SPECTRAL EQUIVALENCE CLASSES OF TORI ARE FINITE

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Abstract. In this paper, we define the geodesic length spectrum for n-dimensional tori and the corresponding spectral equivalences classes. The goal of the paper is to describe how one can prove that each spectral equivalence class is finite by showing that with an appropriate metric, each equivalence class is both discrete and compact. We give a proof of discreteness for all dimensions as well as compactness for the 2-dimensional case. We also give a sketch of the general proof for compactness via Mahler's Compactness Theorem.

Contents

1. Introduction

When studying a set of mathematical objects geometrically, it is useful if one can classify those objects based on some geometric property that is somehow simpler than the objects one is studying. For instance, the radius of a circle (a 1-dimensional torus) allows one to completely classify the circle geometrically (two circles with the same radius will be geometrically congruent). For n-dimensional tori (with $n > 1$), the situation is not quite so simple. However, one can define the length spectrum of a torus to be the set of lengths of geodesic curves (essentially lines on the surface of the torus) along with their multiplicities. The length spectrum does not completely classify n-tori but it does specify certain properties - tori with the same length spectrum have the same volume. Additionally, the size of each spectral equivalence class of tori is finite (up to certain notions of equivalence). Thus, although the length spectrum does not completely classify n-tori, it comes close.

In this paper, we will focus on proving that these spectral equivalence classes are finite. To do this, we want to avoid explicitly counting tori since this can be impractical. Instead, we prove that spectral equivalence classes are discrete and compact. Discreteness allows us to construct a ball around each point in an

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equivalence class so that it intersects no other point in that class. This is an open cover of the spectral equivalence class and so compactness tells us there must be a finite subcover. However, since each ball only contains one element of the set, this means the set is necessarily finite.

2. Definitions

We begin with some basic definitions.

Definition 2.1. We define $GL(n, \mathbb{R})$ to be the group under multiplication of invertible $n \times n$ matrices over the reals.

And similarly:

Definition 2.2. We define $GL(n, \mathbb{Z})$ to be the group of unimodular $n \times n$ matrices over the integers.

Definition 2.3. We define the orthogonal group,

$$O(n) = \{M \in GL(n, \mathbb{R}) : MM^T = I\}.$$

We can view elements of $O(n)$ as specifying linear maps from $\mathbb{R}^n$ to itself that preserve the dot product.

Definition 2.4. An n-dimensional lattice is defined as

$$L = \{x \in \mathbb{R}^n : x = a_1 v_1 + ... + a_n v_n, a_i \in \mathbb{Z}\}$$

where $\{v_1, ... v_n\}$ is a basis for $\mathbb{R}^n$. The space of n-dimensional lattices is denoted L^n.

Example 2.5. The standard euclidean basis for $\mathbb{R}^n$ generates the n-dimensional lattice whose points are just the elements of $\mathbb{Z}^n$.

Intuitively, we view a lattice as a uniformly discrete structure and in fact, we have the following result.

Lemma 2.6. *For any lattice L , there exists an $\epsilon > 0$ so that for any lattice point $p \in L$ there exists a ball of radius ϵ around p that contains no other lattice points.*

Proof. Choose a basis $B = (v_1, v_2, ..., v_n)$ for the lattice L and define the following norm. If $w = a_1 v_1 + a_2 v_2 + ... + a_n v_n$ then $||w||_B = \max(|a_1|, |a_2|, ..., |a_n|)$. Now, since all lattice points have integer coordinates with respect to the basis B, the distance between any two lattice points in this norm is at least 1. But all norms on $\mathbb{R}^n$ are equivalent in the sense that there exist constants $C, D > 0$ so that for all $x \in \mathbb{R}^n$, $C||x||_B \leq ||x|| \leq D||x||_B$ where $||\cdot||$ denotes the standard euclidean norm on $\mathbb{R}^n$. Thus, there exists some $\epsilon > 0$ (namely $\epsilon = C$) so that a ball of radius ϵ around a lattice point p contains no other lattice points. □

Definition 2.7. Let L be an n-dimensional lattice. Define the following equivalence relation $\sim_L$ on $\mathbb{R}^n$ by $x \sim_L y$ if $x - y \in L$. Then the n-dimensional torus, T, is defined as the set of equivalence classes determined by $\sim_L$ and is denoted $\mathbb{R}^n/L$. In other words, an n-dimensional torus is the quotient space $\mathbb{R}^n/L$.

3. Construction of the space of Tori

Associated with the set of n-dimensional tori is a metric space containing these tori as elements. Our next goal is to construct this space.

Let $\mathscr{B}_n$ be the set of ordered bases for $\mathbb{R}^n$. Define the map $B_n : \mathscr{B}_n \to GL(n, \mathbb{R})$ by:

$$B_n(\{\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_n}\}) = [[\mathbf{b_1}]_\mathbf{S}, [\mathbf{b_2}]_\mathbf{S}, ..., [\mathbf{b_n}]_\mathbf{S}] \in \mathbf{GL}(\mathbf{n}, \mathbb{R})$$

where $\mathbf{b_i}$ is the ith basis vector of an ordered basis and $[\cdot]_S$ gives the coordinate column vector with respect to the standard euclidean basis of $\mathbb{R}^n$. Since $\{b_1, ..., b_n\}$ is linearly independent, its image is indeed in $GL(n, \mathbb{R})$. Moreover, note that B_n is clearly injective. The map is also surjective since any invertible matrix has linearly independent column vectors and so corresponds to some ordered basis. Thus, B_n is bijective and so it makes sense to view ordered bases as elements of $GL(n, \mathbb{R})$. Since we can make the identification $M(n, \mathbb{R}) \simeq \mathbb{R}^{(n^2)}$, the map B_n also allows us to view $\mathscr{B}_n$ as a metric space where the distance between ordered bases is given by their corresponding euclidean distance in $\mathbb{R}^{(n^2)}$.

Observe that each ordered basis generates a lattice. If we consider the map $L_n : GL(n, \mathbb{R}) \to L^n$ defined by

$$L_n([[\mathbf{b_1}]_\mathbf{S}^\mathrm{T}, [\mathbf{b_2}]_\mathbf{S}^\mathrm{T}, ..., [\mathbf{b_n}]_\mathbf{S}^\mathrm{T}]) = \{\mathbf{x} \in \mathbb{R}^\mathbf{n}, \mathbf{a_i} \in \mathbb{Z} : \mathbf{x} = \mathbf{a_1 b_1} + ... + \mathbf{a_n b_n}\} \in \mathbf{L^n}$$

where we map each basis to the lattice it produces, we see that L_n is definitely surjective but is not necessarily injective. However, the following result does hold:

Lemma 3.1. *There exists a bijection between $GL(n, \mathbb{R})/GL(n, \mathbb{Z})$ and L^n.*

Proof. Consider the function that maps an equivalence classes of bases in $GL(n, \mathbb{R})/GL(n, \mathbb{Z})$ to the lattice those bases produce. First, to check that this function is well-defined, we must show that if there exists an $A \in GL(n, \mathbb{Z})$ so that for $B, B' \in GL(n, \mathbb{R})$, $BA = B'$ then B and B' produce the same lattice points. But the existence of A means that we can write the basis vectors of B' as linear combinations of the basis vectors of B with integer coefficients. Thus for any point p in the lattice generated by B', we can write p as a linear combination of basis vectors of B with integer coefficients and so the lattice generated by B contains the point p. Thus we have $L_{B'} \subset L_B$. But since $GL(n, \mathbb{Z})$ is a group, there exists an $A^{-1} \in GL(n, \mathbb{Z})$ so that $AA^{-1} = I$. Thus, $BAA^{-1} = B'A^{-1}$ and so $B = B'A^{-1}$. We can use the same argument as before to show that the lattice generated by B' contains all of the lattice points generated by B and so we have shown that the two bases generate the same lattice.

The function we have defined is surjective because by definition, any lattice is generated by some basis. The function is also injective since if B, B' produce the same lattice then the basis vectors of B' must be lattice points of B. This means that we can write each basis vector of B' as a linear combination of the B basis vectors. We can use this to construct an integer matrix A where $BA = B'$. Using the same approach, we can construct A' so that $B'A' = B$. But then we have $B'A'A = B'$ and so multiplying by B'^{-1} (which exists since $B' \in GL(n, \mathbb{R})$) we see that $A'A = I$. Similarly, $AA' = I$ and so A is invertible. Thus $A \in GL(n, \mathbb{Z})$ and so B and B' are equivalent in $GL(n, \mathbb{R})/GL(n, \mathbb{Z})$. Thus we have shown that the function is a bijection. $\square$

The previous lemma means that we can view the space of n-dimensional lattices as $GL(n, \mathbb{R})/GL(n, \mathbb{Z})$. Moreover, we can define a metric on this space by specifying that for $L, L' \in GL(n,\mathbb{R})/GL(n,\mathbb{Z})$, the distance between L and L' is the minimum of the set of euclidean distances in $\mathbb{R}^{(n^2)}$ between elements of the equivalence classes of bases for L and L'.

Associated with each lattice L, we have the torus T_L. However, for geometric reasons, it makes sense to view tori $T_L, T_{L'}$ as equivalent if there exists an $A \in O(n,\mathbb{R})$ that uniquely associates each $v \in L$ with a $v' \in L'$ so that $v' = Av$. In conclusion then, we can view the space of tori as $(GL(n,\mathbb{R})/GL(n,\mathbb{Z}))/O(n,\mathbb{R})$. The distance between two equivalence classes of tori is the minimum euclidean distance between the set of bases associated with each equivalence class. Essentially, this means that two tori T, T' are near each other if there exist corresponding bases B, B' whose coordinates with respect to the standard euclidean basis differ by a small amount.

4. Length Spectrum

Definition 4.1. For any lattice L we define the geodesic length spectrum $\mathscr{L}(L)$ to be the set of ordered pairs (l, m_l) where l is the length of some lattice point in L with the standard norm and m_l is the number of $v \in L$ such that $||v|| = l$.

Notice that the geodesic length spectrum is well-defined for tori since if two lattices produce the same torus, the lengths of their lattice points and therefore their length spectrums are the same. If two tori have the same length spectrum, we say that they are spectrally equivalent.

Definition 4.2. For any torus T, we define $SE(T)$ to be the set of tori that are spectrally equivalent to T.

Clearly if two tori are equivalent, then they must be spectrally equivalent, but the converse is not true in general. For instance, a paper by Milnor [1] shows the existence of two distinct 16-dimensional tori that are spectrally equivalent. However, the following result, known as Jacobi's Identity [2], does hold:

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} = \frac{vol(T_L)}{(4\pi t)^{\frac{n}{2}}} \sum_{v \in L} e^{\frac{-||v||^2}{4t}}, t > 0 \tag{4.3}$$

In the above equation, T_L is a torus associated with the lattice L, $\{\lambda_i\}$ corresponds to the set of eigenvalues of the Laplacian on T_L, $vol(T_L)$ is the absolute value of the determinant of a basis for T_L, and n is the dimension of L.

Now, it is a known result [3] that the set of eigenvalues of the Laplacian on a torus only depends on the length spectrum of that torus. Moreover, by definition, the same is true of the term $\sum_{v \in L} e^{\frac{-||v||^2}{4t}}$. Thus, since all the other terms of the above identity are determined only by the length spectrum of the torus in question, we must have that $vol(T_L)$ is the same for all spectrally equivalent tori.

5. Discreteness

We now state and prove the first important result of the paper.

Proposition 5.1. *For any torus T, the set $SE(T)$ is discrete.*

Proof. Pick any ordered basis $B = (v_1, ..., v_n)$. B specifies a lattice in $\mathbb{R}^n$ containing the points $v_i, v_i + v_j$ for $1 \le i, j \le n$.

From Lemma 2.6, it follows that if we take a ball of some radius around the origin, it can only contain a finite number of lattice points. Thus the set of lengths in the lattice is discrete, and so for each point p of the form just specified (v_i and $v_i + v_j$), there exists a $\delta_p > 0$ such that any lattice point of length different from p differs in length from p by at least δ_p. Since each δ_p is positive and there are only finitely many of them, we have $\delta := \min(\delta_p)_{p\in P} > 0$.

Let $\epsilon = \frac{\delta}{2}$. Now, select any ordered basis B' so that $||B - B'|| < \epsilon$ where we recall that this is an euclidean distance in $\mathbb{R}^{(n^2)}$. By the definition of $||B - B'||$ and the reverse triangle inequality, we have that for any basis vector $v_i \in B$ and the corresponding $v_i' \in B'$,

$$\begin{aligned} |||v_i'|| - ||v_i||| &\leq ||v_i' - v_i|| \\ &\leq ||B' - B|| \\ &< \epsilon \\ &< \delta. \end{aligned}$$

Also for any i, j

$$\begin{aligned} ||v_i' + v_j'|| - ||v_i + v_j|| &\leq ||v_i' + v_j|| + ||v_j' - v_j|| - ||v_i + v_j|| \\ &\leq ||v_i' - v_i|| + ||v_i + v_j|| + ||v_j' - v_j|| - ||v_i + v_j|| \\ &= ||v_i' - v_i|| + ||v_j' - v_j|| \\ &< 2\epsilon. \end{aligned}$$

A similar argument yields

$$||v_i + v_j|| - ||v_i' + v_j'|| < 2\epsilon$$

and so we have

$$|||v_i' + v_j'|| - ||v_i + v_j||| < 2\epsilon = \delta.$$

Next, we observe that the lengths, $||v_i||$ and $||v_i + v_j||$ completely determine the lengths of all the lattice points generated from B. This follows since the lengths are obviously determined by all dot products of the form $v_i \cdot v_i$ and $v_i \cdot v_j$ and these dot products in turn are determined by the above lengths since $||v_i|| = \sqrt{v_i \cdot v_i}$, which determines $v_i \cdot v_i$, and $||v_i + v_j|| = v_i \cdot v_i + v_j \cdot v_j + 2(v_i \cdot v_j)$, which determines $v_i \cdot v_j$ since $v_i \cdot v_i$ and $v_j \cdot v_j$ are already determined by $||v_i||$ and $||v_j||$.

Suppose it is the case that B and B' are ordered bases such that for all i, j where $1 \leq i, j \leq n$, we have $||v_i'|| = ||v_i||$ and $||v_j' + v_i'|| = ||v_j + v_i||$. Then, we can construct a matrix A, that maps each lattice point over B to a lattice point over B' so that if α is the coordinate of the first lattice point in the v_i direction then α is the coordinate of the second lattice point in the v_i' direction. Now since (as argued above) $||v_i||$ and $||v_i + v_j||$ completely determine all dot products and since we have assumed that $||v_i|| = ||v_i'||$ and $||v_i + v_j|| = ||v_i' + v_j'||$, this map will preserve all dot products. Thus, $A \in O(n)$, which means that B and B' produce equivalent tori.

If the above condition does not hold, then we have

$$||v_i|| \neq ||v_i'|| \text{ or } ||v_i + v_j|| \neq ||v_i' + v_j'||$$

for some $1 \leq i, j \leq n$. Consider the case where $||v_i' + v_j'|| \neq ||v_i + v_j||$. Recall that from above we have

$$|||v_i' + v_j'|| - ||v_i + v_j||| \leq \delta.$$

It follows from the definition of δ that $||v_i' + v_j'||$ can correspond to no length in the B lattice and so B and B' are not spectrally equivalent. Similarly, in the case where $||v_i|| \neq ||v_i'||$ then since $|||v_i'|| - ||v_i||| < \delta$, $||v_i'||$ can correspond to no length in B. Thus, either way, B and B' are not spectrally equivalent.

Observe however that the ϵ we chose is only determined by the torus that B generates and not the basis itself. This means we have shown that for each torus $T \in SE(T)$, we can find an $\epsilon > 0$ so that for any $T' \in SE(T)$ where $T' \neq T$, we have that $d(T, T') \geq \epsilon$. Thus $SE(T)$ is discrete. □

6. Compactness

We recall that the goal of the paper is to show that for any torus T, the set $SE(T)$ is finite. As discussed, we need only show that $SE(T)$ is compact and discrete. To show compactness, it suffices to show that $SE(T)$ is closed and bounded.

Proposition 6.1. *Each spectral equivalence class is closed*

Proof. Pick any spectral equivalence class $SE(T)$ and suppose we have a sequence (T_n) of tori with each $T_n \in SE(T)$ that converges to some torus T^*. We need to show that $T^* \in SE(T)$. Since (T_n) converges, by definition there exists a representative ordered basis for each torus in (T_n) such that the sequence of bases (B_n) converges to some basis B in $\mathbb{R}^{(n^2)}$. From the definition of distance between ordered bases, each sequence of corresponding basis vectors must converge and thus, each sequence of corresponding lattice points must also converge.

Now suppose for contradiction that $T^* \notin SE(T)$ so that $\mathscr{L}(T) \neq \mathscr{L}(T^*)$. Then there exists some length l (not necessarily in both spectra) whose multiplicity is not the same in both spectra.

If the multiplicity is higher in $\mathscr{L}(T^*)$ then at least one of the sequences of corresponding lattice points converging to a point with length l must have terms whose lengths are not eventually constant (since otherwise $\mathscr{L}(T)$ would have a multiplicity for length l lattice points greater than or equal to that of $\mathscr{L}(T^*)$). This is a contradiction since it means that for any $\epsilon > 0$, there are infinitely many lattice points of length less than $l + \epsilon$.

If we let the multiplicity of l in $\mathscr{L}(T)$ be s and this multiplicity is higher than in $\mathscr{L}(T^*)$, then we cannot have s distinct sequences of corresponding points that converge to a point of length l. However in each torus of (T_n), we have s lattice points of length l. This means that for any k, there exists a sequence $(p_n)_k$ of corresponding points that converges to some lattice point p with $||p|| \neq l$ but where $||p_k|| = l$. Now we can write $p_k = a_1 \cdot (v_1)_k + a_2 \cdot (v_2)_k + ...a_m \cdot (v_m)_k$ for some a_i where $(v_1)_k, ..., (v_m)_k$ are the basis vectors of B_k. The sequences of corresponding basis vectors all converge, and yet, since the length spectrum is discrete, there is a fixed minimum distance between p_k and p. The only way this can be true for arbitrarily large k is if $\max(|a_i|_{1 \leq i \leq m})$ also becomes arbitrarily large. For each k there are a finite number of a_i and so there must exist a subsequence of the p_k where the corresponding sequence of $|a_j|$ grows arbitrarily large. Now the basis vectors of B_k for any k are linearly independent, which is true if and only if each basis vector has a component orthogonal to the other basis vectors. If we denote this component of the basis vector $(v_j)_k$ as $(v_j^\perp)_k$ then we see that $||p_k|| \geq |a_j| \cdot ||(v_j^\perp)_k||$. Since $|a_j|$ grows arbitrarily large and each p_k has a fixed length l, we have that $||(v_j^\perp)_k||$ must become arbitrarily small and so this subsequence of the $(v_j^\perp)_k$ converges to 0. But the sequences of corresponding basis vectors all converge and so the sequence of corresponding $(v_j^\perp)$ must also converge. Since the subsequence converges to 0, the full sequence must also converge to 0. Thus, in the limit, the orthogonal component of $v_j \in B$ is 0. This means that the set of basis vectors for B is linearly dependent which is a contradiction. Thus, we conclude that $T^* \in SE(T)$.

□

The final result which we need, that $SE(T)$ is bounded, is significantly more difficult to prove than the other results. To prove the full result, we would use Mahler's Compactness Theorem [4], which essentially states that a sequence of lattices that has a fixed volume and fixed length of the shortest nonzero vector must be bounded. Since by section 4 and 2.6, these criteria are met by any sequence of spectrally equivalent tori, we have the result that the closure of $SE(T)$ is compact. But of course from the previous result, this just means that $SE(T)$ is compact. Below, we present an elementary proof of boundedness for 2-dimensional tori.

Proposition 6.2. *For any 2-torus T, $SE(T)$ is bounded in the space of 2-tori.*

Proof. Pick any set of spectrally equivalent tori $SE(T)$. To prove that $SE(T)$ is bounded, it suffices to show that if we pick a representative ordered basis from each equivalence class of 2-tori (up to $GL(n,\mathbb{Z})$ and $O(n)$) then the set of those representatives are bounded in $\mathbb{R}^4$, to do this, we need only show that the coordinates of the representatives are all bounded.

First, from 2.6, we see that there exists a smallest length positive length L in the length spectrum of $SE(T)$. Also, we note that from section 4 we have that the volume is the same for any spectrally equivalent 2-tori and so we can denote the volume of elements of $SE(T)$ to be V.

Now, pick any $\tau \in SE(T)$ and choose an ordered basis $B = (v_1, v_2)$ for τ. Since the basis B produces a lattice with at least one lattice point of length L, there exist $p, q \in \mathbb{Z}$ so that $pv_1 + qv_2 = u$ and $||u|| = L$. Moreover, $\gcd(p,q) = 1$ since if this were not the case, we could use $\frac{p}{\gcd(p,q)}$ and $\frac{q}{\gcd(p,q)}$ as coordinates to produce a vector u' with a smaller norm than u which is impossible. It is an elementary fact of number theory that we can find integers r, s so that $pr - qs = 1$ when $\gcd(p,q) = 1$. Now consider the matrix $A_1 = \begin{pmatrix} p & s \\ q & r \end{pmatrix}$ Since the determinant $pr - qs = 1$, we have that $A_1 \in GL(n,\mathbb{Z})$ and so we can let $B' = BA_1$ so that B' is a new basis for τ. Notice that in this case, the first basis vector of B' is precisely u. Since rotations and reflections are elements of $O(n)$ we can rotate the basis vectors of B' so that the 2nd coordinate of the first basis vector is 0 (in other words, the first basis vector lies along the $x-axis$). If the 2nd basis vector's 2nd coordinate is negative, then we can apply a reflection across the $x-axis$ that leaves the first basis vector unchanged and makes the 2nd coordinate positive. Note that this coordinate is nonzero since otherwise the vectors are linearly dependent. Thus we now have a basis B'' for τ with basis vectors $(L,0), (a,b)$ where $L, b > 0$. Note that the volume of B'' is equal to $|\det(B')| = |bL| = bL$. Finally, there exists an $n \in \mathbb{Z}$ so that $0 \le a + nL < L$. Then we can let $A_2 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where clearly, $A_2 \in GL(n,\mathbb{Z})$ and $M = B''A_2$ is a basis for τ with vectors $(L,0), (a',b)$ where $a' = a + nL$. Now, L is constant, and since volume is fixed, b is constant and equals $\frac{V}{L}$. Finally, $0 \le a' < L$. Thus, all the coordinates are bounded and so we can let M be our representative matrix for

τ. Now, since these bounds only depend on the length spectrum of τ, any other spectrally equivalent torus to τ can yield a basis representative subject to the same bounds. This shows that $SE(T)$ is bounded. $\square$

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