# ON THE CENTER OF THE CYLCOTOMIC HECKE ALGEBRAS OF $G(m, 1, 2)$

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ABSTRACT. We compute the center of the cyclotomic Hecke algebra attached to $G(m, 1, 2)$ and show that if $q \neq 1$ it is equal to the image of the center of the affine Hecke algebra.

## 1. ON THE CENTER OF $\mathcal{K}_2^{\mathbf{v}}$

1.1. **The affine Hecke Algebra.** We consider quotients of the affine Hecke algebra $H_2^{\text{aff}}$ of type $A_2$: this is the algebra over $\mathcal{A} = \mathbb{Z}[q]$ generated by $T, X_1^{\pm 1}, X_2^{\pm 1}$, such that

- there is an injective algebra map $A[X_1^{\pm 1}, X_2^{\pm 1}] \to H_2^{\text{aff}}$.
- $(T - q)(T + 1) = 0$;
- $TX_1T = qX_2$.

Let $S$ be the image of the ring of Laurent polynomials in $H_2^{\text{aff}}$, and let $W$ be the two element group, with $s$ the nontrivial element, acting on $S$ by interchanging $X_1$ and $X_2$. We write the action as $f \mapsto {}^s f$. For convenience of notation we set $Q = q - 1$. The last relation is equivalent to

$$Tf = {}^s fT + Q\frac{f - {}^s f}{1 - X_1X_2^{-1}}, \qquad f \in S. \tag{1.1}$$

It is easy to check from this that the center of $H_2^{\text{aff}}$ is $S^W$ the algebra of symmetric functions in the $X_i^{\pm 1}$.

1.2. **The cyclotomic Hecke algebra.** Now let $A = \mathbb{Z}[q^{\pm 1}, v_1^{\pm 1}, v_2^{\pm 1}, \dots, v_m^{\pm 1}]$ be a Laurent polynomial ring, and extend the scalars of $H_2^{\text{aff}}$ to $A \otimes_{\mathcal{A}} H_2^{\text{aff}}$.

**Definition 1.1.** The cylotomic Hecke algebra $\mathcal{K}_2$ of type $G(m, 1, 2)$ is a quotients of $H_2^{\text{aff}}$: let

$$\begin{aligned} f_v &= (x - v_1)(x - v_2)\dots(x - v_m) \\ &= \sum_{j=0}^{m} (-1)^{m-j} e_{m-j} x^j. \end{aligned} \tag{1.2}$$

where the $e_j$ are the elementary symmetric functions in the $v_i$'s. Let $J_v$ be the two-sided ideal generated by $f_1 = f_v(X_1)$. Then $\mathcal{K}_2 = H_2^{\text{aff}}/J_v$. (Note that our definition coincides with that in [A] up to rescaling, after $q, v_1, v_2, \dots, v_m$ have been inverted except that his "$q$" is a square root of ours).

We say that a polynomial $p$ in $S$ is *$m$-restricted* (or simply *restricted*, when the integer $m$ is understood) if the monomials $X_1^i X_2^j$ occuring with nonzero coefficient in $p$ all satisfy $0 \leq i, j \leq m - 1$. Let $R = R_m$ be the image of $S$ in $\mathcal{K}_2^{\mathbf{v}}$. It is known that this image is isomorphic as an $A$-module to the space of restricted polynomials, and moreover $\mathcal{K}_2^{\mathbf{v}} = R \oplus RT$ as an $A$-module *i.e.* every element of $\mathcal{K}_2^{\mathbf{v}}$ can be written uniquely in the form $f + gT$ where $f$ and $g$ are restricted.

We start with a technical lemma. Let $D$ be the difference operator on $\mathcal{A}[X_1^{\pm 1}, X_2^{\pm 1}]$ given by

$$D(f) = (f - {}^s f)/(1 - X_1 X_2^{-1}),$$

and let ${}^s D$ be the composition of $-D$ with $s$, that is,

$${}^s D(f) = (f - {}^s f)/(1 - X_1^{-1} X_2).$$

**Lemma 1.2.** *Let $R$ be the space of $m$-restricted polynomials in $S$. Then $D$ and ${}^s D$ preserve $R$, and moreover*

$$D(f) = {}^s D(f)$$

*if and only if $f = {}^s f$.*

*Proof.* The proof that $D$ and ${}^s D$ preserve $R$ is a direct calculation: observe that if $f = X_1^i X_2^j$, then we have

$$D(f) = \begin{cases} X_1^i \sum_{k=0}^{j-i-1} X_1^k X_2^{j-k}, & \text{if } j > i; \\ -X_1^j \sum_{k=0}^{i-j-1} X_1^k X_2^{i-k}, & \text{if } j < i. \end{cases}$$

Thus as the highest powers of $X_1$ and $X_2$ occuring in these expressions is $\max\{i, j\}$, it is clear that if $p$ is any $m$-restricted polynomial, so is $D(p)$. Since ${}^s D$ is the composition of $-D$ with $s$, it clearly also preserves restricted polynomials.

Moreover, note that in $D(f)$ where $f$ is the monomial above, $X_1$ never occurs to the power $\max\{i, j\}$, thus for any restricted polynomial $p$, if $D(p) \neq 0$, it has a higher power of $X_2$ occuring than occurs as a power of $X_1$. Thus similarly ${}^s D(p)$, if nonzero, has a higher power of $X_1$ occuring than occurs as a power of $X_2$. It follows that $D(p) = {}^s D(p)$ if and only if $D(p) = 0$, and this occurs only if $p = {}^s p$ as claimed. ☐

**Lemma 1.3.** *Let $z \in \mathcal{K}_2$ and suppose that $z = f + gT$ where $f, g \in R$. Then $z$ commutes with $T$ if and only if $f, g \in R^W$.*

*Proof.* The sufficiency of the condition is clear. To see the necessity, we have

$$\begin{aligned} T(f + gT) &= {}^s f T + Q \frac{f - {}^s f}{1 - X_1 X_2^{-1}} + {}^s g T^2 + Q \frac{g - {}^s g}{1 - X_1 X_2^{-1}} T \\ &= ({}^s f + Q {}^s g + Q \frac{g - {}^s g}{1 - X_1 X_2^{-1}}) T + Q \frac{f - {}^s f}{1 - X_1 X_2^{-1}} + q {}^s g \end{aligned}$$

On the other hand, we have

$$(f + gT)T = (f + Qg)T + qg.$$

Now it is easy to check that all the polynomials involved are reduced, and so it follows from the basis theorem for cyclotomic Hecke algebras that we must have:

$$({}^s f + Q {}^s g + Q \frac{g - {}^s g}{1 - X_1 X_2^{-1}}) = f + Qg,$$

and

$$Q \frac{f - {}^s f}{1 - X_1 X_2^{-1}} + q {}^s g = qg$$

Thus after rearranging the second of these equations becomes:

$$Q\frac{f - {}^s f}{1 - X_1 X_2^{-1}} = q(g - {}^s g).$$

Now note that the right-hand side is an eigenvector for the action of $s$ with eigenvalue $-1$, thus so is the left-hand side, whence we get

$$Q\frac{f - {}^s f}{1 - X_1 X_2^{-1}} = Q\frac{f - {}^s f}{1 - X_2 X_1^{-1}}$$

or equivalently $D(f) = {}^s D(f)$. By the previous Lemma, this is possible only if $f = {}^s f$, since $f \in R$. But then we must also have $g - {}^s g = 0$, and so $f$ and $g$ are symmetric as required. □

Let $\mathcal{Z}$ denote the center of $\mathcal{K}_2$. From the previous lemma, we see that if $z = f + gT \in \mathcal{Z}$, then $f, g \in R^W$. Since $f \in R^W$ is already central, we see that $\mathcal{Z} = \mathcal{Z} \cap R \oplus \mathcal{Z} \cap RT$, and we are reduced to calculating when $gT$ is central. For this we introduce the following operator:

**Definition 1.4.** Let $d\colon R \to R$ be the linear map given on monomials by

$$d(X_1^i X_2^j) = \begin{cases} X_1^i X_2^j & \text{if } i < j; \\ -X_1^j X_2^i & \text{if } i > j \\ 0 & \text{if } i = j. \end{cases}$$

Thus $\ker(d) = R^W$, and since $d^2 = d$, $R = R^W \oplus d(R)$.

**Lemma 1.5.** *$\mathcal{Z} \cap RT$ is a free $A$-module of rank $m$.*

*Proof.* Suppose that $gT \in \mathcal{Z} \cap RT$. We must have $X_1 gT = gTX_1$ and $TgT = gT^2$. Since $T$ is invertible (indeed $T^{-1} = q^{-1}(T - q + 1)$), the second equation is equivalent to $Tg = gT$. By the Lemma 1.3 this implies that $g \in R^W$, and hence the first equation becomes $X_1 gT = TX_1 g$. But then again using the Lemma 1.3, we see that $X_1 g \in R^W$. Let $M$ be the space of such restricted symmetric polynomials:

$$M = \{g \in R^W : X_1 g \in R^W\}.$$

Thus we have shown that $\mathcal{Z} \cap RT = MT$. It is now a linear algebra exercise to show that $M$ is a free $A$-modules of rank $m$.

Now $R = R^W \oplus d(R)$ as an $A$-module, where $d$ is the operator defined before the statement of the lemma. Thus if we let $\phi\colon R^W \to d(R)$ be the map $g \mapsto d(X_1 g)$, we see that $M = \ker(\phi)$. Let $m_{ij} = X_1^i X_2^j + X_1^j X_2^i$ for $i < j$ and $m_{ii} = X_1^i X_2^i$ be the monomial symmetric functions, and let $R^W_{m-1}$ be the span of $\{m_{ij} : 0 \le i \le j < m-1\}$. Then we claim that $\phi\colon R^W_{m-1} \to d(R)$ is an isomorphism of $A$-modules. Indeed for $j < m-1$ we have:

$$\phi(m_{ij}) = \begin{cases} -X_1^i X_2^{j+1}, & \text{if } j - i \le 1; \\ -X_1^i X_2^{j+1} + X_1^{i+1} X_2^j, & \text{if } i < j - 1. \end{cases}$$

If we use reverse lexicographical ordering on the bases $\{m_{ij} : 0 \le i \le j < m-1\}$ and $\{X_1^i X_2^j : 0 \le i < j \le m-1\}$, then the above equations show that the matrix of $\phi_{|R^W_{m-1}}$ with respect to these ordered bases is lower triangular with diagonal entries equal to $-1$, thus $\phi_{|R^W_{m-1}}$ is an isomorphism as claimed.

This immediately implies that $M$ is a free $A$-module of rank $m$. We can be more precise, however, and specify a basis of $M$ by considering the images of $\phi(m_{i(m-1)})$, $(0 \leq i \leq m-1)$: simply set

$$p_i = m_{i(m-1)} - (\phi_{R^W_{m-1}})^{-1}(\phi(m_{i(m-1)}),$$

then $\{p_0, p_1, \dots p_{m-1}\}$ is a basis of $\mathcal{Z} \cap RT$

□

**Example 1.6.** Let $m = 3$. In this case we find that the space $\mathcal{Z} \cap R^W T$ is spanned by $\{p_0, p_1, p_2\}$ where

$$\begin{aligned} p_0 &= (e_3 - e_1(X_1 + X_2) + X_1X_2 + X_1^2 + X_2^2), \\ p_1 &= (e_3 - e_1X_1X_2 + X_1X_2(X_1 + X_2)), \\ p_2 &= (e_3(X_1 + X_2) - e_2X_1X_2 + X_1^2X_2^2). \end{aligned} \tag{1.3}$$

Thus we have shown that the center $\mathcal{Z}$ of $\mathcal{K}_2^{\mathbf{v}}$ (with $m = 3$) is a 9-dimensional free submodule spanned by $R^W$ and $\{p_0T, p_1T, p_2T\}$ (this is exactly as stated in [A, Section 2]).

*Remark* 1.7. It is easy to check directly that the corollary implies that the rank of the center is the number of $m$-multipartitions of 2. This also follows by passing to the field of fractions of $A$, and using the result of [AK] which shows that in the semisimple case (for any $n$), the center has rank equal to the number of $m$-multipartition of $n$.

## 2. ON THE IMAGE OF $Z(H_2^{\mathrm{aff}})$

Next we do some simple computations. Let $h_k = \sum_{j=0}^{k} X_1^j X_2^{k-j}$ be the complete symmetric function in $X_1$ and $X_2$ of degree $k$, and let $\mathcal{I}$ denote the image of the center of $H_2^{\mathrm{aff}}$ in $\mathcal{K}_2^{\mathbf{v}}$.

**Definition 2.1.** Let $f_2 = Tf_1T$. Thus $f_2 \in J_v$.

**Lemma 2.2.** *a) In $H_2^{aff}$, for any $k \geq 2$*

$$TX_1^kT = qX_2^k - Q(X_1X_2)h_{k-2}T$$

*b) We have*

$$qf_v(X_2) = f_2 + QzT,$$

*where* $z = (-1)^{m+1}e_m + (X_1X_2)(\sum_{j=0}^{m-2}(-1)^j e_j h_{m-2-j}) \in R^W$.

*Proof.* The proof of a) is a direct calculation using Equation (1.1). For b) we have using a) we have

$$qX_2^k = TX_1^kT + Q(X_1X_2)h_{k-2}T, \qquad (k \geq 2).$$

Moreover, $qX_2 = TX_1T$, and $q = T^2 - QT$, so that

$$qf_v(X_2) = f_2 + Q\big((-1)^{m+1}e_m + (X_1X_2)(\sum_{j=2}^{m}(-1)^{m-j}e_{m-j}h_{j-2})\big)T,$$

as claimed. □

**Proposition 2.3.** *The elements $QX_1^kz$ lie in $\mathcal{I}$ for all $k \in \mathbb{Z}$, and moreover the elements $\{X_1^{k-1}z := 0 \leq k \leq m-1\}$ are linearly independent.*

*Proof.* From the previous lemma we have

$$\begin{aligned} QX_1^k zT &= qX_1^k f_v(X_2) - X_1^k f_2 \\ &= qX_1^k f_v(X_2) + qX_2^k f_v(X_1) - (qX_2^k f_1 + X_1^k f_2) \\ &\in qX_1^k f_v(X_2) + qX_2^k f_v(X_1) + J_v. \end{aligned}$$

hence we see that $QX_1^k zT \in \mathcal{K}_2^{\mathbf{v}} = H_2^{\text{aff}}/J_v$ is in the image of the center of $H_2^{\text{aff}}$ as required.

It remains to show that the elements $\{X_1^k z : 0 \leq k \leq m-1\}$ are linearly independent. Since $A$ is an integral domain, we see that $X_1^k zT \in \mathcal{Z}$, and hence $X_1^k z \in R^W$ by Lemma 1.3. We have

$$X_1^{k-1}z = (-1)^{m+1}e_m X_1^{k-1} - (X_1^k X_2)(\sum_{j=0}^{m-2}(-1)^{j+1}e_j h_{m-2-j}).$$

Now consider the expression for $X_1^{k-1}z$ as a linear combination of the restriced monomial symmetric functions: by considering the terms involving $X_2^{m-1}$ we see that the coefficient of $m_{j(m-1)}$ is zero unless $j = k$ in which case it is 1. It follows immediately that the $X_1^k z$ are linearly independent. □

**Theorem 2.4.** *Let $B$ denote the localization $A$ where $Q = q-1$ is inverted. Then, over $B$, the center of $H_2^{aff}$ surjects onto the center of $\mathcal{K}_2^{\mathbf{v}}$.*

*Proof.* It is clearly sufficient to show that $p_i T$ lies in $\mathcal{I}$, where $p_i \in R^W$ is as in Lemma 1.5. Since we have inverted $Q$, Proposition 2.3 shows that we have $X_1^k zT \in \mathcal{I}$ for all $k \in \mathbb{Z}$. Now, again by the previous proposition, the coefficient of $m_{j(m-1)}$ in $X_1^{k-1}z$ ($0 \leq k \leq m-1$) in the basis $\{m_{ij}\}$ of restricted monomial symmetric functions is $\delta_{jk}$, and the same is true, by definition, for the $p_k$. Since $\mathcal{I} \subset \mathcal{Z}$, $X_1^k zT$ can be written as a linear combination of the elements $p_i T$, hence it follows immediately that $p_k T = X_1^{k-1}zT$, and we are done. □

**Example 2.5.** We consider again the case $m = 3$, keeping the notation of the previous example. Then $z = p_1$, and it is easy to check that $X_1 p_1 = p_2$, and similarly

$$\begin{aligned} X_1^{-1}p_1 &= e_3 X_1^{-1} - e_1 X_2 + X_2(X_1 + X_2) \\ &= (e_2 - e_1 X_1 + X_1^2) - e_1 X_2 + X_2(X_1 + X_2) \\ &= e_2 - e_1(X_1 + X_2) + (X_1^2 + X_1 X_2 + X_2^2) \\ &= p_0 \end{aligned}$$

so that $X_1^{k-1}z = p_k$ for $0 \leq k \leq 2$.

## 3. A CONJECTURE

We propose the following conjecture (which has probably been made by many people, but I have not been able to find an explicit statement elsewhere, moreover some of the existing literature is somewhat unclear on the status of this conjecture).

**Conjecture 3.1.** Let $H_n^{\text{aff}}$ be the affine Hecke algebra with coefficients extended to $B$, the ring $A = \mathbb{Z}[q^{\pm 1}, v_1^{\pm 1}, \dots, v_m^{\pm 1}]$ with $Q = q-1$ inverted. Let $\psi_m \colon H_n^{\text{aff}} \to \mathcal{K}_n$

be the quotient map. Then

$$\psi_m(Z(H_n^{\text{aff}})) = Z(\mathcal{K}_m).$$

In fact, it may be easier (and as useful) to show this in the case where $H_n^{\text{aff}}$ is defined over a field $F$, and the parameter $q$ is not equal to 1.

*Remark* 3.2. As pointed out in [A], if we specialize to $q = 1$, *i.e.* $Q = 0$, then the image of the center of the affine Hecke algebra does *not* necessarily surject onto the center of the specialized Ariki-Koike algebra (for $\mathcal{K}_2^{\mathbf{v}}$, if we say require $X_1$ to satisfy $X_1^3 - 1 = 0$, then at $q = 1$ this is just the group algebra of the complex reflection group $G(3,1,2)$ which has a 9 dimensional center – the specialization of the center of $\mathcal{K}_2^{\mathbf{v}}$ – whereas the images of $X_1 + X_2$, $X_1X_2$ only generate a 6-dimensional subalgebra).

We list the following evidence for the conjecture:

- [AK] shows that the conjecture holds in the semisimple case (they also show explicitly the conditions on the parameters under which the Ariki-Koike algebras is semisimple).
- This note establishes the case $n = 2$.
- In an orthogonal direction, Francis and Graham [FG] have recently established the conjecture for the (much more substantial) case of the finite Hecke algebra of type $A$, *i.e.* the case $m = 1$.
- Brundan [Br] has recently established the analogous result for the degenerate cyclotomic Hecke algebras [D] – it is probable one can combine this with the work of Lusztig [L] to establish the conjecture in the case where $q$ is not a root of unity (note that this would include cases not covered by the results of [AK]): roughly one must use the fact that if $\chi$ is an element of Spec($\mathcal{Z}$), then the completion of $H_n^{\text{aff}}$ at the ideal $\chi$ is isomorphic to the corresponding completion of the degenerate affine Hecke algebra $\mathcal{H}_n^{\text{aff}}$ at $\chi$. Now any finite dimensional module for $H_n^{\text{aff}}$ in the block corresponding to $\chi$ then corresponds to a module for $\mathcal{H}_n^{\text{aff}}$, and in particular the cyclotomic Hecke algebras and their degenerate versions should have isomorphic block summands, yielding the result.

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