APPROXIMATING THE RANDOM WALK USING THE CENTRAL LIMIT THEOREM

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ABSTRACT. This paper will define the random walk on an integer lattice and will approximate the probability that the random walk is at a certain point after a certain number of steps by using a modified version of the Central Limit Theorem. To accomplish this, we will define the characteristic function of the random walk, find the Taylor expansion of this function, and bound the difference between this function and the estimate. Finally, this paper will demonstrate an application of the LCLT by proving the the simple random walk is recurrent in 1 and 2 dimensions, but transient in all higher dimensions.

CONTENTS

1. INTRODUCTION AND DEFINITIONS

The random walks considered in this paper will be restricted to the lattice $\mathbb{Z}^d$. It can be shown that a random walk on any other lattice is isomorphic to a random walk on $\mathbb{Z}^d$, and so the results of this paper can be extended to random walks on any lattice. Using $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, 0, \ldots, 1)$ as the standard basis of unit vectors for $\mathbb{Z}^d$, we define the lattice $\mathbb{Z}^d$ as all vectors of the form $x = (x^1e_1 + \cdots + x^de_d : x^j \in \mathbb{Z})$ A sequence of points $x_0, x_1, \ldots, x_n \ldots$ in $\mathbb{Z}$ can be written as $x_j = (x_j^1, x_j^2, \ldots, x_j^d)$, where the superscripts denote the vector components, while the subscript denotes the cardinality of the point in the sequence.

Definition 1.1. In this paper, a **random walk** p is the sum of discrete, identically distributed random variables X, where $\mathbb{P}[X = y] = 0$ for all $y \notin \mathbb{Z}^d$ and $\sum_{y \in \mathbb{Z}^d} \mathbb{P}[X = y] = 1$. This sum is written as

$$S_n = S_0 + X_1 + + \cdots + X_n$$

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For the rest of this paper, S_0 is given the trivial distribution $\mathbb{P}[S_0 = 0] = 1$, meaning that the random walk starts at the origin. Note that the random walk can also be defined as a Markov chain with

$$\mathbb{P}[S_{n+1} = z | S_n = y] = \mathbb{P}[X = (z - y)]$$

For the rest of this paper we will use the following notation.

$$p(x) = \mathbb{P}[X = x]$$

$$p_n(x) = \mathbb{P}[S_n = x]$$

The **simple** random walk has the distribution $\mathbb{P}[X_j = e_k] = \mathbb{P}[X_j = -e_k] = 1/(2d)$ for $k = 1, \ldots, d$.

The defintion of the random walk given above is quite general, and before moving on with the paper it is useful to add several more conditions that will make proofs easier down the line. We define the random walk to be

- Aperiodic: $\mathbb{P}[S_n = 0] > 0$ for all n
- Irreducible: $\forall y \in \mathbb{Z}^d, \exists N$ such that if $n \geq N$ then $\mathbb{P}[S_n = y] > 0$
- Symmetric: $\mathbb{P}[X = y] = \mathbb{P}[X = -y], \forall y \in \mathbb{Z}^d$

With these conditions in mind, we write the following.

Definition 1.2. Consider a set of vectors $V = (x_1, \ldots, x_l) \subset \mathbb{Z}^d \setminus [0]$ that spans $\mathbb{Z}^d$; that is $\forall y \in \mathbb{Z}^d, \exists k_1, \ldots, k_l \in \mathbb{Z}$ s.t $k_1 x^1 + \cdots + k_l x^l = y$. Let there be function $\mathcal{K} : x_j \in V \mapsto (0, 1)$. An **aperiodic, irreducible, symmetric** random walk is defined by vector set V and function $\mathcal{K}$ where

$$\mathcal{K}(x_1) + \cdots + \mathcal{K}(x_l) < 1$$

$$\mathcal{K}(0) = 1 - \sum_{j=1}^{l} \mathcal{K}(x_j)$$

$$\mathbb{P}[X = x_j] = \mathbb{P}[X = -x_j] = \frac{1}{2}\mathcal{K}(x_j)$$

We let $\mathcal{P}$ denote the set of all random walks of this form, and say random walk $p \in \mathcal{P}$ if it satisfies these conditions.

It is important to note that the simple random walk defined above is not in $\mathcal{P}$. The simple random walk is symmetric and irreducible, but not aperiodic, since the probability that the walk is at the origin is 0 if n is odd. However, if one considers only even steps then the simple random walk is aperiodic, so the results of this paper can be extended to describe some of the behaviors of periodic walks.

Now that we have defined the type of random walk that will be examined in this paper, we will examine several ideas associated with the expected value of the random variable X.

Definition 1.3. We define the $(j_1, \ldots j_d)$ moment of X as:

$$\mathbb{E}[(X^1)^{j_1} \ldots (X^d)^{j_d}] = \sum_{x \in \mathbb{Z}^d} (x^1)^{j_1} \ldots (x^d)^{j_d} p(x)$$

The **covariance matrix** of X, called Γ, is a $d \times d$ matrix where the (k, l) entry of the matrix is given by:

$$\Gamma_{kl} = \mathbb{E}[(X^k)(X^l)] = \sum_{x \in \mathbb{Z}^d} (x^k)(x^l)p(x) \tag{1.4}$$

This corresponds to the moment where $j_k = j_l = 1$, and all other moment values are equal to 0.

We will now prove several facts about the expected value of different moments of X for $p \in \mathcal{P}$ that will be used later.

Lemma 1.5. *For all $p \in \mathcal{P}$ with covariance matrix Γ, and for all $v \in \mathbb{Z}^d$, the following are true:*

(1) $\mathbb{E}[(X \cdot v)^m] = 0$ *if m is odd*
(2) $\mathbb{E}[(X \cdot v)^2] = v \cdot \Gamma v > 0$

Proof. To prove (1), note that for $\mathbb{E}[(X \cdot v)^m] = \sum_{x \in \mathbb{Z}^d} (x \cdot v)^m p(x)$, we only need to examine $x \in \mathbb{Z}^d$ where both $p(x) \neq 0$ and $x \cdot v \neq 0$, because otherwise $(x \cdot v)^m p(x) = 0$. For each of these remaining x, we know that $p(-x) = p(x)$ by the definition of $\mathcal{P}$. Therefore, if m is odd then $(x \cdot v)^m p(x) + ((-x) \cdot v)^m p(-x) = 0$, which shows that the sum over $\mathbb{Z}^d$ is also 0.

For (2), $\mathbb{E}[(X \cdot v)^2] > 0$ is clear from the fact that the set of vectors V as defined in 1.2 is a generating set for $\mathbb{Z}^d$, which implies there must be some $x_j \in V$ with $x_j \cdot v \neq 0$. For the rest,

$$\mathbb{E}[(X \cdot v)^2] = \sum_{x \in \mathbb{Z}^d} (x \cdot v)^2 p(x) = \sum_{x \in \mathbb{Z}^d} (x^1 v^1 + \cdots + x^d v^d)^2 p(x)$$

$$\mathbb{E}[(X \cdot v)^2] = \sum_{x \in \mathbb{Z}^d} (\sum_{j=1}^{d} (\sum_{k=1}^{d} x^j v^j x^k v^k)) p(x) \tag{1.6}$$

Similarly,

$$v \cdot \Gamma v = \sum_{j=1}^{d} v^j (\Gamma v)^j = \sum_{j=1}^{d} v^j \sum_{k=1}^{d} v^k \mathbb{E}[X^j X^k]$$

$$v \cdot \Gamma v = \sum_{j=1}^{d} v^j \sum_{k=1}^{d} v^k \sum_{x \in \mathbb{Z}^d} x^j x^k p(x)$$

When rearragned, the final expression becomes (1.6), proving the lemma. □

2. Local Central Limit Theorem

The Central Limit Theorem states that if $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables in $\mathbb{R}$ with mean 0 and variance σ^2, then the distribution of

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

approaches a normal distribution with mean 0 and variance σ^2 as n tends towards ∞. The random walk is defined as the sum of independent, identically distributed random variables, and so it seems reasonable to guess that $p \in \mathcal{P}$ in one dimension can be approximated with a normal distribution if n is large enough, as is shown below.

$$p_n(k) = \mathbb{P}[S_n = k] \approx \mathbb{P}[\frac{k}{\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} < \frac{k+1}{\sqrt{n}}]$$
$$\approx \int_{\frac{k}{\sqrt{n}}}^{\frac{k+1}{\sqrt{n}}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy$$

Similarly, it seems reasonable to guess that a joint normal distribution is a good estimate for $p \in \mathcal{P}$ in d-dimensions. If Γ is the covariance matrix for random variable X, then the normalized sums of X, $\frac{S_n}{\sqrt{n}}$, approach the following distribution, which we call $\overline{p}_n$.

$$\overline{p}_n(x) = \frac{1}{(2\pi n)^{d/2}\sqrt{det\Gamma}}\ e^{-\frac{(x\cdot\Gamma^{-1}x)^2}{2n}} = \frac{1}{(2\pi)^d n^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{s\cdot x}{\sqrt{n}}} e^{-\frac{s\cdot\Gamma s}{2}} d^d s \tag{2.1}$$

The middle expression in the equation is easily evaluated. The Local Central Limit Theorem (LCLT) justifies this estimate, and is stated below.

Theorem 2.2. *The* ***Local Central Limit Theorem*** *states that for $p \in \mathcal{P}$, $p_n(x)$ as defined in (1.3) and $\overline{p}_n$ as defined in (2.1), there exists $c \in \mathbb{R}$ such that for all $n > 0$ and $x \in \mathbb{Z}^d$:*

$$|p_n(x) - \overline{p}_n(x)| < \frac{c}{n^{(d+2)/2}} \tag{2.3}$$

To prove this theorem, we will first define the characteristic function of random variable X and find an inversion formula relating $p_n(x)$ to its characteristic function. We will then use the inversion formula and the last expression in equation (2.1) to find and bound the difference between $p_n(x)$ and $\overline{p}_n(x)$.

3. The Characteristic Function

Definition 3.1. The **characteristic function** of $\mathbb{R}^d$-valued random variable $X = (X^1, \ldots X^d)$ is defined as the function $\phi : \mathbb{R}^d \mapsto \mathbb{C}$, where

$$\phi(\theta) = \mathbb{E}[e^{i\theta\cdot X}] = \sum_{x\in\mathbb{R}^d} e^{i\theta\cdot x} p(x) \tag{3.2}$$

In this section, we will prove several properties of the characteristic function that will help to prove the LCLT.

Lemma 3.3. $\phi(0) = 1$, $|\phi(\theta)| \leq 1, \forall\theta \in \mathbb{R}^d$, *and* $\phi(\theta)$ *is continuous* $\forall\theta \in \mathbb{R}^d$.

Proof.

$$\phi(0) = \mathbb{E}[e^{iX\cdot 0}] = \mathbb{E}[1] = 1$$

$$|\phi(\theta)| = \left|\sum_{x\in\mathbb{R}^d} p(x)e^{ix\cdot\theta}\right| \leq \sum_{x\in\mathbb{R}^d} \left|p(x)e^{ix\cdot\theta}\right| \leq \sum_{x\in\mathbb{R}^d} |p(x)| = 1$$

To prove continuity:

$$|\phi(\theta+\theta_1)-\phi(\theta)| = \left|\sum_{x\in\mathbb{R}^d} e^{i(\theta+\theta_1)\cdot x}p(x) - \sum_{x\in\mathbb{R}^d} e^{i\theta\cdot x}p(x)\right|$$

$$= \left|\sum_{x\in\mathbb{R}^d} (e^{ix\cdot\theta})(e^{ix\cdot\theta_1}-1)p(x)\right| \leq \sum_{x\in\mathbb{R}^d} |(e^{ix\cdot\theta_1}-1)p(x)|$$

Since $\lim_{\theta_1\to 0}|e^{ix\theta_1}-1| = 0$ and since $|p(x)| > 0$ for only a finite number of x, $lim_{\theta_1\to 0}|\phi(\theta+\theta_1)-\phi(\theta)| = 0 \ \forall\theta$, showing that $\phi(\theta)$ is continuous everywhere. □

We want to be able to differentiate $\phi(\theta)$ in order to use Taylor expansion to obtain an approximation, and to do this we will define $\phi_u(s)$ as the one-dimensional characteristic function of random variable $X\cdot u$, with $|u| = 1$ and $\theta = su$. Differentiating gives us:

$$\phi_u^{(m)}(s) = i^m\mathbb{E}[(X\cdot u)^m e^{i(X\cdot u)s}] \tag{3.4}$$

Using Taylor expansion around the origin and the remainder given by Taylor's theorem gives

$$\left|\phi_u(s) - \sum_{j=0}^{m}\frac{i^j\mathbb{E}[(X\cdot u)^j]}{j!}s^j\right| \leq \left|\frac{\mathbb{E}[(X\cdot u)^{m+1}]s^{m+1}}{(m+1)!}\right| \tag{3.5}$$

Redefining $\phi(\theta)$ as $\phi_u(s)$, we expand $\phi(\theta)$ about the origin as follows, making use of the first part of Lemma (1.5):

$$\begin{aligned}
\phi(0) &= 1\\
\phi^{(1)}(0) &= i\mathbb{E}[X\cdot u] = 0\\
\phi^{(2)}(0) &= -\frac{\mathbb{E}[(X\cdot u)^2]}{2}\\
\phi^{(3)}(0) &= -i\frac{\mathbb{E}[(X\cdot u)^3]}{6} = 0
\end{aligned}$$

These give us the second order Taylor expansion:

$$\phi(\theta) = 1 - \frac{\mathbb{E}[(X\cdot u)^2]}{2}s^2 + h(\theta)$$

Noting that by Lemma (1.5) and by the definition of s, u, and θ, $\mathbb{E}[(X\cdot u)^2]s^2 = \mathbb{E}[(X\cdot\theta)^2] = \theta\cdot\Gamma\theta$, we obtain:

$$\phi(\theta) = 1 - \frac{\theta\cdot\Gamma\theta}{2} + h(\theta) \tag{3.6}$$

where $h(\theta)$ is bounded as in (3.5) with $m = 3$.

In the end, we are interested in making statements about S_n, not X, and the next lemma gives the characteristic function of S_n.

Lemma 3.7. *The characteristic function of S_n is $\phi_X(\theta)^n$.*

Proof. Since $S_n = X_1 + \cdots + X_n$,

$$\phi_{S_n}(\theta) = \mathbb{E}[e^{iS_n\cdot\theta}] = \mathbb{E}[e^{i(X_1+\cdots+X_n)\cdot\theta}]$$

Since $X_1 \dots X_n$ are independent, this becomes

$$\mathbb{E}[e^{i(X_1+\cdots+X_n)\cdot\theta}] = \mathbb{E}[e^{iX_1\cdot\theta}]\cdots\mathbb{E}[e^{iX_n\cdot\theta}] = \phi_X(\theta)^n$$

□

If X is a $\mathbb{Z}^d$ valued random variable, then its characteristic function has a period of 2π in each dimension, and so we can restrict our study to $\theta \in [-\pi,\pi]^d$ to understand its behavior for all $\theta \in \mathbb{R}^d$.

Until this point, the reason why the characteristic function is useful for proving the LCLT has been murky. However, the following theorem yields a formula for $p_n(x)$ in terms of the characteristic function that is the key to the proof of the LCLT.

Theorem 3.8. *If $X = (X^1,\dots,X^d)$ is a $\mathbb{Z}^d$-valued random variable with characteristic function $\phi(\theta)$, then the following holds:*

$$p(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\phi(\theta)e^{-ix\cdot\theta}d\theta \tag{3.9}$$

Proof.

$$\phi(\theta) = \sum_{y\in\mathbb{Z}^d} e^{iy\cdot\theta}p(y)$$

$$\phi(\theta)e^{-ix\cdot\theta} = \left(\sum_{y\in\mathbb{Z}^d} e^{iy\cdot\theta}p(y)\right)e^{-ix\cdot\theta}$$

$$\int_{[-\pi,\pi]^d}\phi(\theta)e^{-ix\cdot\theta}d\theta = \int_{[-\pi,\pi]^d}\left(\sum_{y\in\mathbb{Z}^d} e^{iy\cdot\theta}p(y)\right)e^{-ix\cdot\theta}d\theta$$

$$\int_{[-\pi,\pi]^d}\phi(\theta)e^{-ix\cdot\theta}d\theta = \sum_{y\in\mathbb{Z}^d}p(y)\int_{[-\pi,\pi]^d}e^{i(y-x)\cdot\theta}d\theta$$

When $y \neq x$, $y - x \neq 2\pi k$ for $k \in \mathbb{Z}^d \setminus [0]$, so

$$\int_{[-\pi,\pi]^d}e^{i(y-x)\cdot\theta}d\theta = 0$$

Therefore

$$\int_{[-\pi,\pi]^d}\phi(\theta)e^{-ix\cdot\theta}d\theta = p(x)\int_{[-\pi,\pi]^d}d\theta = p(x)(2\pi)^d$$

which when rearranged gives the theorem. □

Recalling that $\phi_X(\theta)^n$ is equivalent to the characteristic function of S_n, we get the following corollary.

Corollary 3.10. *If $X = (X^1, \dots, X^d)$ is a $\mathbb{Z}^d$-valued random variable with characteristic function $\phi(\theta)$, and $S_n = X_1 + \cdots + X_n$ then the following holds:*

$$p_n(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi(\theta)^n e^{-ix\cdot\theta} d\theta \tag{3.11}$$

Proof. Same as Theorem 3.8, except that $\phi_X(\theta)^n$ is substituted in for $\phi_{S_n}(\theta)$ in the final step. □

In the final theorem of this section, we prove that $(\phi(\theta))^r$ decays exponentially as θ moves away from the origin, which allows us to restrict our study of the above inversion formula to a small area around the origin.

Lemma 3.12. *For all θ in $[-\pi, \pi]^d$ and $n > 0$, there exists $b > 0$ such that*

$$|\phi(\theta)|^r < e^{-br|\theta|^2}$$

Proof. We begin by proving that $|\phi(\theta)| < 1$ for $\theta \in [-\pi, \pi]^d \setminus [0]$. Note that if $|\phi(\theta)| = 1$, then $|\phi(\theta)^n| = 1$ for all n. Recall that

$$\phi(\theta)^n = \sum_{x\in\mathbb{Z}^d} p_n(x) e^{ix\cdot\theta}$$

Therefore if $|\phi(\theta)| = 1$, then

$$\left| \sum_{x\in\mathbb{Z}^d} p_n(x) e^{ix\cdot\theta} \right| = \sum_{x\in\mathbb{Z}^d} \left| p_n(x) e^{ix\cdot\theta} \right| = \sum_{x\in\mathbb{Z}^d} p_n(x) = 1 \tag{3.13}$$

Also recall that for complex numbers $w_1, \dots w_k$, if $|w_1 + \cdots + w_k| = |w_1| + \cdots + |w_k| = 1$ then there exists γ such that $w_j = r_j e^{\gamma i}$ with each $r_j \geq 0$. By (3.13), each $p_n(x)e^{ix\cdot\theta} = r_j e^{i\gamma}$ for some γ that depends on n. By the way random walk $p \in \mathcal{P}$ was defined, for all x there is an N such that $p_N(x) > 0$. From this we know for all $x \in \mathbb{Z}^d$, there exists γ and N with

$$p_N(x) e^{ix\cdot\theta} = re^{i\gamma}$$

$p_N(x)$ must equal r, so we end up with

$$2k\pi + x \cdot \theta = \gamma$$

The only θ in $[-\pi, \pi]^d$ that satisfies this equation for all $x \in \mathbb{Z}^d$ is 0, so $|\phi(\theta)| < 1$ for $\theta \in [-\pi, \pi]^d \setminus [0]$.

By the previous statement and (3.6), we know we can find a $b > 0$ with $|\phi(\theta)| < |1 - b|\theta|^2|$, since θ is restricted to a bounded set. By the same reasoning, we know from the Taylor expansion of $e^{-b|\theta|^2}$ that $|1 - b|\theta|^2| < e^{-b|\theta|^2|}$. Combining these inequalities and raising to the $r > 0$ power, we get

$$|\phi(\theta)|^r < (e^{-b|\theta|^2})^r$$

which proves the lemma. □

4. Proof of the LCLT

The integral given by the inversion formula of $\phi\left(\frac{\theta}{\sqrt{n}}\right)^n$, which is the characteristic function of the normalized sum $\frac{S_n}{\sqrt{n}}$, looks similar to the right-most expression for $\overline{p}_n(x)$ in (2.1), and we will manipulate the inversion formula to prove the LCLT. An outline of the approach is given below:

- Find the Taylor expansion for $\log \phi(\theta)$ around the origin, and bound the error term.
- Use the Taylor expansion to express $\phi\left(\frac{\theta}{\sqrt{n}}\right)^n$ as an exponential term in the inversion formula.
- Evaluate the inversion integral in a small area around the origin, and estimate the error for areas outside of this region.
- Use information about the third and fourth moments of X implied by the definition of $\mathcal{P}$ to bound the difference between $p_n(x)$ and $\overline{p}_n(x)$ and prove the theorem.

The Taylor expansion of $\log(1-z)$ about the origin is derived as follows: consider the geometric series

$$1 + z + z^2 + \cdots + z^n + \cdots = \sum_{j=0}^{\infty} z^j$$

For $|z| < 1$, this sum converges and satisfies:

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$

Integrating both sides and manipulating yields the expansion of $\log(1 - z)$ in the complex plane, which is:

$$\log(1-z) = -\left(\sum_{j=1}^{\infty} \frac{z^j}{j}\right), |z| < 1 \tag{4.1}$$

By Taylor's Theorem, if $|z| < 1$, there exitst $c > 0$ such that

$$\left|\log(1-z) - \sum_{j=1}^{k} \frac{z^j}{j}\right| < cz^{k+1} \tag{4.2}$$

We use this expansion to prove the following lemma.

Lemma 4.3. *Suppose that we have $p \in \mathcal{P}$ with covariance matrix Γ and covariance matrix ϕ. Then there exists $\epsilon > 0$ such that if $|\theta| < \epsilon\sqrt{n}$ then*

$$\phi\left(\frac{\theta}{\sqrt{n}}\right)^n = e^{-\frac{\theta\cdot\Gamma\theta}{2}}(1 + F_n(\theta)) \tag{4.4}$$

and

$$|F_n(\theta)| < e^{\frac{\theta\cdot\Gamma\theta}{4}} + 1 \tag{4.5}$$

Proof. We find $0 < \delta < 1$ such that if $|\theta| < \delta$ then $|1 - \phi(\theta)| < 1$. From (3.6), we know $1 - \phi(\theta) = \frac{\theta \cdot \Gamma\theta}{2} - h(\theta)$, and using the expansion in (4.1), we can write

$$\log(\phi(\theta)) = \log(1-(1-\phi(\theta))) = -\frac{\theta \cdot \Gamma\theta}{2} + h(\theta) - \frac{(\theta \cdot \Gamma\theta)^2}{8} + \frac{\theta \cdot \Gamma\theta}{2} h(\theta) - \frac{1}{2} h(\theta)^2 + q(\theta)$$

and by (4.2), there exists a $c > 0$ with

$$|q(\theta)| < c|1 - \phi(\theta)|^3 = c\left|\frac{\theta \cdot \Gamma\theta}{2} - h(\theta)\right|^3$$

Similarly,

$$n \log\left(\phi\left(\frac{\theta}{\sqrt{n}}\right)\right) = -\frac{\theta \cdot \Gamma\theta}{2} + nh\left(\frac{\theta}{\sqrt{n}}\right) - \frac{1}{n}\frac{(\theta \cdot \Gamma\theta)^2}{8} + \frac{\theta \cdot \Gamma\theta}{2} h\left(\frac{\theta}{\sqrt{n}}\right) - \frac{1}{2} h\left(\frac{\theta}{\sqrt{n}}\right)^2 + q\left(\frac{\theta}{\sqrt{n}}\right)$$

with q bounded as above. Define $g(\theta, n)$ as follows:

$$g(\theta, n) = n \log\left(\phi\left(\frac{\theta}{\sqrt{n}}\right)\right) + \frac{\theta \cdot \Gamma\theta}{2}$$

so that

$$\phi\left(\frac{\theta}{\sqrt{n}}\right)^n = e^{-\frac{\theta \cdot \Gamma\theta}{2}} e^{g(\theta,n)} \tag{4.6}$$

(3.5) shows that $|h(\theta)|$ is bounded by $c_1|\theta|^4$ for some finite c_1. Because of this and the fact that $|\theta| < \delta < 1$, $\frac{1}{n}\frac{(\theta \cdot \Gamma\theta)^2}{8}$ becomes the dominant error term for $g(\theta, n)$, so there exists c_2 such that

$$|g(\theta, n)| < nh\left(\frac{\theta}{\sqrt{n}}\right) + \frac{c_2|\theta|^4}{n}$$

Both of these terms can be made arbitrarily small by choosing a small enough θ, so there exists $\epsilon, 0 < \epsilon < \delta$, such that if $|\theta| < \epsilon\sqrt{n}$, then

$$|g(\theta, n)| < \frac{\theta \cdot \Gamma\theta}{4}$$

Let $F_n(\theta) = e^{g(\theta,n)} - 1$. It is clear that $|F_n(\theta)| = |e^{g(\theta,n)} - 1| < e^{\frac{\theta \cdot \Gamma\theta}{4}} + 1$, which proves the lemma.

□

For the next theorem, we will use the fact that the function $\int_x^\infty e^{-t^2} dt$ can be written as follows:

$$\int_x^\infty e^{-t^2} dt = \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} - \cdots = \frac{e^{-x^2}}{2x} - \sum_{j=1}^\infty \frac{e^{-x^2}}{2(j+1)x^{2j+1}}$$

This follows from the substitution of e^{-t^2} and repeated use of integration by parts. This expansion motivates the approximation

$$\left|\int_{|x|>r} e^{-t^2} dt\right| < ce^{-\beta|r|^2} \tag{4.7}$$

for $r \in \mathbb{R}^d$, with $c > 0$ and $\beta > 0$. We can now prove the next theorem, which gives the difference between $p_n(x)$ and $\overline{p}_n(x)$ as the sum of a term that decays exponentially and an integral term the depends on $F_n(x)$.

Theorem 4.8. *For $0 \le r \le \epsilon\sqrt{n}$, there exists a $c > 0$ and a $\beta > 0$ such that:*

$$\left| p_n(x) - (\overline{p}_n(x) + \frac{1}{(2\pi)^d n^{d/2}} \int_{|\theta|\le r} e^{-\frac{ix\cdot\theta}{\sqrt{n}}} e^{-\frac{\theta\cdot\Gamma\theta}{2}} F_n(\theta) d\theta) \right| < cn^{-d/2} e^{-\beta r^2} \tag{4.9}$$

Proof. Recalling the inversion formula in (3.11), we use the substitution $\theta = \frac{s}{\sqrt{n}}$ to write $p_n(x)$ as:

$$p_n(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi(\theta)^n e^{-ix\cdot\theta} d\theta = \frac{1}{(2\pi)^d n^{d/2}} \int_{[-\sqrt{n}\pi,\sqrt{n}\pi]^d} \phi\left(\frac{s}{\sqrt{n}}\right)^n e^{-iz\cdot s} ds$$

with $z = x/\sqrt{n}$. Using lemma 3.12, we know that there exists a $b > 0$ such that $|\phi(s/\sqrt{n})|^n < e^{-b|s|^2}$. Combining this with (4.7), we get:

$$\left| p_n(x) - \frac{1}{(2\pi)^d n^{d/2}} \int_{|s|<\epsilon\sqrt{n}} \phi(\frac{s}{\sqrt{n}})^n e^{-iz\cdot s} ds \right| < \frac{1}{(2\pi)^d n^{d/2}} \int_{|s|>\epsilon\sqrt{n}} e^{-b|s|^2} ds < c_1 n^{-d/2} e^{-\beta_1 n}$$

(4.7) and the definition of $\overline{p}_n(x)$ in 2.1 gives us:

$$\left| \overline{p}_n(x) - \frac{1}{(2\pi)^d n^{d/2}} \int_{|s|<\epsilon\sqrt{n}} e^{-s\cdot\Gamma s} e^{-iz\cdot s} ds \right| < \frac{1}{(2\pi)^d n^{d/2}} \int_{|s|>\epsilon\sqrt{n}} e^{-b|s|^2} ds < c_2 n^{-d/2} e^{-\beta_2 n}$$

By the previous lemma, we know that for $|s| < \epsilon\sqrt{n}$, we can write $\phi(s/\sqrt{n})^n = e^{-s\cdot\Gamma s}(1 + F_n(s))$, so combining these inequalities gives us

$$\left| p_n(x) - (\overline{p}_n(x) + \frac{1}{(2\pi)^d n^{d/2}} \int_{|\theta|<\epsilon\sqrt{n}} e^{-iz\cdot s} e^{-\frac{\theta\cdot\Gamma\theta}{2}} F_n(\theta) d\theta) \right| < c_1 n^{-d/2} e^{-\beta_1 n} + c_2 n^{-d/2} e^{-\beta_2 n}$$

This equation shows the result holds for $r = \epsilon\sqrt{n}$. For smaller values of r, we use the fact that $|F_n(s)| < e^{-\frac{s\cdot\Gamma s}{4}} + 1$ to show

$$\left| \int_{r<|s|<\epsilon\sqrt{n}} e^{-iz\cdot s} e^{-\frac{s\cdot\Gamma s}{2}} F_n(s) ds \right| < 2 \int_{|s|>r} e^{-\frac{s\cdot\Gamma s}{8}} ds < c_3 e^{-\beta_3 r^2}$$

Since in this case $r < \epsilon\sqrt{n}$, this becomes the dominant error term, proving the theorem. □

We need one more lemma bounding $F_n(x)$ before we can finish the proof of the LCLT.

Lemma 4.10. *There exists a $c > 0$ such that if $|\theta| < n^{1/8}$, then*

$$|F_n(s)| < c\frac{|\theta|^4}{n} \tag{4.11}$$

Proof. Recall that from lemma 4.3, for some $c > 0$ we have

$$|g(\theta, n)| < nh(\frac{\theta}{\sqrt{n}}) + \frac{c|\theta|^4}{n} \tag{4.12}$$

and that $|h(\theta)| < c|\theta|^4$ by Taylor's theorem. Therefore, for a different $c > 0$, we have

$$|g(\theta, n)| < c\frac{|\theta|^4}{n}$$

We want to be able to bound $F_n(\theta)$ using $g(\theta, n)$. If $|\theta| < n^{1/8}$, then $|g(\theta, n)| < \frac{c}{\sqrt{n}} \leq c$ for all n. By the Taylor expansion of e^z, we know that if z is restricted to a bounded set, $|e^z - 1| < cz$, and since $g(\theta, n)$ is restricted to a bounded set for $|\theta| < n^{1/8}$, we can write

$$|F_n(\theta)| = |e^{g(\theta,n)} - 1| < c_1|g(\theta, n)| < c_1|nh(\frac{\theta}{\sqrt{n}}) + \frac{c_2|\theta|^4}{n}| < c_3\frac{|\theta|^4}{n} \tag{4.13}$$

□

We are now in position to finish the proof of the LCLT.

Theorem 4.14. *For all $p \in \mathcal{P}$, there exists a $c > 0$ with*

$$|p_n(x) - \overline{p}_n(x)| < \frac{c}{n^{(d+2)/2}} \tag{4.15}$$

Proof. Let $r = min(\epsilon\sqrt{n}, n^{1/8})$. By the previous lemma, we know that for such an r,

$$\left|\int_{|\theta|\leq r} e^{-\frac{ix\cdot\theta}{\sqrt{n}}} e^{-\frac{\theta\cdot\Gamma\theta}{2}} F_n(\theta)d\theta)\right| < \frac{c_1}{n}\int_{\mathbb{R}^d} |\theta|^4 e^{-\frac{\theta\cdot\Gamma\theta}{2}} d\theta < \frac{c_2}{n}$$

Letting $\frac{c}{n^{(d+2)/2}} = \frac{c_2}{n}\frac{1}{(2\pi)^d n^{d/2}}$, using (4.9) we can now write

$$|p_n(x) - \overline{p}_n(x)| < \frac{c}{n^{(d+2)/2}} + c_3 e^{-\beta n^{1/4}}$$

Since $e^{-\beta n^{1/4}}$ decays faster than any power of n, this proves the theorem. □

5. Application of the LCLT: Recurrence and Transience

One simple but insightful application of the LCLT is to prove that random walks $p \in \mathcal{P}$ visit every point in the lattice infinitely often for $d = 1, 2$, but tend towards infinity if n is large enough for $d \geq 3$. We start with some definitions.

Definition 5.1. A random walk is called **reccurrent** if it visits every point in the lattice infinitely often. In other words, $\sum_{j=1}^n p_j(x)$ increases without bound as n goes towards ∞ for all $x \in \mathbb{Z}^d$. A random walk is called **transient** if $\sum_{j=1}^\infty p_j(x)$ is finite for all x.

For any Markov chain, if $\mathbb{P}[S_{n+j} = z|S_n = y] > 0$ and if $\mathbb{P}[S_{n+k} = y|S_n = z] > 0$ for all n and for some j and k greater than 0, then if one of those states is reccurent, the other is as well. The same applies for transience. Since we defined random walks $p \in \mathcal{P}$ so that this is true for all y and z in $\mathbb{Z}^d$, any random walk must be either reccurrent or transient.

Theorem 5.2. *Consider random walk $p \in \mathcal{P}$. The for $d = 1, 2$, the p is recurrent. For $d \geq 3$, p is transient.*

Proof. Note that it suffices to prove the theorem only for $x = 0$. Recalling formula (2.1), we note that $\overline{p}_n(0) = \frac{1}{(2\pi n)^{d/2}\sqrt{det\Gamma}}$. By the LCLT, we also know that $\lim_{n\to\infty} \frac{p_n(0)}{\overline{p}_n(0)} = 1$. This implies that there exists $0 < c_1 < c_2$ and a sufficiently large N such that if $n > N$, $c_1 n^{d/2} < p_n(0) < c_2 n^{d/2}$. Therefore,

$$\sum_N^n c_1 n^{d/2} < \sum_N^n p_n(0) < \sum_N^n c_2 n^{d/2} \tag{5.3}$$

If $d = 1, 2$, $\lim_{n\to\infty} \sum_N^n p_n(0) > \lim_{n\to\infty} \sum_N^n c_1 n^{d/2}$, which diverges, proving that the random walk is recurrent. If $d \geq 3$, $\lim_{n\to\infty} \sum_N^n p_n(0) < \lim_{n\to\infty} \sum_N^n c_2 n^{d/2}$, which converges, proving that the random walk is transient. □

By considering only even numbered steps, it is clear that this theorem also applies to periodic random walks such as the simple random walk. It is interesting to note that the transience or recurrence of a random walk is independent of the probability distribution of X. As an example, consider a three-dimensional random walk with $\mathcal{K}(x^1) = \mathcal{K}(x^2) >> \mathcal{K}(x^3)$. (Here we use the function $\mathcal{K}$ in definition 1.2.) Despite the fact that the probability of travelling in one direction on the lattice is minute compared to the other two, the random walk is transient, even though the probability distribution of X is not that different from the distribution one would get from a recurrent two-dimensional walk.

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