THE TOPOLOGY OF BAUMSLAG-SOLITAR REPRESENTATIONS

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Abstract. Let $\Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle$ be a Baumslag–Solitar group and let $G$ be a complex reductive algebraic group with maximal compact subgroup $K < G$. We show that, when $p$ and $q$ are relatively prime with distinct absolute values, there is a strong deformation retraction retraction of $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$.

1. Introduction

Let $\Gamma$ be a finitely-generated group (fix a finite generating set $S \subset \Gamma$) and let $G$ be a (topological) group. Identifying a group homomorphism $\rho: \Gamma \to G$ with its restriction $\rho|_S$ identifies $\text{Hom}(\Gamma, G)$ with the subset of $G^S$ cut out by the relations defining $\Gamma$. In particular, $\text{Hom}(\Gamma, G)$ is a closed subset and we equip it with the subspace topology. In the case under consideration, $G$ will be a complex linear algebraic group and this identification also endows $\text{Hom}(\Gamma, G)$ with the structure of a complex affine algebraic variety.

For any topological subgroup $K < G$, the inclusion of $K$ in $G$ induces a topological inclusion $\text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G)$. When $K$ is compact, $\text{Hom}(\Gamma, K)$ is itself compact and thus more amenable to topological analysis. In particular, individual elements of $\text{Hom}(\Gamma, K)$ are also often easier to analyze. For example, every unitary representation of $\Gamma$ is completely reducible whereas this is usually not the case for general linear representations. For this reason, it is remarkable that in some cases the inclusion above is in fact a homotopy equivalence, due to the existence of a deformation retraction from $\text{Hom}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$. This should be contrasted with the fact that, for general finitely-generated groups, rigidity results (e.g. those of Selberg [12]) ensure that this inclusion is not even a bijection of connected components.

Nevertheless, recent success in producing such retractions has been achieved using a mixture of topological and algebraic tools for classical families of finitely-generated groups. Notable positive results include the case of free-abelian groups by Pettet–Souto [10], torsion-free expanding nilpotent groups by Silberman–Souto [13] and general nilpotent groups by Bergeron [2].

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In this paper, we turn our attention to a class of groups that has often served as a testing ground for ideas in geometric group theory: the Baumslag-Solitar groups

\[ \text{BS}(p, q) := \langle a, b \mid ab^p a^{-1} = b^q \rangle. \]

It is, in some sense, their “bad behaviour” that motivated their discovery in the work of Baumslag–Solitar [1] as the first examples (for some \( p \) and \( q \)) of non-Hopfian residually finite groups. More recently, they also proved to be interesting cases in Gromov’s proposed classification of finitely-generated groups up to quasi-isometry as exhibited by Farb–Mosher [6, 5] and Whyte [15].

From our point of view, analyzing the topology of their representation varieties, we think of the Baumslag-Solitar groups as the simplest interesting infinite groups which are not lattices in Lie groups. Our main result establishes the following:

**Theorem 1.1.** Let \( p \) and \( q \) be (non-zero) relatively prime integers with distinct absolute values and consider the Baumslag-Solitar group

\[ \Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle. \]

If \( G \) is the group of complex points of a reductive algebraic group and \( K < G \) is a maximal compact subgroup, then there is a strong deformation retraction of \( \text{Hom}(\Gamma, G) \) onto \( \text{Hom}(\Gamma, K) \). In particular, the two spaces are homotopy equivalent.

We conclude this introduction with a short outline of the proof:

1. An application of Kempf–Ness theory provides a deformation retraction of \( \text{Hom}(\Gamma, G) \) onto a subset \( \text{Hom}_{\text{KN}}(\Gamma, G) \) which, in particular, consists of completely reducible representations.
2. Let \( B \) denote the subgroup of \( \Gamma \) generated by \( b \). We produce an integer \( O \) (depending only on \( \Gamma \) and \( G \)) such that, for every representation \( \rho \) in \( \text{Hom}_{\text{KN}}(\Gamma, G) \), the abelian group \( \rho(B) \) is finite and of order at most \( O \).
3. Letting \( \tilde{F} \) (respectively \( \tilde{F}_K \)) denote the manifold of abelian subgroups of \( G \) (respectively of \( K \)) of order at most \( O \), we show that

\[ \text{Hom}_{\text{KN}}(\Gamma, G) \to \tilde{F}; \quad \rho \mapsto \rho(B) \]

is a fiber bundle onto a union of connected components. Since the inclusion of \( \tilde{F}_K \) into \( \tilde{F} \) is a homotopy equivalence, it follows that there is a retraction of \( \text{Hom}_{\text{KN}}(\Gamma, G) \) onto the subset consisting of those \( \rho \) with \( \rho(B) \subset K \).
4. Each fiber of the bundle is the domain of a natural covering map onto a subset of the semisimple elements of a reductive group \( G' \) which admits a deformation retraction into a compact subgroup \( K' \subset G' \). Lifting this retraction to the fibers, we obtain the desired retraction of \( \text{Hom}_{\text{KN}}(\Gamma, G) \) onto \( \text{Hom}(\Gamma, K) \).
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2. Background on Reductive Algebraic Groups

An (affine) algebraic group is, properly, an (affine) variety endowed with a group law for which the group operations are morphisms of varieties, i.e., polynomial maps. For our purposes, we will use the terms “affine variety” to mean an affine algebraic set (the zero locus of a family of complex polynomials in several variables) and “algebraic group” to mean the set of complex points of the affine algebraic group. It turns out that affine algebraic groups are linear so we may (and always will) identify them with Zariski closed subgroups of $SL_n\mathbb{C}$ (see, for instance, Springer [14]).

Given a compact Lie group $K$, the Peter-Weyl Theorem provides a faithful embedding $K \to \operatorname{GL}_n\mathbb{R}$ for some $n$. Identifying $K$ with its image realizes it as a real algebraic subgroup of $\operatorname{GL}_n\mathbb{R}$. We then define the complexification $G := K_\mathbb{C}$ to be the vanishing locus in $\operatorname{GL}_n\mathbb{C}$ of the ideal defining $K$. The group $G$ is a complex algebraic group which is independent up to isomorphism of the embedding provided by the Peter-Weyl Theorem. A complex linear algebraic group $G$ is reductive if and only if it is the complexification of a compact Lie group $K$ as above. In this case, $K$ is always a maximal compact subgroup $G$. We summarize some well-known facts in the following (see, for instance, Helgarson [7]):

**Proposition 2.1.** Let $S(G)$ denote the set of maximal compact subgroups of $G$. This set admits the structure of a connected non-positively curved symmetric space in such a way that the action

$$G \acts S(G), \quad (g, K) \mapsto gKg^{-1}$$

is a smooth isometric action. In particular:

1. any two maximal compact subgroups of $G$ are conjugate by an element in the identity component $G^\circ$, and
2. if a compact subgroup $H \subset G$ normalizes a maximal compact subgroup $K \in S(G)$, then $H \subset K$.

Let us denote the centralizer and normalizer of a subgroup $H \subset G$ by

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}, \quad \text{and} \quad Z_G(H) := \{g \in G \mid \forall h \in H : ghg^{-1} = h\}.$$

We will also make use of the subset $N_{G^\circ}(H) := G^\circ \cap N_G(H)$. We record the following possibly non-standard facts for later use:

**Lemma 2.2** (Pettet-Souto [10], Lemma 4.4). Let $G$ and $K$ be as above. If $H \subset K$ is a subgroup, then the natural inclusion between normalizers

$$N_K(H) \to N_G(H)$$
are homotopy equivalences. □

Corollary 2.3. If $A \subset K$ is abelian, then $N_K(A)$ is maximal compact in $N_G(A)$.

Proof. We first observe that $N_G(A)$ is reductive. Indeed, since $A$ is diagonalizable, the identity component of the normalizer $N_G(A)^o$ coincides with the identity component of the centralizer $Z_G(A)^o$ which is itself reductive in light of a result of Richardson [11, Theorem 6.1].

Now, the compact subgroup $N_K(A) \subset N_G(A)$ is contained in some maximal compact subgroup $\kappa \subset N_G(A)$. Here, it follows from Lemma 2.2 and the fact that $N_G(A)$ is reductive that the inclusions $N_K(A) \hookrightarrow N_G(A)$ and $\kappa \hookrightarrow N_G(A)$ are homotopy equivalences. This implies that the inclusion of $N_K(A)$ into $\kappa$ is also a homotopy equivalence. Since the latter two are closed manifolds, this implies that the inclusion is surjective, meaning that $\kappa = N_K(A)$ after all. □

Let $V$ be a complex vector space and consider a $G$-variety $X \subset V$, i.e., suppose that $G$ acts on $X \subset V$ via an embedding $G \subset \text{GL}(V)$. If $K \subset G$ is a maximal compact subgroup, we may assume without loss of generality that $V$ is equipped with a $K$-invariant Hermitian norm $\|\cdot\|$. The restriction of this norm to $X$ provides a gateway to analyze its topology through a package of results known as Kempf–Ness Theory (see Bergeron [2, Section 3] for an introduction and Schwarz [4, Chapter 5] for the details). To state the results we need, write $\mathcal{M}$ for the Kempf–Ness Set of minimal vectors in $X$:

$$\mathcal{M} := \{ x \in X : \forall g \in G \text{ we have } \|x\| \leq \|g \cdot x\| \}.$$

Then:

1. there is a strong deformation retraction of $X$ onto $\mathcal{M}$, and
2. the $G$-orbit of every $x \in \mathcal{M}$ is closed.

3. The Main Theorem

Let $p$ and $q$ be relatively prime integers with $|p| \neq |q|$. We consider the Baumslag-Solitar group

$$\Gamma = \langle a, b \mid ab^p a^{-1} = b^q \rangle.$$

Let us also fix once and for all a complex reductive algebraic group $G$ along with a maximal compact subgroup $K \subset G$.

As explained in the introduction, the image of $\text{Hom}(\Gamma, G)$ in $G \times G$ under the injective map

$$\rho \mapsto (\rho(a), \rho(b))$$
is the closed subset cut out by the relation
\[ \rho(a)\rho(b)^q\rho(a)^{-1} = \rho(b)^q. \]

This simultaneously endows \( \text{Hom}(\Gamma, G) \) with compatible structures of an affine algebraic variety (when \( G \) is viewed as an affine algebraic group) and a Hausdorff topological space (when \( G \) is viewed as a Lie group). It is well known (see, for instance, Lubotzky–Magid [8]) and otherwise easy to see that, in both points of view, the geometric structures obtained from distinct presentations of the same group \( \Gamma \) are isomorphic. The choice of presentation is thus immaterial.

If \( K \subset G \) is a maximal compact subgroup, the inclusion \( K \hookrightarrow G \) induces a natural inclusion \( \text{Hom}(\Gamma, K) \hookrightarrow \text{Hom}(\Gamma, G) \) and we would like to show that there is a deformation retraction of the latter space onto the former.

3.1. Kempf–Ness Theory. In this section, we take the first step of our argument, obtaining an intermediate deformation retract between \( \text{Hom}(\Gamma, K) \) and \( \text{Hom}(\Gamma, G) \).

For this, observe that the action of \( G \) on itself by conjugation induces an action of \( G \) on \( \text{Hom}(\Gamma, G) \) via \( (g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1} \). In terms of our embedding this is simply the restriction of the diagonal action of \( G \) on \( G \times G \) to the subspace/subvariety \( \text{Hom}(\Gamma, G) \). Fixing an embedding of \( G \) in \( \text{SL}_n\mathbb{C} \), this allows us to view \( \text{Hom}(\Gamma, G) \subset (M_n\mathbb{C})^2 \cong \mathbb{C}^{2n^2} \) as a \( G \)-variety. Endowing the ambient vector space with a \( K \)-invariant norm (for example, we may assume the embedding maps \( K \) into \( \text{SU}(n) \) and take the Hilbert–Schmidt norm on \( M_n(\mathbb{C}) \)), we obtain a retraction of \( \text{Hom}(\Gamma, G) \) onto the associated Kempf–Ness set \( \text{Hom}_{KN}(\Gamma, G) \). Concretely, this is the set of all homomorphisms \( \rho: \Gamma \rightarrow G \) satisfying the following condition for every \( g \in G \):

\[ \|gp(a)g^{-1}\|_{HS}^2 + \|gp(b)g^{-1}\|_{HS}^2 \geq \|\rho(a)\|_{HS}^2 + \|\rho(b)\|_{HS}^2. \]

3.2. Manifolds of Finite Abelian Groups. Keeping the notation as above, let \( B \) denote the subgroup of \( \Gamma \) generated by \( b \). Our goal in this section is to realize the Kempf–Ness set \( \text{Hom}_{KN}(\Gamma, G) \) as the total space of a bundle over a manifold of finite abelian subgroups of \( G \). The following proposition should be compared with Pettet-Souto [10, Section 3.4] and Bergeron-Silberman [3, Proposition 2.1].

Proposition 3.1. The set of abelian groups

\[ \mathcal{F} := \{ \rho(B) \subset G : \rho \in \text{Hom}_{KN}(\Gamma, G) \} \]

is invariant under conjugation by \( G \) and admits a manifold structure with finitely many connected components such that the \( G \)-action is smooth and the projection

\[ \pi: \text{Hom}_{KN}(\Gamma, G) \rightarrow \mathcal{F}, \quad \pi(\rho) := \rho(B) \]
is a locally trivial fiber bundle. Moreover, if \( H \in \mathcal{F} \), the image of a representation \( \rho \in \pi^{-1}(H) \) is contained in the normalizer \( N_G(H) \).

The following two lemmas and their proofs are inspired by the work of McLaury [9] on irreducible linear representations of Baumslag–Solitar groups. We quote Theorem 3.2 of his paper as Lemma 3.2 below but give a full proof since there are minor errors in the argument given in [9]. Lemma 3.3 then generalizes Theorems 3.3 and 5.1 of [9] to our context of representations in general reductive groups lying in the Kempf–Ness set. These lemmas are the only places in the paper where we directly use the hypothesis that \( p \) and \( q \) are relatively prime.

**Lemma 3.2.** Let \( G \) be a linear algebraic group and let \( \Gamma \) be a Baumslag–Solitar group with \( p \) and \( q \) relatively prime. If \( \rho \in \text{Hom}(\Gamma, G) \), then \( \rho(\Gamma) \) is solvable.

**Proof.** Let \( L = \overline{\rho(\Gamma)} \) be the Zariski closure of the image of \( \Gamma \) and let \( H = \overline{\rho(b)} \) be the Zariski closure of \( \rho(b) \). The subgroups \( H_1 = \langle \rho(b^p) \rangle \) and \( H_2 = \langle \rho(b^q) \rangle \) are then of finite index in \( H \) dividing \( p \) and \( q \) respectively. In particular, we have an identity \( H^o = H_1^o = H_2^o \) of (algebraic) connected components. Since \( \rho(a) \) conjugates \( \rho(b^p) \) to \( \rho(b^q) \), it also conjugates \( H_1 \) to \( H_2 \) and therefore normalizes \( H^o \). Since \( H^o \) is also normalized by \( \rho(b) \in H \), we conclude that \( H^o \) is normalized by \( \rho(\Gamma) \) and is hence normal in \( L \).

The finite subgroups \( H_1/H^o \) and \( H_2/H^o \) of \( L/H^o \) are conjugate by the image of \( \rho(a) \) in \( L/H^o \). In particular, they have the same order. Now, since these subgroups are both contained in the finite group \( H/H^o \), they also have the same index there – in other words \( [H:H_1] = [H:H_2] \). However, as noted above, these indices divide \( p \) and \( q \) respectively. Since \( \gcd(p, q) = 1 \), we conclude that \( H = H_1 = H_2 \) and hence that \( H \) itself is normalized by \( \rho(a) \). Given that \( H \) also contains \( \rho(b) \) it now follows that \( H \) is normal in \( L \).

Finally, since \( \rho(b) \in H \), the image of \( \rho(a) \) in \( L/H \) is a (Zariski-) topological generator. Having shown that both \( H \) and \( L/H \) are abelian, we conclude that \( L \) and its subgroup \( \rho(\Gamma) \) are solvable. \( \square \)

**Lemma 3.3.** If \( \rho \in \text{Hom}_{KN}(\Gamma, G) \), then \( \rho(\Gamma) \) is virtually abelian and consists of semisimple elements. Furthermore, the abelian group \( \rho(B) \) is normal in \( \rho(\Gamma) \), finite, and its order is bounded by a constant \( O = O(\Gamma, G) \) that is independent of \( \rho \).

**Proof.** Recall that, since \( \rho \) lies in the Kempf–Ness set, its orbit under conjugation must be closed. A result of Richardson [11, Theorem 3.6] now implies that the Zariski closure \( L \) of the image of \( \rho \) is reductive. It follows that \( L^o \) is a connected reductive solvable group, in other words an algebraic torus (possibly zero-dimensional). This makes \( L \) and its subgroup \( \rho(\Gamma) \) virtually abelian and ensures that every element of \( L^o \) is semisimple. Since every element of \( L \) has a positive power in \( L^o \), the Jordan
decomposition shows that every element of $L$ (in particular, every element of $\rho(\Gamma)$) is semisimple as well.

Now, $H = \rho(B)$ is a commutative subgroup consisting of semisimple elements. As such, the group $N_G(H)/Z_G(H)$ is finite (see, for instance, Springer [14, Cor. 3.2.9]), and it follows that conjugation by $\rho(a)$ is an automorphism of $H$ of finite order, say $r$. It follows that $\rho(a^r)$ commutes with $H$ and, in particular, with $\rho(b)$. On the other hand, $ab^pa^{-1} = b^q$ implies that $ab^ka^{-1} = b^{qk/p}$ whenever $k$ is divisible by $p$. Repeatedly applying this we obtain the identity $a^rb^{pr}a^{-r} = b^{q^r}$ and, upon applying $\rho$, that $\rho(b)^{pr} = \rho(b)^{q^r}$. Since $p$ and $q$ are distinct, we conclude that $\rho(b)$ has finite order (dividing $|p^r - q^r|$) and that $\rho(B)$ is finite. In turn, given that $\rho(B)$ is finite, it is also Zariski closed and $H = \rho(B)$. It follows that $\rho(B)$ is normalized by $L$, hence by $\rho(\Gamma)$. As before, $\rho(\Gamma)/\rho(B)$ is cyclic, generated by the image of $a$.

We now rely on our linear embedding $G \subset S\!L_n\!\mathbb{C}$. Since $L$ is reductive, this linear representation of $L$ decomposes as a direct sum of irreducible representations. Since $\rho(\Gamma)$ is Zariski-dense in $L$, those subspaces are also irreducible as representations of $\Gamma$. The calculations of McLaury [9, Thm. 5.1] now show that every eigenvalue of $b$ in this representation is a root of unity of order dividing $\Gamma$. The calculations of McLaury [9, Thm. 5.1] now show that every eigenvalue of $b$ in this representation is a root of unity of order dividing $\prod_{k=1}^n |p^k - q^k|$. Since $\rho(b)$ is semisimple it follows that $\rho(b)$ has order dividing $O$ and, in particular, that $|\rho(B)| \leq O$.

**Proof of Proposition 3.1**. Before we study $\mathcal{F}$, we first consider the slightly larger set

$$\tilde{\mathcal{F}} := \{H < G : H \text{ is abelian of order at most } O\}.$$ 

Observe that $G^\circ$ (the identity component of $G$) acts by conjugation on $\tilde{\mathcal{F}}$ with closed stabilizers. As such, we can endow $\tilde{\mathcal{F}}$ with the orbifold structure with respect to which each $G^\circ$-orbit is a connected homogeneous $G^\circ$-manifold. Concretely, if we define the “connected normalizer” as $N_{G^\circ}(H) := N_G(H) \cap G^\circ$, then the connected component of $H \in \tilde{\mathcal{F}}$ will be its $G^\circ$ orbit, topologized via the identification with $G^\circ/N_{G^\circ}(H)$. There being only finitely many $G$-orbits in $\tilde{\mathcal{F}}$, we necessarily give it the topology of the disjoint union of the orbits.

A homomorphism $\rho : B \to G$ need not extend to the full group $\Gamma$ so the map

$$\pi : \text{Hom}_{KN}(\Gamma, G) \to \tilde{\mathcal{F}}, \quad \pi(\rho) = \rho(B)$$

may not be surjective. Recall, however, that we have defined $\mathcal{F} = \pi(\text{Hom}_{KN}(\Gamma, G))$ and observe that the $G^\circ$-equivariance of $\pi$ implies that $\mathcal{F}$ is the union of those components of $\tilde{\mathcal{F}}$ it intersects, giving it the desired manifold structure.

For the bundle structure, let $\mathcal{Z} \subset \mathcal{F}$ denote the connected component of a finite abelian subgroup $H \in \mathcal{F}$, let $\mathcal{H} := \pi^{-1}(\mathcal{Z}) \subset \text{Hom}_{KN}(\Gamma, G)$ and let $\mathcal{H}_H := \pi^{-1}(H)$. We can then identify $\mathcal{H}$ and the restriction $\pi$ there with the twisted product

$$(G^\circ \times \mathcal{H}_H)/N_{G^\circ}(H) \to G^\circ/N_{G^\circ}(H)$$
where $N_{G^o}(H)$ acts on $G^o$ (resp. $\mathcal{H}_H$) by right multiplication (resp. conjugation). This shows that $\pi$ is a locally trivial fibre bundle. □

We will also need to consider the analogous manifold of abelian subgroups in the maximal compact subgroup $K \subset G$. To this end, let

$$\tilde{\mathcal{F}}_K := \{H \in \tilde{\mathcal{F}} \mid H \subset K\}$$

be the manifold of abelian subgroups of $K$ of order at most $O$. It follows, as before, that the conjugation action of $K$ on $\tilde{\mathcal{F}}_K$ has finitely many orbits and we can endow $\tilde{\mathcal{F}}_K$ with the structure of a homogeneous manifold. Concretely, the connected component of $H \in \tilde{\mathcal{F}}_K$ is identified with $K^o/N_K^o(H)$. We record the following two facts about manifolds of finite abelian subgroups:

**Proposition 3.4.** The manifold $\tilde{\mathcal{F}}_K$ is a deformation retract of $\tilde{\mathcal{F}}$.

**Proof.** The argument is analogous to the one in Pettet–Souto [10, Proposition 4.1]. We first verify that $\tilde{\mathcal{F}}_K \hookrightarrow \tilde{\mathcal{F}}$ induces a bijection at the level of connected components. To see this, recall that any $H \in \mathcal{F}$, being compact, can be conjugated into $K$ by an element of $G^o$, which shows that $\pi_0(\tilde{\mathcal{F}}_K) \to \pi_0(\tilde{\mathcal{F}})$ is surjective. Conversely, Pettet–Souto [10, Lemma 3.4] asserts that two subgroups of $K$ which are conjugate by an element $G^o$ are also conjugate by an element of $K^o$, in other words that our map of connected components is injective.

Let $Z$ denote a component of $\tilde{\mathcal{F}}$ and let $Z_K$ denote the unique component of $\tilde{\mathcal{F}}_K$ contained in $Z$. To complete the proof, it suffices to show that $Z_K \hookrightarrow Z$ is a homotopy equivalence. Fix a subgroup $H \in Z_K$ and use it to identify $Z_K$ and $Z$ with $K^o/N_K^o(H)$ and $G^o/N_{G^o}(H)$ respectively. We now have a commutative diagram

$$
\begin{array}{ccc}
N_{K^o}(H) & \longrightarrow & K^o \\
\downarrow & & \downarrow \\
N_{G^o}(H) & \longrightarrow & G^o
\end{array}
\quad \quad
\begin{array}{ccc}
& & Z_K \\
& & \downarrow \\
& & Z
\end{array}
$$

where the two rows are principal bundles. Since the first two columns are homotopy equivalences (that was Lemma 2.2), it follows that the last column is also a homotopy equivalence. □

**Proposition 3.5** (Cartan’s Lemma). There is a continuous map $\kappa: \tilde{\mathcal{F}} \to \mathcal{S}(G)$ with:

1. $H \subset \kappa(H)$ for all $H \in \tilde{\mathcal{F}}$, and
2. $\kappa(H) = K$ for all $H \in \tilde{\mathcal{F}}_K$.

**Proof.** Let $d(\cdot, \cdot)$ denote the Riemannian metric on the simply connected non-positively curved complete metric space $\mathcal{S}(G)$. The strict convexity of $d(\cdot, \cdot)^2$ implies that the
function 
\[ f_H : S(G) \to \mathbb{R}_+, \quad f_H(\kappa) = \sum_{g \in H} d(\kappa, gKg^{-1})^2 \]
is strictly convex. Since it is also proper, the function \( f_H \) attains its minimum at a unique point \( \kappa = \kappa(H) \in S(G) \). It follows from the \( H \)-invariance of \( f_H \) that this minimum is also \( H \)-invariant and, hence, that \( H \) is contained in the maximal compact subgroup stabilizing \( \kappa(H) \). Notice also that if \( H \subset K \), then the non-negative function \( f_H \) vanishes at \( K \), showing that \( \kappa(H) = K \). Finally, it follows from the non-positive curvature that the minimum of \( f_H \) depends continuously on \( H \), that is, that
\[ \kappa : \tilde{F} \to S(G), \quad H \mapsto \kappa(H) \]
is continuous. \( \square \)

3.3. The Covering Map. Before going further, we need another bundle. For any finite abelian subgroup \( H \subset G \) let
\[ G_H := N_G(H)/H \]
and let us denote the corresponding natural projection by
\[ p : N_G(H) \to G_H. \]
Notice, as in the proof of Corollary 2.3, that \( N_G(H) \) and thus \( G_H \) are reductive.

Recall from the proof of Proposition 3.1 that if \( \rho \in \pi^{-1}(H) \) then \( \rho(B) = H \) and \( \rho(\Gamma) \subset N_G(H) \). Thus, given any such \( \rho \) we obtain a representation
\[ p \circ \rho : \Gamma \to G_H \]
with image landing in the subset \((G_H)_s\) of semisimple elements of \( G_H \). By construction, the subgroup \( B \) of \( \Gamma \) is contained in the kernel of \( p \circ \rho \) so this map is uniquely determined by the image of the element \( a \in \Gamma \). Altogether, we have a map
\[ p_* : \pi^{-1}(H) \to (G_H)_s \]
with finite fibers because the projection \( N_G(H) \to G_H \) has finite kernel. In particular, \( p_*(\pi^{-1}(H)) \) is contained in the set \((G_H)_s\) of semisimple elements of \( G_H \).

The action by conjugation of \( N_{G^0}(H) \) on \( N_G(H) \) induces an action
\[ (G_H)_s \curvearrowright N_{G^0}(H), \quad (p(g), h) \mapsto p(ghh^{-1}). \]
This can now be used to construct the twisted product
\[ \mathcal{G} := (G^0 \times (G_H)_s)/N_{G^0}(H) \xrightarrow{\pi_0} G^0/N_{G^0}(H) = \mathcal{Z}. \]
Here, as in the previous section, \( \mathcal{Z} \) denotes the component of \( \mathcal{F} \) containing \( H \). The map \( p_* \) is \( N_{G^0}(H) \)-equivariant and, in particular, induces a morphism of fiber bundles from \( \mathcal{H} \) to \( \mathcal{G} \) where \( \mathcal{H} = \pi^{-1}(\mathcal{Z}) \) is a component of the bundle defined in Proposition 3.1. In fact, can say a bit more:
Proposition 3.6. The bundle map $p_*: \mathcal{H} \to \mathcal{G}$ is a finite cover onto a union of connected components of $\mathcal{G}$.

Proof. Since $p_*$ is a morphism of fiber bundles, it suffices to show that the map is a covering onto a union of connected components at the level of fibers. In other words, for every $H \in \mathcal{Z}$, we need to verify our claim for the map

$$p_*: \pi^{-1}(H) \to \pi^{-1}_G(H)$$

where $\pi^{-1}(H)$ consists of homomorphisms $\rho: \Gamma \to N_G(H)_s$ with $\rho(B) = H$ and $\pi^{-1}_G(H) = (G_H)_s = (N_G(H)/H)_s$. Since all the objects in the game are semi-analytic sets, it suffices to show that $p_*$ is locally injective and has the path-lifting property.

We start by showing that $p_*$ is locally injective. For this, since $H$ is finite and thus discrete in $G$, we may choose $U \subset N_G(H)$ to be a neighbourhood of the identity such that $U^{-1} = U$ and $U \cap H = \{1\}$. Using our identification of $\text{Hom}(\Gamma, G)$ with a subset of $G^2$ via the images of $a$ and $b$, the intersection of the set $\rho(a)U \times \rho(b)U$ with $\text{Hom}(\Gamma, G)$ defines a neighbourhood of $\rho \in \text{Hom}(\Gamma, G)$. Consider now representations $\rho$ and $\psi$ in $\pi^{-1}(H)$, and suppose that $\psi(a) \in \rho(a)U$ and $\psi(b) \in \rho(b)U$. Suppose also that $p_*(\rho) = p_*(\psi)$. It now follows that $\rho(a)^{-1}\psi(a)$ and $\rho(b)^{-1}\psi(b)$ are in $H$ and hence that $\rho(a)^{-1}\psi(a)$ and $\rho(b)^{-1}\psi(b)$ are in $U \cap H = \{1\}$. In other words, $\rho = \psi$ and $p_*$ is locally injective.

To verify that $p_*$ has the path-lifting property it is convenient to identify the fiber $\pi^{-1}_G(H)$ with its section $\{\text{Id}\} \times (G_H)_s \subset G \times (G_H)_s$ and to view $\pi^{-1}(H)$ as a subset of $N_G(H) \times H$. Let $\rho \in \pi^{-1}(H)$ and consider a path

$$\gamma: [0,1] \to \{\text{Id}\} \times (G_H)_s \subset G \times G_H; \quad \gamma(t) = (\gamma_1(t), \gamma_2(t))$$

with $\gamma(0) = p_*(\rho)$. This path always admits a unique lift $\tilde{\gamma}$ to the finite cover

$$N_G(H) \times N_G(H) \to G_H \times G_H$$

with $\tilde{\gamma}(0) = (\rho(a), \rho(b))$. To complete the proof we need to show that

$$\tilde{\gamma}(t) \in \pi^{-1}(H) \subset N_G(H) \times N_G(H),$$

i.e., we need to check that the relations of $\Gamma$ are preserved within the path and that the corresponding homomorphisms map $B$ onto $H$.

Let $w$ be a word in the letters $a$ and $b$ which is trivial in $\Gamma$. Substituting $\gamma_i(t)$ (respectively $\tilde{\gamma}_i(t)$) for $a$ and $b$ in $w$ yields a continuous path $w(t)$ in $G_H$ (respectively $N_G(H)$) such that $w(t)$ is the image of $\tilde{w}(t)$ under the quotient map. By construction, $w(t)$ is the identity element of $G_H$ for all $t$. It follows that $\tilde{w}(t) \in H$ for every $t$ and, since $H$ is discrete, this is a constant map as well. Given that $\tilde{w}(0)$ is trivial (it corresponds to the homomorphism $\rho$), we conclude that $\tilde{w}(t)$ is trivial for all $t$ as well. This shows that $\tilde{\gamma}(t)$ is a path of homomorphisms $\rho_t: \Gamma \to N_G(H)_s$. The image
of these homomorphisms is always contained in $N_G(H)$, because $\gamma_i(t)$ is semisimple and the quotient map is finite-to-one.

Finally, we show that the homomorphisms $\rho_t$ determined by $\tilde{\gamma}(t)$ send the subgroup $\langle b \rangle = B \subset \Gamma$ onto $H$. This follows, once again, because the homomorphisms to $G_H$ determined by $\gamma(t)$ map $b$ to the identity in $G_H$ and, as above, this implies that $\rho_t(b)$ is a fixed element of $H$. Since $\rho_0(b)$ is a generator of $H$ this then holds for $\rho_t(b)$. □

3.4. Putting the Puzzle Back Together. Recall that we denote the manifold of maximal compact subgroups of $G$ by $S(G)$ and, from Proposition 3.5, that we have a continuous map $\kappa : F \to S(G)$. We use this map to define sub-bundles of $\pi : \text{Hom}_{KN}(\Gamma, G) \to F$ and $\pi_G : \mathcal{G} \to \mathcal{Z}$ which will be intermediate steps in our deformations of $\text{Hom}_{KN}(\Gamma, G)$ onto $\text{Hom}(\Gamma, K)$.

First, consider the set
\[ \mathcal{K} := \{ \rho \in \text{Hom}_{KN}(\Gamma, G) \mid \rho(\Gamma) \subset \kappa(\pi(\rho)) \}. \]
Observe that, by the continuity of $\kappa$, the restriction of $\pi$ to $\mathcal{K}$ yields a sub-bundle $\pi_{\mathcal{K}} : \mathcal{K} \to F$. Since $\kappa(H) = K$ for every $H \in \mathcal{F}_K$, it follows that
\[ \text{Hom}(\Gamma, K) = \pi_{\mathcal{K}}^{-1}(\mathcal{F}_K). \]
Recalling from Proposition 3.4 that $\mathcal{F}_K$ is a deformation retract of $\mathcal{F}$, it follows moreover that $\text{Hom}(\Gamma, K)$ is a deformation retract of $\mathcal{K}$.

Second, observe by Corollary 2.3 that $N_{\kappa(H)}(H)$ is a maximal compact subgroup of $N_G(H)$. This ensures that
\[ K_H := N_{\kappa(H)}(H)/H \]
is also a maximal compact subgroup of $G_H$. We can now consider the subset $\mathcal{C} \subset \mathcal{G}$ defined fiber-wise by specifying that
\[ \mathcal{C} \cap \pi_{\mathcal{G}}^{-1}(H) = K_H \]
for every $H \in \mathcal{Z}$. It follows, once again, from the continuity of $\kappa$ that $K_H$ varies continuously so we obtain a sub-bundle
\[ \pi_{\mathcal{C}} : \mathcal{C} \to \mathcal{Z} \]
of $\pi_{\mathcal{G}} : \mathcal{G} \to \mathcal{Z}$. In fact, the following holds:

Lemma 3.7. Keeping the notation as above, we have $p_{\pi}^{-1}(\mathcal{C}) = \mathcal{K} \cap \pi^{-1}(\mathcal{Z})$.

Proof. Suppose $\rho : \Gamma \to G$ is a representation with $p_{\pi}(\rho) \in \mathcal{C}$. This means that
\[ (p \circ \rho)(\Gamma) = p(\rho(\Gamma)) \subset K_{\pi(\rho)}. \]
In particular, we see that $\rho(\Gamma) \subset p^{-1}(K_{\pi(\rho)}) = \kappa(\pi(\rho))$ and therefore $\rho \in \mathcal{K}$. □
In order to alleviate the notation (with the hope that no confusion shall arise), from now on we will use $G$ (respectively $C$) to denote the union of the corresponding bundles (respectively sub-bundles) over all connected components of $F$ rather than restricting our attention to $Z \subset F$.

We can then consider the globally induced morphism of fiber bundles

$$p_\ast: \text{Hom}_{KN}(\Gamma,G) \to G$$

where, applying Lemma 3.7 component by component, we have $p_\ast^{-1}(C) = K$. Similarly, applying Proposition 3.6 component by component, we have that $p_\ast$ is a covering map onto a union of connected components of $G$. To summarize, we have:

**Proposition 3.8.** The bundle $\pi_G: G \to F$ contains a sub-bundle $\pi_C: C \to F$ such that, for every $H \in F$, we have:

1. $\pi_G^{-1}(H) = (G_H)_s$, and
2. $\pi_K^{-1}(H) = K_H$.

Here, $K_H$ is a maximal compact subgroup of the reductive group $G_H$ and $(G_H)_s$ denotes the set of semisimple elements in $G_H$. Moreover, the bundle morphism

$$p_\ast: \text{Hom}_{KN}(\Gamma,G) \to G$$

is a finite cover onto a union of connected components of $G$ with $p_\ast^{-1}(C) = K$. □

3.5. **Proof of the Main Theorem.** With Proposition 3.8 at our disposition, we are now ready to complete the proof of our main theorem. After our applications of Kempf–Ness theory in Section 3.1, it suffices to show that $\text{Hom}(\Gamma, K)$ is a deformation retract of $\text{Hom}_{KN}(\Gamma, G)$.

To begin, recall from Propositions 3.1 and 3.8 that we have bundles

$$\pi: \text{Hom}_{KN}(\Gamma,G) \to F, \text{ and } \pi_G: G \to F,$$

along with respective sub-bundles

$$\pi: K \to F, \text{ and } \pi_C: C \to F.$$

Recall, moreover, that they are related by a covering map $p_\ast: \text{Hom}_{KN}(\Gamma,G) \to G$. Our deformation retraction now proceed in three steps:

**Step one:** For every $H \in F$, we have a retraction of the fiber $\pi_G^{-1}(H) = (G_H)_s$ onto the fiber $\pi_C^{-1}(H) = K_H$. Since everything in sight is a fiber bundle, this implies that there is a fiber-preserving retraction of $\mathcal{G}$ onto $\mathcal{C}$.

**Step two:** Use the covering map $p_\ast: \text{Hom}_{KN}(\Gamma,G) \to G$ to lift the retraction from step one to a fiber-preserving retraction of $\text{Hom}_{KN}(\Gamma,G)$ onto the sub-bundle $K$.

**Step three:** Lift the deformation retraction of $F$ onto $F_K$ given by Lemma 3.4 to a retraction of $K$ onto $K \cap \pi^{-1}(F_K) = \text{Hom}(\Gamma, K)$.

This completes the proof of our main theorem. □
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