this implies $B = g^{-1}B_0 g$. □

Exercise 12.3. Suppose that H is a maximal solvable subgroup of G. Does H have to lie in a Borel subgroup? (It is always true that $B = N_G(B)$, though we will not prove it, so there is no solvable disconnected group which contains a Borel subgroup. That however, is *not* an answer to the question!)

Definition 12.4. Let G be a smooth connected algebraic group. A *maximal torus* in G is a torus in G maximal in the partial order given by containment.

Corollary 12.5. *Let G be a smooth connected algebraic group over an algebraically closed field. Then the maximal tori in G are all conjugate in G. Moreover the set*

$$\{(T, B) : B \text{ a Borel subgroup}, T \text{ a maximal torus in } B\},$$

is acted on transitively by G.

Proof. Since a torus is abelian, and hence solvable, any such torus T will lie in a Borel subgroup B. But then since T is maximal in G it must be maximal in B, and then the result follows from the previous Proposition and the conjugacy of maximal tori in solvable groups. The "moreover" follows similarly. □

Remark 12.6. Tori in groups over non-algebraically closed fields do not have to be conjugate, though the previous result shows that they will become so once we base-change to $\overline{\mathsf{k}}$. Hence the classification of conjugacy classes of tori in G is an essentially "arithmetic" question.

Lemma 12.7. *Let G be a connected group variety, and let B be a Borel subgroup of G.*

(1) *If G is nontrivial, so is B.*
(2) *B is of finite index in $N_G(B)$, and $N_G(B)$ is self-normalizing.*

Proof. If B is trivial, then $G \cong G/B$ is both affine and projective, hence G is trivial. For the second part, note that B is clearly a Borel subgroup of $N_G(B)$. If B' is a Borel subgroup of $N_G(B)/B$, then the preimage of B' in $N_G(B)$ is a Borel subgroup of $N_G(B)$ containing B, and hence equals B. Thus B' is trivial, and it follows $N_G(B)/B$ is finite. To see that $N_G(B)$ is self-normalizing, note that if g normalizes $N_G(B)$, it normalizes $N_G(B)^0 = B$, and hence $g \in N_G(B)$. □

Borel subgroups allow us to study algebraic groups via solvable groups. The next results give examples of this.

Theorem 12.8. *Let G be a group variety and B a Borel subgroup.Then the centre of B is central in G and moreover we have*

$$Z(B) = Z(G)^0.$$

Proof. Suppose that $z \in B$ is central in B, and consider the map $c\colon G \to G$ given by $g \mapsto gzg^{-1}$. Since this map is constant on B-cosets, it descends to a map $G/B \to G$, which must be constant, since G is affine while G/B is complete. It follows that c is constant, and hence z is central in G as required. Since $Z(G)^0$ lies is abelian and hence solvable, it lies in a Borel subgroup of G. But then since Borel subgroups are all conjugate, and $Z(G^0)$ is normal, it in fact lies in all Borel subgroups. It follows immediately that $Z(G)^0 = Z(B)$ as required. □

Lemma 12.9. *Let G be a connected group variety with B a Borel subgroup. If B is nilpotent then so is G.*

Proof. Use induction on $\dim(G)$. If $\dim(G) = 0$ then we are done trivially. Otherwise, $\dim(G) > 0$ and by the above theorem, we know $\dim(B) > 0$. If B is nilpotent, the last nontrivial term in the lower central series for B, N say, will be a connected and nontrivial central subgroup of B, and hence by the above theorem, of G. But then by induction G/N is nilpotent, and so G is a central extension of a nilpotent group, and hence also nilpotent. □

13. REDUCTIVE GROUPS

In this final section we want to describe the combinatorics that classify reductive group over algebraically closed fields, and show how it arises naturally from *enhanced flag variety* $\{(B,T) : B \text{ a Borel}, T \text{a maximal torus}, T \subset B\}$ of G.

Definition 13.1. Let T be a torus. The characters of T form a free abelian group $X(T)$. The *cocharacter* group of T is

$$Y(T) = \mathrm{Hom}(\mathbb{G}_m, T),$$

the group homomorphisms from $\mathbb{G}_m$ to T. The set $Y(T)$ is a group since T is abelian. Composition gives a natural pairing

$$\langle \cdot, \cdot \rangle \colon Y(T) \times X(T) \to \mathbb{Z},$$

where if $\nu \in Y(T)$ and $\chi \in X(T)$, then $\nu^* \circ \chi^*(t) = t^{\langle \nu, \chi \rangle}$ (here t is a coordinate on $\mathbb{G}_m$). It is clear that this pairing is perfect, so that each of the groups $X(T)$ and $Y(T)$ can be identified with dual of the other.

Now suppose that G is a smooth connected reductive algebraic group. We have seen that the set of pairs (B,T) of a Borel subgroup of G and a maximal torus of G lying in B are all conjugate. Fix one such pair. Then if $\mathfrak{g} = \mathrm{Lie}(G)$, the adjoint action makes $\mathfrak{g}$ into a G-representation. Restricting this representation to T, we see that we may write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in X(T) \setminus \{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ is the subspace of $\mathfrak{g}$ on which T acts trivially, and $\mathfrak{g}_\alpha$ is the subspace on which T acts via the character χ. The collection

$$\Phi = \{\alpha \in X(T) : \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\}\}$$

is called the set of *roots* of G.

If $N_G(T)$ is the normalizer of T in G, then by the rigidity of tori we must have $N_G(T)^0 = Z_G(T)^0$, the identity component of the centralizer of T in G. In fact one can show that $Z_G(T)$ is connected (this follows from the fact that G is the union of its Borel subgroups, and the corresponding result for solvable groups which we already established) so that we have a finite group $W = N_G(T)/Z_G(T)$, which acts (faithfully) by automorphisms on T. The finite group $W = W(G,T)$ is called the *Weyl group* of G. Clearly W acts on $X(T)$, and on the set Φ.

These are almost all the data one needs in order to characterize a reductive group. The missing ingredient is the set of *coroots* of G. Let $\alpha \in \Phi$. Then set $T_\alpha = \ker(\alpha)^0$ the identity component of the kernel of α, a torus in T. The group $G_\alpha = Z_G(T_\alpha)$ is connected and one can show its Weyl group $W(G_\alpha, T) < W(G,T)$ has order 2. Let s_α be the nontrivial element. Then s_α acts on $X(T)$, and it can be shown that this action is given by

$$s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$$

(where we write the group operation additively) for a unique $\check{\alpha} \in Y(T)$. The *coroots* of G are the set

$$\Phi^\vee = \{\check{\alpha} \in Y(T) : \alpha \in \Phi\}.$$

Since $s_\alpha^2 = 1$, we have $\langle \alpha, \check{\alpha} \rangle = 2$.

This data can be abstracted as follows:

Definition 13.2. A *abstract root datum* is a quadruple $(X, Y, \Phi, \Phi^\vee)$ such that

(1) X and Y are finitely generated free abelian group, with a perfect pairing
$$\langle \cdot, \cdot \rangle \colon Y \times X \to \mathbb{Z}.$$
(2) The set Φ is a finite subset of $X \backslash \{0\}$ and $\Phi^\vee$ is a finite subset of $Y \backslash \{0\}$, and there is a bijection $\alpha \leftrightarrow \check{\alpha}$ between Φ and $\Phi^\vee$ such that $\langle \check{\alpha}, \alpha \rangle = 2$.
(3) For each $\alpha \in \Phi$ define $s_\alpha \colon X \to X$ by
$$s_\alpha(\chi) = \eta - \langle \check{\alpha}, \chi \rangle \alpha,$$
and $s_{\check{\alpha}} \colon Y \to Y$ by
$$s_{\check{\alpha}}(\eta) = \eta - \langle \eta, \alpha \rangle \check{\alpha}.$$
(4) The automorphisms s_α and $s_{\check{\alpha}}$ preserve Φ and $\Phi^\vee$ respectively.
(5) The datum is *reduced*: if $\alpha \in \Phi$ then $2\alpha \notin \Phi$.

It is a theorem of Chevalley that a smooth reductive group G over an algebraically closed field is classified by its *root datum* $(X(T), Y(T), \Phi, \Phi^\vee)$. Indeed given any abstract root datum there is a reductive algebraic group realizing that root datum, and and isomorphism between reductive groups arises from an isomorphism of root datum (with the obvious definition of an isomorphism of abstract root datum). For a proof of the isomorphism theorem, see [St99] (this proof is simpler than the ones in the existing textbooks). It is then a relatively straightforward task to classify, in a fashion almost identical to the classification of finite reflection groups, the possible abstract root data, and thus obtain an explicit classification of reductive algebraic groups.

Example 13.3. Let G be a reductive algebraic group with root datum $(X, Y, \Phi, \Phi^\vee)$. Then there is a natural isomorphism of the root datum with itself which is the identity on the groups X and Y, but which sends $\alpha \to -\alpha$ and $\check{\alpha} \to -\check{\alpha}$. The isomorphism theorem for reductive groups then shows that this corresponds an automorphism of G fixing T. Moreover all such maps, known as "Chevalley involutions" are conjugate via elements of T. For $G = \mathrm{GL}_n$ such a map is given by $g \mapsto g^{-t}$, the transpose map composed with inversion.

Example 13.4. Let $G = \mathrm{GL}_n$. Then it follows from the Lie-Kolchin theorem that a Borel subgroup is the stabilizer B of a flag in $\mathcal{F}(\mathsf{k}^n)$ – thus we may choose the subgroup of upper triangular matrices U_n. A maximal torus is then clearly given by taking the diagonal matrices, which we will denote by D_n. Let $\epsilon_i \colon D_n \to \mathbb{G}_m$ be the projection to the i-th diagonal entry, so that $X(T) = \bigoplus \mathbb{Z}\epsilon_i$. Similarly, let η_i be the map $\mathbb{G}_m \to D_n$ given by the i-th diagonal entry, $Y(T) = \bigoplus \mathbb{Z}\eta_i$. The pairing is then given by $\langle \eta_i, \epsilon_j \rangle = \delta_{ij}$.

The Lie algebra of GL_n is $\mathrm{Mat}_n(\mathsf{k})$, with the adjoint action given by conjugation of matrices. It is then easy to see that matrices $E_{ij}(\lambda)$ with λ in the (i, j)th entry and 0 everywhere else, give subrepresentions for the action of D_n on $\mathrm{Mat}_n(\mathsf{k})$, with associated character $\epsilon_i - \epsilon_j$. Thus the roots of GL_n are the set

$$\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq n\}.$$

The Weyl group W of G is $N_G(D_n)/Z_G(D_n) = N_G(D_n)/D_n$ is the symmetric group on n letters, since $N_G(D_n)$ is obviously the group of "monomial matrices" – the matrices with exactly one nonzero entry in each row and column. Finally, the coroots $\Phi^\vee$ are given by $\{\eta_i - \eta_j : 1 \leq i, j \leq n\}$, as one can check by calculating the groups G_α explicitly.

Remark 13.5. Notice that the abstract data $(X, Y, \Phi, \Phi^\vee)$ has a natural involution interchanging X and Y, so that any reductive group naturally has another reductive group $G^\vee$ attached to it. This duality turns out to be fundamental in the study of reductive groups. $G^\vee$ is often known as the Langlands dual group of G.

APPENDIX A. THE SHEAF OF REGULAR FUNCTIONS ON AN AFFINE VARIETY

Let X be a topological space. A k-algebra presheaf $\mathcal{F}$ on X is a contravariant functor $\mathcal{F}\colon Top_X^{\mathrm{op}} \to \mathsf{k}$-algebras. Thus to each open set U we attach a k-algebra $\mathcal{F}(U)$, and if $U \subset V$ are open sets, there is a "restriction map" $\rho_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$. We say $\mathcal{F}$ is a sheaf if it is locally determined:

Definition A.1. Let $\mathcal{F}$ be a presheaf on a topological space X. Then $\mathcal{F}$ is a sheaf if whenever U is an open set and $U = \bigcup_{i\in I} U_i$ is an open covering of U, we have

(1) $\forall f \in \mathcal{F}(U)$, if $\rho_{U,U_i}(f) = 0$ for all i, then $f = 0$.
(2) if $f_i \in \mathcal{F}(U_i)$ agree on intersections, that is
$$\rho_{U_i,U_i\cap U_j}(f_i) = \rho_{U_j,U_i\cap U_j}(f_j),$$
then there is a section $f \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(f) = f_i$.

Note the first condition implies that the section f in the second must be unique if it exists. Here If $U \subset V$ we denote the associated restriction map as $\rho_{V,U}\mathcal{F}(V) \to \mathcal{F}(U)$.

Example A.2. (1) If $X = \mathbb{R}^n$, then there is a sheaf $\mathcal{C}^\infty$ of smooth functions on X:
$$\mathcal{C}^\infty(U) = \{f\colon U \to \mathbb{R} : f \text{ infinitely differentiable}\}$$
where the restriction maps are given by restricting functions. Similarly if $X = \mathbb{C}^n$ there is a sheaf of holomorphic functions on X. The crucial point is that to know a function is smooth (or holomorphic) it is enough to know it is smooth (holomorphic) in some neighborhood of every point.
(2) Let M be a smooth manifold. Then for any open set $U \subset M$ we have an algebra of smooth functions $\mathcal{C}_M^\infty(U)$ just as for $\mathbb{R}^n$. Giving the pair consisting of the topological space M and the sheaf C_M^∞ is equivalent to giving the smooth structure on M (indeed you can develop the theory of smooth manifolds this way, rather than via atlases etc. but this is rarely done, because there are so many smooth functions).
(3) If we set $\mathcal{G}(U) = \{f\colon U \to \mathbb{R} : f \text{ is constant}\}$ then $\mathcal{G}$ defines a presheaf on $\mathbb{R}^n$, however it is *not* a sheaf.

We defined an affine variety to be a set X with an algebra of functions $\mathsf{k}[X]$. From this we can make X into a topological space using the Zariski topology. It is then possible to build a sheaf $\mathcal{O}_X$ of "regular functions" on X. Indeed if $U_g = \{x \in X : g(x) \neq 0\}$ is a basic open set then we know U_g also has the structure of an affine variety, with regular functions $\mathsf{k}[X]_g$, that is, ratios f/g^n where $f \in \mathsf{k}[X]$ and $n \geq 0$.

Remark A.3. This gives a natural candidate for the sheaf $\mathcal{O}_X$: given an open set U, let $\mathcal{O}_X(U)$ be the functions $f\colon U \to \mathsf{k}$ for which given any point x in U there is a neighborhood V of x on which $f = g/h$, $(g, h \in \mathsf{k}[X])$, where h is nonzero on V[19]. We however will proceed in a slightly different fashion, though one should keep in mind that this is exactly the sheaf we will obtain.

[19]To be clear: this gives a completely rigorous definition of $\mathcal{O}_X$, it's drawback is that it doesn't tell you much about what $\mathcal{O}_X$ looks like – see below.

We first show that the assignment $\mathcal{O}_X(U_g) = \mathsf{k}[G]_g$ is a "partially defined sheaf" on X, that is, $\mathcal{O}_X$ satisfies the conditions for a sheaf in Definition A.1 when all the open sets are basic.

Lemma A.4. *Let $U = U_g$ be a basic open set, and $U = \bigcup_{i \in I} U_i$ be a covering by basic open sets, $U_i = U_{f_i}$ say. Then the sheaf conditions for $\mathcal{O}(U)$, $\mathcal{O}(U_i)$ are satisfied for the covering $\{U_i\}$.*

Proof. The first condition is clear, simply because our sections are functions, so if a function vanishes on each open set of a cover, it obviously vanishes on the union. The harder part of the lemma is to show that if we have functions $\{g_i \in \mathsf{k}[U_i] : i \in I\}$ which agree on the overlaps, then they come from a function in $\mathsf{k}[X]_g$. Since X is quasi-compact[20], there is a finite subset I_0 of I which still covers U, and we suppose that $I_0 = \{1, 2, \ldots, n\}$.

Write $g_i = h_i/f_i^m$ for $h_i \in \mathsf{k}[X]$ (since there are only finitely many g_is we can pick the same m for all). We are given that $g_{i|U_i \cap U_j} = g_{j|U_i \cap U_j}$, thus it follows that

$$(h_i/f_i^m)(f_i f_j)^k = (h_j/f_j^m)(f_i f_j)^k$$

in $\mathsf{k}[X]$, for large enough k. Fix l large enough so that $l \geq k$ and $(h_i/f_i^m)(f_i f_j)^l = (h_j/f_j^m)(f_i f_j)^l$ for all i, j, and let $\tilde{h}_i = h_i f_i^{l-m}$. Thus we have

$$\tilde{h}_i f_j^l = \tilde{h}_j f_i^l, \forall i, j.$$

and $g_i = \tilde{h}_i/f_i^l$. Since the U_i cover X, it follows that $V(g) = V(f_i : 1 \leq i \leq n)$, and hence by the Nullstellensatz we can write:

$$g^m = \sum_{i=1}^{n} a_i f_i,$$

for some $a_i \in \mathsf{k}[X]$ and m sufficiently large. Now raising this identity to a sufficiently large power we may write

$$g^M = \sum_{i=1}^{n} b_i f_i^l, \quad b_i \in \mathsf{k}[X]$$

Set $h = g^{-M} \sum_{i=1}^{n} b_i \tilde{h}_i \in \mathsf{k}[X]_g$. We claim that $h_{|U_i} = g_i$. To see this note

$$g^M \tilde{h}_i = \sum_{j=1}^{n} b_j \tilde{h}_i f_j^l = \sum_{i=1}^{n} b_j \tilde{h}_j f_i^l = h f_i^l,$$

and hence $h_{|U_i} = g_i$ as required (it readily follows that h also agrees with the g_i for $i \in I \backslash I_0$). $\square$

Now we extend $\mathcal{O}_X$ to a sheaf on X. Given the previous lemma, this is actually an exercise in pure sheaf theory. Consider an abstraction of the situation we have: let X be a topological space with a basis of open sets $\mathcal{U}$[21]. A "$\mathcal{U}$-sheaf" $\mathcal{G}$ on X is an assignment $\mathcal{G}(U)$ for each $U \in \mathcal{U}$ equipped with restriction maps $\rho_{U_2,U_1} : \mathcal{G}(U_2) \to \mathcal{G}(U_1)$ for inclusions $U_1 \subset U_2$ of basic open sets, such that the sheaf property is satisfied for open sets in $\mathcal{U}$ (that is, (1) and (2) of Definition A.1 hold when all

[20] That is, compact but not Hausdorff – this is easy to see, since any ideal is finitely generated (Hilbert's basis theorem).

[21] Recall that a basis of open sets is a collection of open sets $\mathcal{U}$ in X such that any open set V contains an open set U in the collection $\mathcal{U}$ *i.e.* every open set is a union of ones in $\mathcal{U}$.

the open sets U and U_i are in $\mathcal{U}$, where $U_i \cap U_j$ in (2) is replaced by any $V \in \mathcal{U}$ contained in $U_i \cap U_j$[22]). Any sheaf on X clearly yields a $\mathcal{U}$-sheaf just by restricting $\mathcal{F}$ to the basic open sets.

Lemma A.5. *A sheaf $\mathcal{F}$ on X is uniquely determined by its associated $\mathcal{U}$-sheaf, indeed any $\mathcal{U}$-sheaf $\mathcal{G}$ extends uniquely to a sheaf $\tilde{\mathcal{G}}$ on X.*

Proof. Let $\mathcal{G}$ be a $\mathcal{U}$-sheaf, and let U be an open set in X. Then set

$$\begin{aligned}\tilde{\mathcal{G}}(U) &= \varprojlim\{\mathcal{G}(V) : V \in \mathcal{U}, V \subset U\}\\ &= \{(\alpha_V) : V \in \mathcal{U}, V \subset U, \alpha_V \in \mathcal{G}(V),\ \text{if } V \subset W \text{ then } \rho_{W,V}(\alpha_W) = \alpha_V\}.\end{aligned}$$

Evidently $\tilde{\mathcal{G}}(U) = \mathcal{G}(U)$ for $U \in \mathcal{U}$. Also, the restriction maps given on $\mathcal{U}$ induce restriction maps $\mathcal{G}(U_1) \to \mathcal{G}(U_2)$ for arbitrary open sets U_1, U_2. One then checks that the sheaf property follows from the assumptions. Uniqueness also follows easily from the sheaf property. □

By Lemma A.4 we may apply Lemma A.5 to an affine variety with $\mathcal{U}$ the collection $\{U_g : g \in \mathsf{k}[X]\}$, and obtain a sheaf $\mathcal{O}_X$ from the $\mathcal{U}$-sheaf $\{\mathsf{k}[U_g]\}$. It also follows that for any open set U we have

$$\mathcal{O}_X(U) = \{f\colon U \to \mathsf{k} : \forall x \in U, \exists g \in \mathsf{k}[X], x \in U_g \subset U, \text{ and } f_{|U_g} \in \mathsf{k}[X]_f\}. \tag{A.1}$$

In other words $\mathcal{O}_X$ is exactly the sheaf proposed in Remark A.3. We could simply have defined $\mathcal{O}_X$ to be this sheaf, but although this may seem simpler and more direct, it doesn't actually save work – with this definition one then has to check that $\mathcal{O}_X(U_f) = \mathsf{k}[U_f]$.

Remark A.6. It follows from this that an affine variety is a topological space equipped with a sheaf of k-algebras $\mathcal{O}_X$. Such a structure is known as a *ringed space*. Any open subset U of a ringed space X is again a ringed space – the sheaf $\mathcal{O}_U$ is just the restriction of the sheaf $\mathcal{O}_X$ to U. Thus, for example, $\mathbb{A}^2$ is a ringed space, so $\mathbb{A}^2 - \{(0,0)\}$ is a ringed space also. It is, however, *not* isomorphic, as a ringed space, to an affine variety, as we showed before (but now the argument should make sense!)

APPENDIX B. ABSTRACT VARIETIES

Now that we have a given an affine variety the structure of a ringed space, it is not hard to define a variety.

Definition B.1. Let $(X, \mathcal{O})$ be a ringed space. We say X is a prevariety if there is a finite open covering $\{U_i\}_{i\in I}$ such that each $(U_i, \mathcal{O}_{|U_i})$ is isomorphic as a ringed space to an affine variety. X is a variety if it is a prevariety and the image of the map

$$\Delta\colon X \to X \times X, \quad \Delta(x) = (x, x)$$

is closed in $X \times X$.

[22]For the basic open sets U_g we have $U_g \cap U_h = U_{gh}$, so in fact the more complicated version of sheaf axiom (2) is not necessary.

Remark B.2. This last condition requires some care: the product of prevarieties exists in a natural way, but just as for affine varieties, the topology on the product is stronger than the product topology. One should think of this condition as an analogue of the Hausdorff axiom for manifolds: a manifold is a second countable topological space which is locally homeomorphic to $\mathbb{R}^n$, which is also Hausdorff.

Remark B.3. Once you have absorbed this definition, it is not much harder to define a scheme – in the language of functors which I am using, you need to define a notion of "open subfunctor" and then a scheme is a k-functor which is both "locally" and affine k-functor and satisfies a sort of sheaf property. (Showing that an affine k-functor satisfies this property is the analogue of Lemma A.4.) Alternatively in more traditional language, the main task is to construct a sheaf of rings $\mathcal{O}$ on $\mathrm{Spec}(R)$ the set of prime ideals of a k-algebra R.

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