# TOOLS FROM HARMONIC ANALYSIS

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Abstract. The Fourier transform can be thought of as a map that decomposes a function into oscillatory functions. In this paper, we will apply this decomposition to help us gain valuable insights into the behavior of our original function. Some particular properties of a function that the Fourier transform will help us examine include smoothness, localization, and its $L^2$ norm. We will conclude with a section on the uncertainty principle, which says though these transformations are useful there is a limit to the amount of information they can convey.

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## 1. Basic Properties of the Fourier Transformation

Let $f \in L^1(\mathbb{R}^n)$. Then we define its Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx$$

We can generalize this definition in the following way. Let $M(\mathbb{R}^n)$ be the space of finite complex-valued measures on $\mathbb{R}^n$ with the norm $\|\mu\| = |\mu|(\mathbb{R}^n)$ where $|\mu|$ is the total variation. We then define

$$\hat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} d\mu(x)$$

There are several basic formulas for Fourier transform that we will find useful throughout this paper. In particular:

Let $\Gamma(x) = e^{-\pi |x|^2}$. Then

$$\hat{\Gamma}(\xi) = e^{-\pi |\xi|^2} \tag{1.1}$$

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Let $\tau \in \mathbb{R}^n$. Define $f_\tau(x) = f(x-\tau)$. Then

$$\hat{f}_\tau(\xi) = e^{-2\pi i \tau\cdot\xi}\hat{f}(\xi) \tag{1.2}$$

Let $e_\tau(x) = e^{2\pi i x\cdot\tau}$ Then

$$\widehat{e_\tau f}(\xi) = \hat{f}(\xi-\tau) \tag{1.3}$$

Let $T$ be an invertible linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Further, let $T^{-t}$ be the inverse transpose of $T$. Then

$$\widehat{f\circ T} = |\det(T)|^{-1}\hat{f}\circ T^{-t} \tag{1.4}$$

Let $T$ be a dilation (i.e. $Tx = rx$ for some $r > 0$). Then as a special case of (1.4) we have

$$\widehat{f(rx)} = r^{-n}\hat{f}(r^{-1}\xi) \tag{1.5}$$

Finally, Let $\tilde{f}(x) = \overline{f(-x)}$. Then

$$\hat{\tilde{f}} = \bar{\hat{f}} \tag{1.6}$$

Two important properties we will use to relate $f$ to $\hat{f}$ are those of smoothness and localization. We say our function $f$ is smooth if it is infinitely differentiable. In order to define localization, we first let $D(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. With this in mind, we now say that $f$ is localized in space if for some $R \geq 0$ we have $\text{supp} f \subset D(0,R)$.

These concepts of smoothness and localization will relate $f$ and $\hat{f}$ in the following way. Suppose our function $f$ is localized in space, then $\hat{f}$ should be smooth. Conversely if $f$ is smooth than $\hat{f}$ should be localized. We will shortly prove rigorous version of both of these principals, starting first by showing that if $f$ is localized in space then $f$ is smooth.

To aid us in this persuit, we will make use of multiindex notation. Specifically, a we define a multiindex to be a vector $\alpha \in \mathbb{R}^n$ whose components are nonnegative integers. If $\alpha$ is a multiindex then by definition

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \tag{1.7}$$

$$x^\alpha = \Pi_{j=1}^n x_j^{\alpha_j} \tag{1.8}$$

The length of $\alpha$, denoted by $|\alpha|$, is defined as

$$|\alpha| = \sum_j \alpha_j \tag{1.9}$$

**Theorem 1.10.** *Let $\mu \in M(\mathbb{R}^n)$ and supp $(\mu)$ be compact. Then $\hat{\mu}$ is in $C^\infty$ and*

$$D^\alpha \hat{\mu} = ((-2\pi i x)^\alpha \mu)\widehat{\ } \tag{1.11}$$

*Further, if supp $\mu \subset D(0,R)$ then*

$$\|D^\alpha \hat{\mu}\|_\infty \leq (2\pi R)^{|\alpha|} \|\mu\| \tag{1.12}$$

*Proof.* Note that for any $\alpha$ the measure $(2\pi i x)^\alpha \mu$ is again a finite measure with compact support. Hence, if we can prove that $\hat{\mu}$ is $C^1$ and that (1.8) holds with $|\alpha| = 1$, then the fact that $\hat{\mu} \in C^\infty$ and (1.8) will follow by induction.
Fix $j \in \{1, ..., n\}$ and let $e_j$ be the $jth$ standard basis vector. Fix $\xi \in \mathbb{R}^n$ and consider the difference quotient

$$\begin{aligned}\Delta(h) &= \frac{\hat{\mu}(\xi + he_j) - \hat{\mu}(\xi)}{h} \\ &= \int \frac{e^{-2\pi i h x_j} - 1}{h} e^{-2\pi i \xi \cdot x} d\mu(x)\end{aligned}$$

As $h \to 0$ the quantity $\frac{e^{-2\pi i h x_j}-1}{h}$ converges pointwise to $-2\pi i x_j$. Further, $|\frac{e^{-2\pi i h x_j}-1}{h}| \leq 2\pi|x_j|$ for each $h$. Hence, the integrand $\frac{e^{-2\pi i h x_j}-1}{h} e^{-2\pi i \xi \cdot x}$ is dominated by $|2\pi x_j|$, which is a bounded function on the support of $\mu$. Thus, by applying the dominated convergence theorem we see that

$$\begin{aligned}\lim_{h\to 0} \Delta(h) &= \int \lim_{h \to 0} \frac{e^{-2\pi i h x_j} - 1}{h} e^{-2\pi i \xi \cdot x} d\mu(x) \\ &= \int -2\pi i x_j e^{-2\pi i \xi \cdot x} d\mu(x)\end{aligned}$$

Hence, we see that (1.8) holds when $|\alpha| = 1$. And so by induction (1.8) holds in general.
To obtain (1.9) we first note if $\mu \in M(\mathbb{R}^n)$ then the following holds

$$\|\hat{\mu}\|_\infty \leq \|\mu\|_{M(\mathbb{R}^n)}$$

For given any $\xi$ we have

$$\begin{aligned}|\hat{\mu}(\xi)| &= |\int e^{-2\pi i x \cdot \xi} d\mu(x)| \\ &\leq \int |e^{-2\pi i x \cdot \xi}| d|\mu|(x) \\ &= \|\mu\|\end{aligned}$$

Using this fact and (1.8) we see that
$|(2\pi i x)^\alpha \mu| \leq (2\pi R)^{|\alpha|} \|\mu\|$. And so we obtain (1.9).

$\square$

Having seen that $\mu$ being localized implies $\hat{\mu}$ is smooth, we now wish to show that $\mu$ being smooth implies $\hat{\mu}$ is localized. More specifically, we wish to prove the following result.

**Theorem 1.13.** *Suppose that $f$ is $C^N$ and that $D^\alpha f \in L^1$ for all $\alpha$ with $0 \leq |\alpha| \leq N$. Then when $|a| \leq N$, we have*

$$\widehat{D^\alpha} f(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi) \tag{1.14}$$

*And furthermore*

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^{-N} \tag{1.15}$$

*For a suitable constant $C$*

In order to prove this result, we will first need a preliminary lemma. We begin by letting $\phi : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ function with the following properties

(1) $\phi(x) = 1$ if $|x| \leq 1$

(2) $\phi(x) = 0$ if $|x| \geq 2$

(3) $0 \leq \phi \leq (1)$

(4) $\phi$ is radial

Now define $\phi_k = \phi(\frac{x}{k})$; in other words, rescale the function $\phi$ to live on a scale $k$ instead of a scale 1. If $\alpha$ is multiindex, then there is a constant $C_\alpha$ such that $|D^\alpha \phi_k| \leq \frac{C_\alpha}{k^{|\alpha|}}$ uniformly in $\alpha$. Furthermore, if $\alpha \neq 0$ then the support of $D^\alpha \phi$ is contained in the region $k \leq |x| \leq 2k$ With this appropriately rescaled function in mind we have the following lemma.

**Lemma 1.16.** *If $f$ is $C^N, D^\alpha f \in L^1$, for all $\alpha$ with $|\alpha| \leq N$ and if we let $f_k = \phi_k f$ then $\lim_{k\to\infty} \|D^\alpha f_k - D^\alpha f\|_1 = 0$ for all $\alpha$ with $|\alpha| \leq N$*

*Proof.* It is immediately seen that

$$\lim_{k\to\infty} \|\phi_k D^\alpha f - D^\alpha f\|_1 = 0$$

and so in this case it is sufficient to show that

$$\lim_{k\to\infty} \|D^\alpha(\phi_k f) - \phi_k D^\alpha f\|_1 = 0 \tag{1.17}$$

By an application of the Leibniz rule we have

$$D^\alpha(\phi_k f) - \phi_k D^\alpha f = \sum_{0<\beta\leq\alpha} c_\beta D^{\alpha-\beta} f D^\beta \phi_k$$

Where the various $c_\beta$'s are constants. Hence we have

$$\begin{aligned} \|D^\alpha(\phi_k f) - \phi_k D^\alpha f\|_1 &\leq C \sum_{0<\beta\leq\alpha} \|D^\beta \phi_k\|_\infty \|D^{\alpha-\beta} f\|_{L^1(\{x:|x|\geq k\})} \\ &\leq Ck^{-1} \sum_{0<\beta\leq\alpha} \|D^{\alpha-\beta}\|_{L^1(\{x:|x|\geq k\})} \end{aligned}$$

The last line goes to zero since the $L^1$ norms are taken only over the region $|x| \geq k$ □

We are now in a position to give a proof of Theorem 1.10

*Proof.* If $f$ is $C^1$ with compact support, then by integration by parts we have

$$\int \frac{\partial f}{\partial x_j}(x) e^{-2\pi i x\cdot\xi} dx = 2\pi i \xi_j \int e^{-2\pi i x\cdot\xi} f(x) dx$$

And so (1.11) holds when $|\alpha| = 1$. We then use induction to prove (1.11) for all $\alpha$. To finish the proof, we need now remove the compact support assumption. Let $f_k$ be as in Lemma 1.12. Then (1.11) holds for $f_k$. We now let $k \to \infty$ so that we may pass from our series of functions $f_k$ to our desired function $f$. On one hand, $\widehat{D^\alpha f_k}$ converges uniformly to $D^\alpha f$ as $k \to \infty$ by Lemma 1.13. On the other hand, $\hat{f}_k$ converges uniformly to $\hat{f}$ and so $(2\pi i\xi)^\alpha \hat{f}_k$ converges to $(2\pi i \xi)^\alpha \hat{f}$ pointwise. This proves (1.11)

To prove (1.12) observe that (1.11) together with the boundedness of the fourier transform implies that $\xi^\alpha \hat{f} \in L^\infty$ if $|\alpha| \leq N$. By the appropriate estimation utilizing the fact that

$$C_N^{-1}(1+|\xi|)^N \leq \sum_{\alpha\leq N} |\xi^\alpha \leq C_N(1+|\xi|)^N$$

and so (1.12) follows. □

## 2. Plancherel's Theorem and the Fourier Inversion Formula

The aim of this section will be to develop both the Plancherel theorem and the Fourier inversion formula. The Plancherel theorem will ultimately allow us to relate the $L^2$ norm of a function to the $L^2$ norm of its Fourier Transform. This ability to compare norms will become essential when discussing the uncertainty principle.

Before we get to the main results of this section we first take care of a couple of essential definitions.

**Definition 2.1.** Let $\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$ where $\alpha$ and $\beta$ are multi-indices. We define the Schwartz space $S(\mathbb{R}^n)$ to be the space of $C^\infty$ functions that decay faster than any polynomial. To be precise,

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) |\ \|f\|_{\alpha,\beta} \leq \infty \ \ \forall \alpha, \beta\} \tag{2.2}$$

As we will soon see, the Schwartz space has many useful properties. For example, the Fourier transformation of a Schwartz function is again a Schwartz function. Further, the Schwartz space is dense in $L^2$.

The next definition of interest is that of convolution. Given two functions $\phi$ and $f$ we define their convolution $\phi * f(x)$ as follows.

**Definition 2.3.** $\phi * f(x) = \int \phi(y) f(x-y) dy$

Some basic facts about convolution:

(1) Note by making the change of variables $y \to x - y$ we immediately see that $f * \phi = \phi * f$.

(2) Further, we can quickly see that supp $(\phi * f) \subset$ supp $(\phi)+$ supp $f$ Where $E+F=\{x+y|x\in E, y\in F\}$

(3) If $\phi \in L^1$ and $f \in L^p$ $1 \leq p \leq \infty$ then the integral in definition 2.2 is an absolutely convergent Lebesgue integral for almost every $x$ and

$$\|\phi * f\|_p \leq \|\phi\|_1 \|f\|_p \tag{2.4}$$

(4) If $\phi \in L^p$ and $f \in L^{p'}, \frac{1}{p}+\frac{1}{p'}=1$ then by Hölder's inequality the integral in definition 2.2 is an absolutely convergent Lebesgue integral for every $x$ and

$$\|\phi * f\|_\infty \leq \|\phi\|_p \|f\|_{p'} \tag{2.5}$$

(5) Let $\phi \in C_0^\infty$ and $f \in L^1_{loc}$ Then $\phi * f$ is $C^\infty$ and

$$D^\alpha(\phi * f) = (D^\alpha \phi) * f$$

(6) From (5) it quickly follows that if $f, g \in S$ then $f * g \in S$

(7) Finally, as one of convolutions most important properties we have

$$\widehat{f * g} = \hat{f}\hat{g} \tag{2.6}$$

$$\widehat{fg} = \hat{f} * \hat{g} \tag{2.7}$$

Where (2.5) is an application of Fubini's Theorem and (2.6) will be proved shortly via the inversion theorem.

We now move to one of the most useful theorems in this section, that of Fourier Inversion.

**Theorem 2.8.** *Suppose that $f \in L^1$ and assume that $\hat{f}$ is also in $L^1$. Then for a.e. $x$*

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \tag{2.9}$$

*Equivalently, we also have the following result*

$$\hat{\hat{f}}(x) = f(-x) \tag{2.10}$$

Our proof of the inversion theorem will largely rely on three pieces of information.

(1) Recall from (1.1) the function $\Gamma(x) = e^{-\pi|x|^2}$. This function gamma has the property that $\hat{\Gamma} = \Gamma$. Therefore, we automatically know it satisfies (2.9). We can extend this one function to a family of functions $\Gamma_\epsilon(x) = e^{-\pi\epsilon^2|x|^2}$. From this we have

$$\hat{\Gamma}_\epsilon(\xi) = \epsilon^{-n} e^{-\pi\frac{|\xi|^2}{\epsilon^2}} \tag{2.11}$$

If we apply this relation again with $\epsilon$ replaced by $\epsilon^{-1}$ we can verify that $\Gamma_\epsilon$ satisfies (2.9).

(2) Let $\phi \in S$ and $\int \phi = 1$. Also, define $\phi^\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$. Then if $f$ is a continuous function which goes to zero at $\infty$ then $\phi^\epsilon * f \to f$ uniformly as $\epsilon \to 0$.

(3) Suppose that $\mu \in M(\mathbb{R}^n)$ and $v \in M(\mathbb{R}^n)$. Then we have the following duality relation.

$$\int \hat{\mu} dv = \int \hat{v} d\mu \tag{2.12}$$

We get this result via Fubini's theorem as follows

$$\int \hat{\mu} dv = \int \int e^{-2\pi i \xi \cdot x} d\mu(x) dv(\xi) = \int \int e^{-2\pi i \xi \cdot x} dv(\xi) d\mu(x) = \int \hat{v} d\mu$$

We note as a particular case of this result that we have for $f, g \in L^1$

$$\int \hat{f}(x) g(x) dx = \int f(x) \hat{g}(x) dx \tag{2.13}$$

With this information in hand, it is now possible to prove the inversion theorem.

*Proof.* Consider the integral in (2.8) only this time include a dampening factor to get

$$I_\epsilon(x) = \int \hat{f}(\xi) e^{-\pi\epsilon^2|\xi|^2} e^{2\pi i \xi \cdot x} d\xi \tag{2.14}$$

We will evaluate the limit as $\epsilon \to 0$ in the following two ways.
1) As $\epsilon \to 0$ $I_\epsilon(x) \to \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$ for each fixed $x$. This follows from the dominated convergence theorem since $\hat{f} \in L^1$.
2) With $x$ and $\epsilon$ fixed define $g(\xi) = e^{-\pi\epsilon^2|\xi|^2} e^2 \pi i \xi \cdot x$. We then have

$$I_\epsilon(x) = \int f(y) \hat{g}(y) dy \tag{2.15}$$

by (2.11). We can now evaluate $\hat{g}$ using the fact that $g(\xi) = e_x(\xi)\Gamma_\epsilon(\xi)$ and (2.10) along with (1.3). Thus we have

$$\hat{g}(y) = \hat{\Gamma}_\epsilon(y - x) = \Gamma^\epsilon(x - y)$$

Where $\Gamma^\epsilon(y) = \epsilon^{-n}\Gamma(\frac{y}{\epsilon})$ is an approximate identity similar to $\phi^\epsilon(x)$ defined above. Note we also needed to apply the fact that the Gaussian is even. Accordingly,

$$I_\epsilon = f * \Gamma^\epsilon \tag{2.16}$$

and so we conclude that

$$I_\epsilon \to f \tag{2.17}$$

in $L^1$ as $\epsilon \to 0$.
Hence, we have seen that the functions $I_\epsilon$ converge pointwise to $\int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$ and converges in $L^1$ to $f$. This is only possible when (2.8) holds. □

With the powerful inversion theorem in hand, we now get a bounty of results for very little effort. Let us start by supplying the proof of (2.6) that was previously promised.

*Proof.* Recall we wish to show that given $f, g \in S$ we have $\widehat{fg} = \hat{f} * \hat{g}$. We then have $\widehat{\hat{f} * \hat{g}}(-x) = \hat{\hat{f}}(-x)\hat{\hat{g}}(-x) = f(x)g(x)$ as desired. □

Another important result that the inversion theorem allows is the following

**Theorem 2.18.** *The Fourier Transformation maps $S$ onto $S$*

*Proof.* Given $f \in S$ let $F(x) = f(-x)$ and let $g = \hat{F}$. Then $g \in S$. For if $h \in S$ then $h \in L^1$ and so $\hat{h}$ is bounded. Thus if $h \in S$ then $\widehat{D^\alpha x^\beta h}$ is bounded for any given $\alpha$ and $\beta$ since $D^\alpha x^\beta h$ is again in $S$. However, Theorems 1.7 and 1.10 imply that

$$\widehat{D^\alpha x^\beta} h(\xi) = (2\pi i)^{|\alpha|}(-2\pi i)^{-|\beta|}\xi^\alpha D^\beta \hat{h}(\xi)$$

and so we see that $\xi^\alpha D^\beta \hat{f}$ is again bounded, which means $\hat{f} \in S$. From here, we can see that the inversion theorem (and specifically 2.9) implies

$$\hat{g}(x) = \hat{\hat{F}}(x) = F(-x) = f(x)$$

and hence the desired result has been shown. □

As a final application of the inversion theorem, we now prove the second major theorem of this section, that of Plancherel.

**Theorem 2.19.** *If $u, v \in S$ then*

$$\int \hat{u}\bar{\hat{v}} = \int u\bar{v} \tag{2.20}$$

*Proof.* by the inversion theorem we have

$$\int u(x)\bar{v}(x)dx = \int \hat{\hat{u}}(-x)\bar{v}(x)dx = \int \hat{\hat{u}}(x)\bar{v}(-x)dx$$

Or, in other words

$$\int \hat{u}\bar{\hat{v}} = \hat{\hat{u}}\tilde{\bar{v}}$$

We apply the duality relation to the right side to obtain

$$\int u\bar{v} = \hat{u}\hat{\tilde{\bar{v}}}$$

from here, 2.19 follows from (1.6) □

Before we end this section we note that the Plancherel Theorem can be extended from $S$ to all of $L^2$ in the following way

**Theorem 2.21.** *There is a unique bounded operator $\mathcal{F} : L^2 \to L^2$ such that $\mathcal{F}f = \hat{f}$ when $f \in S$. In addition $\mathcal{F}$ is a unitary operator and if $f \in L^1 \cap L^2$ then $\mathcal{F}f = \hat{f}$*

*Proof.* Existence and uniqueness follows directly from Plancherel's Theorem.
To see that $\|\mathcal{F}f\|_2 = \|f\|_2$ simply take (2.19) with $u = v = f$ and take absolute values to get

$$\int |f\bar{f}| = \int |f|^2 = \int |\hat{f}\bar{\hat{f}}| = \int |\hat{f}|^2$$

which is exactly to say that $f$ and $\mathcal{F}f$ have identical $L^2$ norms. Given this isometry property, we see that the range of $\mathcal{F}$ must be closed. The fact that $\mathcal{F}$ is unitary will follow if we show the range is dense. To get this result we fix some $h \in L^1_{loc}$. Then we note that there is some fixed sequence $\{g_k\} \subset C_0^\infty$ such that if $p \in [1, \infty)$ and $h \in L^p$ then $g_k \to h$ in $L^p$. We now combine the existence of this sequence $g_k$ with the fact Fourier transform map $S$ onto $S$ to arrive at the desired result. It now only remains to prove that if $f \in L^1 \cap L^2$ then $\mathcal{F}f = \hat{f}$. For $f \in S$ this result is true by definition. So suppose that $f \in L^1 \cap L^2$. Then there is a sequence $\{g_k\} \subset S$ which converges to $f$ in both $L^1$ and in $L^2$, further by the boundedness of the Fourier transform $\hat{g}_k$ converges to $\hat{f}$ uniformly. On the other hand, $\hat{g}_k$ converges to $\mathcal{F}f$ in $L^2$ by the boundedness of the operator $\mathcal{F}$. It now follows that $\mathcal{F}f = \hat{f}$ $\square$

## 3. The Hausdorff-Young and Young Inequalities

We begin this section by noting the Riesz-Thorin Interpolation Theorem. We will use this result to help prove the Hausdorff-Young and Young inequalities, which will aid in bounding the $L^p$ norm of a function's Fourier transformation.

**Theorem 3.1.** *Let $T$ be a linear operator with domain $L^{p_0} + L^{p_1}$, $1 \le p_0 \le p_1 \le \infty$. Assume that $f \in L^{p_0}$ implies*

$$\|Tf\|_{q_0} \le A_0 \|f\|_{p_0} \tag{3.2}$$

*$f \in L^{p_1}$ implies*

$$\|Tf\|_{q_1} \le A_1 \|f\|_{p_1} \tag{3.3}$$

*for some $1 \le q_0, q_1 \le \infty$. Suppose that for a certain $\theta \in (0, 1)$*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{3.4}$$

*and*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{3.5}$$

*Then $f \in L^p$ implies*

$$\|Tf\|_q \le A_0^{1-\theta} A_1^{\theta} \|f\|_p$$

This result essentially says that if we can bound a linear operator is bounded for $L^{p_0} \to L^{q_0}$ and $L^{p_0} \to L^{q_1}$ then it is also bounded in every intermediate space.

$T$ in two different $L^p$ spaces $L^{p_1}, L^{p_2}$ with $p_1 \le p_2$ then we know the Operator is also bounded in any intermediate $L^p$ space $L^{p_n}$ where $p_1 \le p_n \le p_2$

We will now develop the Hausdorff-Young inequality. In doing so we will adopt the convention that the indices $p$ and $p'$ must satisfy

$$\frac{1}{p} + \frac{1}{p'} = 1$$

**Theorem 3.6.** *if* $1 \leq p \leq 2$ *then*

$$\|\hat{f}\|_{p'} \leq \|f\|_p \tag{3.7}$$

*Proof.* We will interpolate between the cases $p = 1$ and $p = 2$, which we already know. Apply theorem 3.1 with $p_0 = 1, q_0 = \infty, p_1 = q_1 = 2, A_0 = A_1 = 1$. The hypothesis of (3.2) and (3.3) now follow from the boundedness of the Fourier transform and theorem 2.20 respectively. For given $p, q$ existence of $\theta \in (0,1)$ for which (3.4) and (3.5) hold is equivalent to $1 < p < 2$ and $q = p'$ . The result follows. □

Having proved the Hausdorff-Young inequality, we will now make good use of the Riesz Interpolation Theorem one more time in order to prove Young's Inequality.

**Theorem 3.8.** *Let* $\phi \in L^p, \psi \in L^r$ *where* $1 \leq p, r \leq \infty$ *and* $\frac{1}{p} + \frac{1}{r} \geq 1$. *Let* $\frac{1}{q} = \frac{1}{p} - \frac{1}{r'}$. *Then the integral defining* $\phi * \psi$ *is absolutely convergent for a.e.* $x$ *and*

$$\|\phi * \psi\|_q \leq \|\phi\|_p \|\psi\|_r$$

*Proof.* define $T\psi = \phi * \psi$. Inequalities (2.3) and (2.4) imply that

$$T : L^1 + L^{p'} \to L^p + L^\infty$$

with

$$\|T\psi\|_p \leq \|\phi\|_p \|\psi\|_1$$
$$\|T\psi\|_\infty \leq \|\phi\|_p \|\psi\|_{p'}$$

If $\frac{1}{q} = \frac{1}{p} - \frac{1}{r'}$ then there is $\theta \in [0,1]$ with

$$\frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{p'}$$
$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{\infty}$$

The result now follows from theorem 3.1 □

## 4. The Uncertainty Principle

The Uncertainty Principle is not the name of a single theorem. Rather, it is the name given to a heuristic principle that a function $f$ and its Fourier Transform $\hat{f}$ can not both be localized to small sets. In this section, we will see two versions of this principle, Bernstein's inequality and the Heisenberg Uncertainty Principle. Both are quantitative manifestations of the idea that if $f$ is concentrated on a scale $R > 0$, then $\hat{f}$ can not be concentrated on a scale much less than $R^{-1}$.

We will begin by proving several different versions of Bernstein's Inequality. For our purposes, we can think of this inequality as saying that if a measure $\mu$ is supported on an ellipsoid $E$, than for many purposes $\hat{\mu}$ may be regarded as being constant on any dual ellipsoid $E^*$. The simplest rigorous manifestation of this principle is give by the $L^2$ Bernstein inequality.

**Theorem 4.1.** *Assume that $f \in L^2$ and $\hat{f}$ is supported in $D(0,R)$. Then $f$ is $C^\infty$ and there is an estimate*

$$\|D^\alpha f\|_2 \leq (2\pi R)^\alpha \|f\|_2 \tag{4.2}$$

*Proof.* This follows almost immediately from the Plancherel theorem. We note first that the Fourier inversion formula

$$f(x) = \int \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \tag{4.3}$$

is valid. Namely, note that the support assumption implies that $\hat{f} \in L^1$, so that the right side is the Fourier transformation of an $L^1 \cap L^2$ function. By theorem 2.20, it is equal to $f$. Theorem 1.7 applied to $\hat{f}$ now implies that $f$ is $C^\infty$ and that $D^\alpha f$ is obtained by differentiation under the integral sign in (4.3). The estimate (4.2) holds since

$$\|D^\alpha f\|_2 = \|\widehat{D^\alpha f}\|_2 = \|(2\pi i \xi)^\alpha \hat{f}\|_2 \leq (2\pi R)^{|\alpha|} \|\hat{f}\|_2 = (2\pi R)^{|\alpha|} \|f\|_2$$

□

We can develop a corresponding statement for $L^p$ norms, but without the Plancherel theorem we will be forced to employ different machinery.

**Lemma 4.4.** *There is a fixed Schwartz function $\phi$ such that if $f \in L^1 + L^2$ and $\hat{f}$ is supported in $D(0,R)$ then*

$$f = \phi^{R^{-1}} * f$$

*Proof.* take $\phi \in S$ so that $\hat{\phi}$ is equal to 1 on $D(0,1)$. Thus, $\widehat{\phi^{R^{-1}}}(\xi) = \hat{\phi}(R^{-1}\xi)$ is equal to 1 on $D(0,R)$ so $(\phi^{R^{-1}} * f - f)\hat{}$ vanishes identically. Hence, $\phi^{R^{-1}} * f = f$ as desired. □

With this lemma in hand we can now state and prove Bernstein's inequality for a disk.

**Theorem 4.5.** *Suppose that $f \in L^1 + L^2$ and $\hat{f}$ is supported in $D(0,R)$. Then for any $\alpha$ and $p \in [1,\infty]$*

$$\|D^\alpha f\|_p \leq (CR)^{|\alpha|} \|f\|_p \tag{4.6}$$

*and for any $1 \leq p \leq q \leq \infty$*

$$\|f\|_q \leq C R^{n(1/p - 1/q} \|f\|_p \tag{4.7}$$

*Proof.* The function $\psi = \phi^{R^{-1}}$ satisfies

$$\|\psi\|_r = C R^{\frac{n}{r'}} \tag{4.8}$$

for any $r \in [1,\infty]$, where $C = \|\phi\|_r$. Also, by the chain rule we have

$$\|\nabla \psi\|_1 = R \|\phi\|_1 \tag{4.9}$$

We know that $f = \psi * f$. in the case of the first derivatives, (4.6) therefore follows from (4.8) and (2.3). The general case of (4.6) then follows by induction. For (4.7) let $r$ satisfy $\frac{1}{q} = \frac{1}{p} - \frac{1}{r'}$. Apply Young's inequality to obtain

$$\begin{aligned} \|f\|_q &= \|\psi * f\|_q \\ &\leq \|\psi\|_r \|f\|_p \end{aligned}$$

$$\leq R^{f\frac{n}{r'}}\|f\|_p$$
$$= R^{n(1/p-1/q)}\|f\|_p$$

□

There is no reason that we should restrict the $L^p \to L^q$ bound to balls. Using change of variables, it is possible to extend this bound from balls to ellipsoids. We define an ellipsoid in $\mathbb{R}^n$ to be a set of the form

$$E = \{x \in \mathbb{R}^n : \sum_j \frac{|(x-a)\cdot e_j|^2}{r_j^2} \leq 1\} \tag{4.10}$$

for some $a \in \mathbb{R}^n$ (the center), some choice of orthonormal basis $\{e_j\}$ (the axes) and some choice of positive numbers $r_j$ (the axis length). If $E$ and $E^*$ are two ellipsoids, then we say that $E^*$ is dual to $E$ if $E^*$ has the same axes as $E$ and reciprocal axis lengths. Hence, if $E$ is given by (4.10) then $E^*$ will be of the form

$$\{x \in \mathbb{R}^n : \sum_j r_j^2 |(x-b)\cdot e_j|^2 \leq 1\} \tag{4.11}$$

for some choice of center point $b$.

With this formulation of the dual ellipsoid in mind, we can now extend Bernstein's inequality from a disk to an ellipsoid.

**Theorem 4.12.** *Suppose* $1 \leq p \leq q \leq \infty$. *Further, suppose that* $f \in L^1 + L^2$ *and* $\hat{f}$ *is supported in an ellipsoid* $E$. *Then*

$$\|f\|_q \leq |E|^{\frac{1}{p}-\frac{1}{q}}\|f\|_p \tag{4.13}$$

*Proof.* Let $k$ be the center of $E$. Let $T$ be a linear map taking the unit ball onto $E-k$. Let $S = T^{-t}$. Thus, $T = S^{-t}$ also. Let $f_1(x) = e^{-2\pi i k\cdot x}f(x)$ and $g = f_1 \circ S$ so that

$$\hat{g}(\xi) = |\det S|^{-1}\hat{f}_1(S^{-t}(\xi)) = |\det S|^{-1}\hat{f}(S^{-t}(\xi+k)) = |\det T|\hat{f}(T(\xi)+k)$$

Thus $\hat{g}$ and so by Theorem 4.5 we have $\|g\|_q \leq \|g\|_p$. On the other hand,

$$\|g\|_q = |\det S|^{-\frac{1}{q}}\|f\|_q = |\det T|^{\frac{1}{q}}\|f\|_q = |E|^{\frac{1}{q}}\|f\|_q$$

and likewise with $q$ replaced by $p$. And so,

$$|E|^{\frac{1}{q}}\|f\|_q \leq |E|^{\frac{1}{p}}\|f\|_p$$

and we are done □

These first two Bernstein inequalities are concerned with the behavior of $f$ in some global space $L^p$ given that $\hat{f}$ is supported over some region. We would also like to develop a related statement to capture the behavior of $f$ at a single point. This pointwise statement roughly says that if supp $\hat{f} \subset E$ then for any dual ellipsoid $E^*$, the values of f on $E^*$ are roughly controlled by its average over $E^*$. To develop a rigorous formulation of this principle, Let $N$ be some number. Also, let $\phi_k = (1+|x|^2)^{-N}$. Suppose an ellipsoid $R^*$ is given. Define $\phi_{E^*}(x) = \phi(T(x-k))$, where $k$ is the center of $E^*$ and $T$ is a self adjoint linear map taking $E^*-k$ onto the unit ball. If $T_1$ and $T_2$ are two such maps then $T_1 \circ T_2^{-1}$ is an orthogonal transformation, and so $\phi_{E^*}$ is well defined. By looking at the formulation

$$\phi_{E^*}(x) = (1 + \sum_j \frac{|(x-k)\cdot e_j|^2}{r_j^2})^{-N}$$

we come to see that $\phi_{E^*}$ is roughly equal to 1 on $E^*$ and decays rapidly as one moves away form $E^*$. More rapidly, in fact, than $\frac{1}{x^p}$ for $p \leq N$. With this in mind, we have the following rigorous version of a "pointwise" version of Bernstein's inequality.

**Theorem 4.14.** *Suppose that $f \in L^1 + L^2$ and $\hat{f}$ is supported in an ellipsoid $E$. Then for any dual ellipsoid $E^*$ and any $z \in E^*$*

$$|f(z)| \leq C_N \frac{1}{E^*} \int |f(x)|\phi_{E^*} dx \tag{4.15}$$

*Proof.* Assume first that $E$ is the unit ball, and $E^*$ is also the unit ball. Then $f$ is the convolution of itself with a fixed Schwartz function $\psi$. With this in mind we have

$$\begin{aligned} |f(z)| &\leq \int |f(x)||\psi(z-x)|dx \\ &\leq C_N \int |f(x)||1+|z-x|^2)^{-N} \\ &\leq C_N \int |f(x)|(1+|x|^2)^{-N} \end{aligned}$$

Where we used the Schwartz space bounds for $\psi$ and that $1+|z-x|^2 \geq |1+|x|^2$ uniformly in $x$ when $|z| \leq 1$. This proves (4.15) when $E = E^*$ =unit ball
Suppose now that $E$ is centered at zero but $E$ and $E^*$ are otherwise arbitrary. Let $k$ and $T$ be as above and consider

$$g(x) = f(T^{-1}x + k))$$

Its Fourier transformation is supported on $T^{-1}E$, and if $T$ maps $E^*$ onto the unit ball, then $T^{-1}$ maps $E$ onto the unit ball. Hence, if $y \in D(0,1)$

$$|g(y)| \leq \int \phi(x)|g(x)|dx$$

so that by changing variables we then have

$$f(T^{-1}z + k) \leq \int \phi(x)|f(T^{-1}x + k)|dx = |\det T| \int \phi_{E^*}(x)|f(x)|dx$$

Since $|\det T| = \frac{1}{|E^*|}$ we get (4.15) as desired.

□

We note that in a certain sense Theorem 4.14 is the "best we can do." Though it may sound appealing, it is not possible to make a stronger conclusion along the lines of when $x \in E^*$ then $|f(x)|$ is bounded by the average of $f$ over the double of $E^*$. The following example demonstrates this fact.
Let $E = E^*$ =unit interval. Consider a fixed Schwartz function $g$ with $g(0) \neq 0$ whose Fourier transformation is supported in the unit interval $[-1,1]$. Now consider the functions

$$f_N(x) = (1 - \frac{x^2}{4})^N g(x)$$

Since $\hat{f}_N$ are linear combinations of $\hat{g}$ and its derivatives, they have the same support as $\hat{g}$. Further still, they converge pointwise boundedly to zero on $[-2, 2]$, except at the origin. It follows then that there can be no estimate of the value of $f_N$ at the origin by its average over $[-2, 2]$.

While Bernstein's inequalities are useful, they are certainly not the most famous manifestation of the uncertainty principle. That distinction belongs to Heisenberg's uncertainty principle, which we will now prove to conclude this section.

**Theorem 4.16.** *for any $f \in S(\mathbb{R})$ one has*

$$\|f\|_2^2 \leq 4\pi\|(x - x_o)f_{\|2\|}(\xi - \xi_o)\hat{f}\|_2 \tag{4.17}$$

*Proof.* Without loss of generality let $x_o = \xi_o = 0$. Define $D = \frac{1}{2\pi i}\frac{d}{dx}$, and let $(Xf)(x) = xf(x)$. Then the commutator

$$[D, X] = DX - XD = \frac{1}{2\pi i}$$

and so for any $f \in S(\mathbb{R})$ we have

$$\begin{aligned}\|f\|_2^2 &= 2\pi i\langle [D, X]f, f\rangle = 2\pi i(\langle Xf, Df\rangle - \langle Df, Xf\rangle) \\ &= 4\pi \mathrm{Im}\langle Df, Xf\rangle \leq 4\pi\|Df\|_2\|Xf\|_2\end{aligned}$$

and we are done $\square$

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## References

[1] Thomas H. Wolff Lectures on Harmonic Analysis American Mathematical Society. 2003.