AN APPLICATION OF THE VAN KAMPEN THEOREM

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Abstract. In this paper, we prove the van Kampen Theorem and illustrate its applications to cell complexes; in particular, we compute the fundamental group for every compact surface and show that every group is the fundamental group of some space. Working towards this aim, we first introduce the categorical language needed for our statement of the van Kampen Theorem and then construct the fundamental group and groupoid for a space. Finally, we utilize this background to prove the theorem and carry out the desired applications.

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1. Introduction

One tool mathematicians have developed to solve topological problems involves associating computable invariants to spaces. This paper introduces the fundamental group, one of the most easily definable algebraic invariants. The fundamental group is based on the notion of path homotopies in a space and in particular loops around a given basepoint. It is a powerful invariant which can be utilized to give relatively elementary proofs of the Fundamental Theorem of Algebra and Brouwer Fixed-Point Theorem (See Chapter 1 of [3]). However, an invariant is only useful if we can compute it for a variety of spaces and the van Kampen Theorem provides the tools to do this for a large class of spaces.

Specifically, the van Kampen Theorem allows us to express the fundamental group of a space in terms of the fundamental groups of simpler subspaces. Therefore, if a space admits a reasonable decomposition, the van Kampen Theorem allows us to compute its fundamental group. Utilizing this fact we can compute the fundamental group for many spaces; in fact, combining this theorem with the classification of surfaces will allow us to compute the fundamental group for every connected compact surface. This paper aims to provide a thorough discussion of the van Kampen Theorem including a number of its applications. We adopt a "ground up" approach to minimize any formal prerequisites necessary to understand the

material and consequently begin with the language and structures needed for our statement of the theorem.

2. Categorical language

The fundamental group is an algebraic structure associated with a topological space: Given a space X with chosen basepoint $x \in X$, we associate the group $\pi_1(X, x)$ to the pair (X, x). As a result, this paper often concerns mappings from topological spaces into groups. Category theory provides the language to discuss these types of mappings rigorously and therefore we will introduce the basic categorical language utilized in our development of the van Kampen Theorem. The treatment here closely follows J. P. May's *A Concise Course in Algebraic Topology.*

Categories

Definition 2.1. A category $\mathscr{C}$ consists of a class of objects $\mathrm{ob}(\mathscr{C})$, a set $\mathscr{C}(X, Y)$ of morphisms between any two objects $X, Y \in \mathrm{ob}(\mathscr{C})$, an identity morphism $\mathrm{id}_X \in \mathscr{C}(X, X)$ for each $X \in \mathrm{ob}(\mathscr{C})$, and a composition law $\circ : \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$ such that the following properties hold:

- If $f : X \to Y$, then $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$.
- If $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 2.2. Let $\mathscr{C}$ be a category. If $\mathrm{ob}(\mathscr{C})$ forms a set, we say the category is small.

Definition 2.3. We call a morphism $f : X \to Y$ an isomorphism if it has an inverse; that is, if there exists a mapping $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Definition 2.4. Let $\mathscr{C}$ be a category. A skeleton of $\mathscr{C}$ is a subcategory $\mathrm{sk}\mathscr{C}$ with one object from each isomorphism class of objects in $\mathrm{ob}(\mathscr{C})$ and with morphisms $\mathrm{sk}\mathscr{C}(X, Y) = \mathscr{C}(X, Y)$.

Definition 2.5. A groupoid is a category, all of whose morphisms are isomorphisms.

Remark. The morphisms of a groupoid with only one object satisfy the group axioms. Thus we regard a groupoid with only one object as a group.

Example 2.6. The following are categories utilized in this paper:

- Let **Top** be the category whose objects are topological spaces and whose morphisms are continuous maps.
- Let $\mathbf{Top}_*$ be the category whose objects are based topological spaces and whose morphisms are continuous maps preserving basepoints.
- Let **Grp** be the category whose objects are groups and whose morphisms are group homomorphisms.
- Let **Gpd** be the category whose objects are groupoids and whose morphisms are functors.

Functors and natural transformations

Definition 2.7. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \to \mathscr{D}$ associates to each object $X \in \mathrm{ob}(\mathscr{C})$ an object $F(X) \in \mathrm{ob}(\mathscr{D})$ and to each morphism $f \in \mathscr{C}(X, Y)$ a morphism $F(f) \in \mathscr{D}(F(X), F(Y))$ in such a way that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition 2.8. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $F, G : \mathscr{C} \to \mathscr{D}$ functors. A natural transformation $\alpha : F \to G$ consists of morphisms $\alpha_X : F(X) \to G(X)$ for each $X \in \mathrm{ob}(\mathscr{C})$ which make the following diagram commute for all $f \in \mathscr{C}(X, Y)$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y). \end{tikzcd}$$

If each morphism α_X is an isomorphism, then we call α a natural isomorphism.

Definition 2.9. Given categories $\mathscr{C}$ and $\mathscr{D}$, we say a functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that there are natural isomorphisms $\varepsilon : GF \to \mathrm{id}_{\mathscr{C}}$ and $\eta : \mathrm{id}_{\mathscr{D}} \to FG$.

Proposition 2.10. *Let $\mathscr{C}$ be a category and $\mathrm{sk}\mathscr{C}$ a skeleton of $\mathscr{C}$. Then there is an equivalence of categories between $\mathscr{C}$ and $\mathrm{sk}\mathscr{C}$.*

Proof. Let $F : \mathrm{sk}\mathscr{C} \to \mathscr{C}$ be the inclusion functor. Define $G : \mathscr{C} \to \mathrm{sk}\mathscr{C}$ by mapping $X \in \mathscr{C}$ to its isomorphism class representative $G(X) \in \mathrm{sk}\mathscr{C}$, choosing an isomorphism $\alpha_X : X \to G(X)$, and defining $G(f) = \alpha_Y \circ f \circ \alpha_X^{-1}$ for each morphism $f \in \mathscr{C}(X, Y)$. Choosing α_X to be the identity morphism whenever $X \in \mathrm{ob}(\mathrm{sk}\mathscr{C})$ forces the equality $G \circ F = \mathrm{id}_{\mathrm{sk}\mathscr{C}}$. Therefore, it suffices to show that the α_X determine a natural isomorphism $\mathrm{id}_{\mathscr{C}} \to F \circ G$. Let $f \in \mathscr{C}(X, Y)$. Since $(F \circ G)(f) \circ \alpha_X = G(f) \circ \alpha_X = \alpha_Y \circ f = \alpha_Y \circ \mathrm{id}_{\mathscr{C}}(f)$, naturality follows. As each α_X is an isomorphism, this completes the proof. $\square$

Remark. A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if and only if F is full, faithful, and essentially surjective.[1] The previous theorem follows definitionally utilizing this alternative formulation of an equivalence of categories.

Colimits

Definition 2.11. Let $\mathscr{D}$ be a small category and $\mathscr{C}$ any category. A diagram of type $\mathscr{D}$ in $\mathscr{C}$ is a functor $F : \mathscr{D} \to \mathscr{C}$. A morphism of diagrams is a natural transformation.

Definition 2.12. Let $N \in \mathrm{ob}(\mathscr{C})$. The constant diagram $\underline{N}$ is the functor which sends every object of $\mathscr{D}$ to N and every morphism of $\mathscr{D}$ to the identity morphism id_N.

Definition 2.13. Let $F : \mathscr{D} \to \mathscr{C}$ be a diagram. The colimit of F is an object $\mathrm{colim}\, F \in \mathrm{ob}(\mathscr{C})$ together with a morphism of diagrams $\iota : F \to \underline{\mathrm{colim}\, F}$ such that if $\eta : F \to \underline{N}$ is a morphism of diagrams, then there is a unique map $\tilde{\eta} : \mathrm{colim}\, F \to N$

[1] A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be full (resp. faithful) if the morphism $\mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y))$ induced by F is surjective (resp. injective). If every object of $\mathscr{D}$ is isomorphic to some object $F(X) \in \mathrm{ob}(\mathscr{D})$ where $X \in \mathrm{ob}(\mathscr{C})$, we say F is essentially surjective.

such that for any morphism $f : X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\iota_X"] \arrow[ddr, bend right, "\eta_X"'] & & F(Y) \arrow[dl, "\iota_Y"'] \arrow[ddl, bend left, "\eta_Y"] \\
& \operatorname{colim} F \arrow[d, dashed, "\tilde{\eta}"] & \\
& N. &
\end{tikzcd}$$

Definition 2.14. Suppose $\mathscr{C}$ is a category, $f \in \mathscr{C}(Z, X)$ and $g \in \mathscr{C}(Z, Y)$. The pushout of f and g consists of an object $P \in \operatorname{ob}(\mathscr{C})$ and morphisms $\iota_1 \in \mathscr{C}(X, P)$ and $\iota_2 \in \mathscr{C}(Y, P)$ satisfying $\iota_1 \circ f = \iota_2 \circ g$ such that the following holds: for any two morphisms $\eta_1 \in \mathscr{C}(X, N)$ and $\eta_2 \in \mathscr{C}(Y, N)$ satisfying $\eta_1 \circ f = \eta_2 \circ g$, there exists a unique map $\tilde{\eta} \in \mathscr{C}(P, N)$ which makes the following diagram commute:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\iota_1"] \arrow[ddr, bend left, "\eta_1"] & \\
Y \arrow[r, "\iota_2"'] \arrow[drr, bend right, "\eta_2"'] & P \arrow[dr, dashed, "\tilde{\eta}"] & \\
& & N.
\end{tikzcd}$$

Thus a pushout is the colimit of a diagram $F(X) \longleftarrow F(Z) \longrightarrow F(Y)$.

Definition 2.15. Let $\mathscr{C}$ be a category and suppose $X_j \in \operatorname{ob}(\mathscr{C})$ for each $j \in J$, where J is some index set. The coproduct of the X_j consists of an object $X \in \operatorname{ob}(\mathscr{C})$ and a morphism $\iota_j \in \mathscr{C}(X_j, X)$ for each $j \in J$ such that given any family of morphisms $\eta_j \in \mathscr{C}(X_j, N)$ $(j \in J)$, there exists a unique map $\tilde{\eta} \in \mathscr{C}(X, N)$ which makes the following diagram commute:

$$\begin{tikzcd}
X_j \arrow[d, "\iota_j"'] \arrow[dr, "\eta_j"] & \\
X \arrow[r, "\tilde{\eta}"'] & N.
\end{tikzcd}$$

While we defined the coproduct in an arbitrary category $\mathscr{C}$, in this paper we will primarily concern ourselves with coproducts in **Grp**, the category of groups. Therefore, we will deviate from the strictly categorical definitions of this section to introduce an algebraic concept: the free product of groups.

Definition 2.16. Let I be an index set and suppose each G_i $(i \in I)$ is a group. Choose a presentation $G_i = \langle R_i | S_i \rangle$ for each group.[2] We define the free product of these groups by $*_{i \in I} G_i = \langle \coprod_{i \in I} R_i | \coprod_{i \in I} S_i \rangle$.

Theorem 2.17. *The coproduct in the category of groups is the free product.*

Proof. The relevant universal properties follow immediately and their verification is therefore left to the reader. □

[2]Consult Chapter 1 Section 12 of [2] for a proof that every group has a presentation in terms of generators and relations

3. The fundamental groupoid and fundamental group

The last section introduced the language needed to understand our statement of the van Kampen Theorem. This theorem provides information about the functors we introduce in this section, specifically the fundamental groupoid and fundamental group of a space. The algebraic structures these functors yield are defined in terms of path homotopies, so this will serve as our starting point.

Path homotopies

Definition 3.1. Let X be a topological space and suppose $f, g : I \to X$ are paths from x to y. We say $f \sim g$ (read "f is homotopic to g") if there exists a continuous map $h : I \times I \to X$ such that $h(s, 0) = f(s)$, $h(s, 1) = g(s)$, $h(0, t) = x$ and $h(1, t) = y$.[3]

Remark. All paths are assumed continuous with domain $I = [0, 1]$. If $f : I \to X$ is a path from x to y, we write $f : x \to y$. If $h : I \times I \to X$ is a homotopy between $f, g : I \to X$, we write $h : f \simeq g$.

Proposition 3.2. *Let $\sim$ be as above. Then $\sim$ is an equivalence relation.*

Proof. The homotopy $h(s, t) = f(s)$ proves $f \sim f$. If $f \sim g$ and $h : f \simeq g$ is a basepoint-preserving homotopy, setting $k(s, t) = h(s, 1 - t)$ proves $g \sim f$. If $f \sim g$ and $g \sim p$, let $h : f \simeq g$ and $k : g \simeq p$ be the respective homotopies. Setting

$$j(s,t) = \begin{cases} h(s, 2t) & : 0 \le t \le \frac{1}{2} \\ k(s, 2t-1) & : \frac{1}{2} < t \le 1 \end{cases}$$

proves $f \sim p$. □

The fundamental groupoid and fundamental group

Definition 3.3. Given paths $f : x \to y$ and $g : y \to z$, define the path $g \cdot f : x \to z$ by

$$(g \cdot f)(s) = \begin{cases} f(2s) & : 0 \le s \le \frac{1}{2} \\ g(2s-1) & : \frac{1}{2} < s \le 1 \end{cases}.$$

Define the path $f^{-1} : y \to x$ by $f^{-1}(s) = f(1 - s)$. Define the constant loop at x by $c_x(s) = x$.

Definition 3.4. Suppose $f : x \to y$ and $g : y \to z$ are paths. We define composition of equivalence classes of paths by $[g] \cdot [f] = [g \cdot f]$.

Proposition 3.5. *Composition of equivalence classes of paths is well-defined.*

Proof. Let $f_1 \in [f]$ and $g_1 \in [g]$. It suffices to show $g \cdot f \simeq g_1 \cdot f_1$. By hypothesis, there exist $F : f \simeq f_1$ and $G : g \simeq g_1$ which fix basepoints. Defining

$$H(s,t) = \begin{cases} F(2s, t) & : 0 \le s \le \frac{1}{2} \\ G(2s-1, t) & : \frac{1}{2} < s \le 1 \end{cases},$$

we obtain the desired path homotopy. □

[3] In general, homotopies do not preserve basepoints; for example, in our discussion of homotopy equivalences we lift this requirement. However, if we do not force path homotopies to preserve basepoints, then every path is homotopic to a constant loop.

Proposition 3.6. *Suppose $f : x \to y$, $g : y \to z$, and $h : z \to w$ are paths. Then $[h \cdot (g \cdot f)] = [(h \cdot g) \cdot f]$, $[c_y \cdot f] = [f \cdot c_x] = [f]$, $[f^{-1} \cdot f] = [c_x]$, and $[f \cdot f^{-1}] = [c_y]$.*

Proof. Define $k : I \times I \to X$ by

$$k(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & : 0 \leq s \leq \frac{t}{4} + \frac{1}{4} \\ g(4s - 1 - t) & : \frac{t}{4} + \frac{1}{4} < s \leq \frac{t}{4} + \frac{1}{2} \\ h\left(1 - \frac{4(1-s)}{2-t}\right) & : \frac{t}{4} + \frac{1}{2} < s \leq 1 \end{cases},$$

showing $h \cdot (g \cdot f) \simeq (h \cdot g) \cdot f$ and establishing the first equality. To prove the relation $f \simeq f \cdot c_x$, set

$$j(s,t) = \begin{cases} c_x(0) & : 0 \leq s \leq \frac{t}{2} \\ f\left(1 - \frac{2(1-s)}{2-t}\right) & : \frac{t}{2} < s \leq 1 \end{cases},$$

giving the desired homotopy. The relation $c_y \cdot f \simeq f$ follows similarly. Finally, defining

$$i(s,t) = \begin{cases} f(2s) & : 0 \leq s \leq \frac{t}{2} \\ f(t) & : \frac{t}{2} < s < 1 - \frac{t}{2} \\ f^{-1}(2s-1) & : 1 - \frac{t}{2} \leq s \leq 1 \end{cases},$$

we obtain the relation $f^{-1} \cdot f \simeq c_x$. Since $(f^{-1})^{-1} = f$, this shows $f \cdot f^{-1} = (f^{-1})^{-1} \cdot f^{-1} \simeq c_y$, establishing the theorem. □

Definition 3.7 (The fundamental groupoid)**.** Let $\Pi(X)$ be the category whose objects are points of X and whose morphisms $x \to y$ are the equivalence classes of paths. Choosing the composition of equivalence classes as our composition law, Proposition 3.6 shows $\Pi(X)$ is a groupoid.

Definition 3.8 (The fundamental group)**.** Let $\pi_1(X, x)$ be the collection of equivalence classes of paths $x \to x$. Since $\pi_1(X, x)$ is a groupoid with only one object, it follows that $\pi_1(X, x)$ is a group.

The following theorem establishes the functoriality of the fundamental groupoid and fundamental group and its corollary provides a connection between the fundamental group and fundamental groupoid. These results will be crucial in our proof of the van Kampen Theorem.

Proposition 3.9. *The fundamental groupoid is a functor $\Pi : \mathbf{Top} \to \mathbf{Gpd}$ from the category of topological spaces to the category of groupoids. The fundamental group is a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ from the category of based topological spaces to the category of groups.*

Proof. Define $\Pi : \mathbf{Top} \to \mathbf{Gpd}$ as follows: For $X \in \mathrm{ob}(\mathbf{Top})$ set $\Pi : X \mapsto \Pi(X)$ and for $f \in \mathbf{Top}(X, Y)$ define $\Pi(f) : \Pi(X) \to \Pi(Y)$ by $\Pi(f)(x) = f(x)$ for $x \in \mathrm{ob}(\Pi(X))$ and $\Pi(f)[p] = [f \circ p]$ for $[p] \in \Pi(X)(x, y)$. Clearly Π is well-defined on objects, and thus we only need to check Π is well-defined on morphisms. Suppose $q \in [p]$ and let $h : p \simeq q$ be the required homotopy. The map $f \circ h$ gives a basepoint-preserving homotopy, proving $f \circ p \simeq f \circ q$. This shows Π is well-defined on morphisms.

To check Π takes continuous maps into maps of groupoids, note that since $\mathrm{id}_x = [c_x]$, we have $\Pi(f)(\mathrm{id}_x) = [f \circ c_x] = [c_{f(x)}] = \mathrm{id}_{f(x)}$. Appealing to our definition of

$p \cdot q$ for paths, we see

$$(\Pi(f)[p \cdot q])(s) = \begin{cases} [f \circ q](2s) & : 0 \leq s \leq \frac{1}{2} \\ [f \circ p](2s-1) & : \frac{1}{2} < s \leq 1 \end{cases} = (\Pi(f)[p] \cdot \Pi(f)[q])(s),$$

showing $\Pi(f)[p \cdot q] = \Pi(f)[p] \cdot \Pi(f)[q]$. Therefore, Π takes continuous maps to functors, and since functors take isomorphisms to isomorphisms, this proves Π maps into the target category.

To verify the functoriality axioms for Π, note that $\Pi(\mathrm{id}_X)(x) = x$ and $\Pi(\mathrm{id}_X)[f] = [f]$, so $\Pi(\mathrm{id}_X) = \mathrm{id}_{\Pi(X)}$. Moreover, we have

$$\Pi(g \circ f)[p] = [(g \circ f) \circ p] = [g \circ (f \circ p)] = \Pi(g)(\Pi(f)[p]) = (\Pi(g) \circ \Pi(f))[p]$$

This proves Π is a functor. The proof for the fundamental group follows similarly from this argument. □

Corollary. *Let X be a path-connected space. Then given $x \in X$, the inclusion $\pi_1(X, x) \to \Pi(X)$ is an equivalence of categories.*

Proof. Since X is path-connected, any two objects in $\Pi(X)$ are isomorphic. Therefore $\pi_1(X, x)$ is a skeleton of $\Pi(X)$. Applying Propositions 2.10 and 3.9, the result follows immediately. □

4. The van Kampen Theorem

Here we utilize the results developed in the preceding sections to give a statement and proof of the van Kampen Theorem. We begin first with the statement for the fundamental groupoid and then prove the theorem for the fundamental group of a space.

Theorem 4.1 (van Kampen)**.** *Let $\mathscr{O} = \{U\}$ be a cover of a topological space X by open subsets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe*

$$\Pi|_{\mathscr{O}} : \mathscr{O} \to \mathbf{Gpd}$$

gives a diagram of groupoids. The groupoid $\Pi(X)$ is the colimit of this diagram. That is $\Pi(X) \cong \mathrm{colim}_{U \in \mathscr{O}} \Pi(U)$.

Proof. Define $\iota : \Pi|_{\mathscr{O}} \to \underline{\Pi(X)}$ by determining the functor $\iota_U : \Pi(U) \to \Pi(X)$ via the inclusion $U \to X$. Let $\mathscr{C}$ be a groupoid and $\eta : \Pi|_{\mathscr{O}} \to \underline{\mathscr{C}}$ a morphism of diagrams. Define $\tilde{\eta} : \Pi(X) \to \mathscr{C}$ such that $\tilde{\eta}(x) = \eta_U(x)$ for $x \in \mathrm{ob}(\Pi(U)) \subset \mathrm{ob}(\Pi(X))$. For a path $f : x \to y$ contained entirely in some U, set $\tilde{\eta}[f] = \eta|_U[f]$. More generally, for any path $f : x \to y$ we can write $f = \Pi_{i=1}^n f_i$ where each f_i is contained entirely in some U. In this case, set $\tilde{\eta}[f] = \Pi_{i=1}^n \tilde{\eta}[f_i]$. Since $\tilde{\eta}|_{\Pi(U)} = \eta_U$, the map $\tilde{\eta}$ clearly makes the colimit diagram commute as needed. Note that making this diagram commute forces each of our choices for $\tilde{\eta}$ and therefore establishes its uniqueness.

To complete the proof we need only show $\tilde{\eta}$ is well-defined. To do this, suppose first that $x \in U_1, U_2$ for $U_1, U_2 \in \mathscr{O}$. Let $j_1 : U_1 \cap U_2 \to U_1$ be the inclusion. Then $\eta_{U_1 \cap U_2} = \eta_{U_1} \circ j_1$, so $\eta_{U_1 \cap U_2}(x) = \eta_{U_1}(x)$. Similarly, the inclusion $j_2 : U_1 \cap U_2 \to U_2$ yields the equality $\eta_{U_1 \cap U_2}(x) = \eta_{U_2}(x)$. This shows the way $\tilde{\eta}$ maps objects is well-defined. By replacing x with paths $f : x \to y$ such that $\mathrm{im}(f) \subset U_1, U_2$, the same argument shows $\tilde{\eta}$ is well-defined for paths contained in some U. Now suppose $f \sim g$, so there exists a basepoint-preserving homotopy $h : f \simeq g$. Divide $I \times I$

into finitely many rectangles such that each subrectangle maps into some U (this is possible since $I \times I$ is compact). Choose this subdivision such that it refines the subdivision of $I \times \{0\}$ and $I \times \{1\}$ used to decompose f and g respectively. Number the subrectangles sequentially from left to right, row by row. This produces an $n \times m$ grid of the following form:

$I \times I$

21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

With reference to the diagram above, define γ_r to be the path in $I \times I$ which travels on the grid lines subdividing $I \times I$ and which separates the first r rectangles from the rest. Note that $\tilde{\eta}[\gamma_r] = \tilde{\eta}[\gamma_{r+1}]$ as follows:

- The path decomposition of each curve differs only on the $(r+1)$st rectangle.
- This rectangle is mapped into some $U \in \mathscr{O}$ and the path components of γ_r and γ_{r+1} which differ lie on the boundary of this rectangle.
- These path components are clearly homotopic. Since both of these are contained in some U, the morphism $\tilde{\eta}$ yields the same map for both path components.

Proceeding rectangle by rectangle now, we see $\tilde{\eta}[f] = \tilde{\eta}[\gamma_0] = \tilde{\eta}[\gamma_{nm}] = \tilde{\eta}[g]$, proving $\tilde{\eta}$ is well-defined. □

Theorem 4.2 (van Kampen)**.** *Let X be path-connected and choose a basepoint $x \in X$. Let $\mathscr{O}$ be a cover of X by path-connected subsets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$ and x is in each $U \in \mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe*

$$\pi_1|_{\mathscr{O}} : \mathscr{O} \to \mathbf{Grp}$$

gives a diagram of groups. The group $\pi_1(X, x)$ is the colimit of this diagram. That is $\pi_1(X, x) \cong \operatorname{colim}_{U \in \mathscr{O}} \pi_1(U, x)$.

Proof. Step 1: The theorem holds when $\mathscr{O}$ is finite.

Define $\iota : \pi_1|_{\mathscr{O}} \to \underline{\pi_1(X, x)}$ by determining $\iota_U : \pi_1(U, x) \to \pi_1(X, x)$ via the inclusion $U \to X$. Let G be a group and $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$ a morphism of diagrams. The inclusion functor $J : \pi_1(X, x) \to \Pi(X)$ is an equivalence of categories. By Proposition 2.10, we determine an inverse equivalence $F : \Pi(X) \to \pi_1(X, x)$ by a choice of path classes $y \to x$. By making appropriate choices of path classes, we can ensure we obtain compatible inverse equivalences for each inclusion functor $J_U : \pi_1(U, x) \to \Pi(U)$. To do this, we choose our path classes as follows:

- For every $y \in X$ we need to choose an isomorphism $\alpha_y : y \to x$. Let U_y be the intersection of all $U \in \mathscr{O}$ containing y. Since $U_y \in \mathscr{O}$, choose α_y such that $\operatorname{im}(\alpha_y) \subset U_y$. With these choices we determine the path class of $f : y \to z$ by the map $f \mapsto \alpha_z \cdot f \cdot \alpha_y^{-1}$.
- The choices above guarantee compatible inverses to each inclusion $J_U : \pi_1(U, x) \to \Pi(U)$. By choosing $\alpha_x = c_x$ we force the relations $F \circ J = \mathrm{id}_{\pi_1(X,x)}$ and $F_U \circ J_U = \mathrm{id}_{\pi_1(U,x)}$ for each $U \in \mathscr{O}$.

Keeping these equivalences in mind, note that the functors

$$\Pi(U) \xrightarrow{F_U} \pi_1(U,x) \xrightarrow{\eta_U} G$$

specify a diagram of groupoids $\Pi|_{\mathscr{O}} \to \underline{G}$. Applying Theorem 4.1, we obtain a unique map $\zeta : \Pi(X) \to G$ such that $\zeta|_{\Pi(U)} = \eta_U \circ F_U$.

Now, set $\tilde{\eta} = \zeta \circ J$. The diagram

$$\begin{tikzcd}
\pi_1(U,x) \arrow[r, "J_U"] & \Pi(U) \arrow[rr, "\zeta"] \arrow[dr, "F_U"'] & & G \\
& & \pi_1(U,x) \arrow[ur, "\eta_U"'] &
\end{tikzcd}$$

shows $\tilde{\eta}|_{\pi_1(U,x)} = \zeta \circ J_U = \eta_U \circ F_U \circ J_U = \eta_U$ since $F_U \circ J_U = \mathrm{id}_{\pi_1(U,x)}$. Therefore, the map $\tilde{\eta}$ makes the colimit diagram commute. To complete the proof of this case, we just need to check uniqueness. To do this, suppose $\tilde{\xi} : \pi_1(X,x) \to G$ is a map such that $\tilde{\xi}|_{\pi_1(U,x)} = \eta_U$. Then $\tilde{\xi} \circ F|_{\pi_1(U,x)} = \eta_U \circ F_U$. Since the map ζ is unique, this tells us $\zeta = \tilde{\xi} \circ F$. And finally, because $F \circ J = \mathrm{id}_{\pi_1(X,x)}$, we have $\tilde{\xi} = \zeta \circ J = \tilde{\eta}$.

To prove the general case, let $\mathscr{F}$ be the collection of finite subsets of $\mathscr{O}$. For $\mathscr{L} \in \mathscr{F}$ we define $U_{\mathscr{L}} = \bigcup_{U \in \mathscr{L}} U$. Now, let $\mathscr{O}\mathscr{F}$ be the category whose objects are the $U_{\mathscr{L}}$ (for $\mathscr{L} \in \mathscr{F}$) and whose morphisms $U_{\mathscr{L}} \to U_{\mathscr{T}}$ are inclusions.

Step 2: $\mathrm{colim}_{U_{\mathscr{L}} \in \mathscr{O}\mathscr{F}} \pi_1(U_{\mathscr{L}}, x) \cong \pi_1(X,x)$.

Define $\iota : \pi_1|_{\mathscr{O}\mathscr{F}} \to \underline{\pi(X,x)}$ by determining $\iota_{U_{\mathscr{L}}} : \pi_1(U_{\mathscr{L}}, x) \to \pi(X,x)$ via the inclusion $U_{\mathscr{L}} \to X$. Let G be a group and $\eta : \pi_1|_{\mathscr{O}\mathscr{F}} \to \underline{G}$ a morphism of diagrams. Now, given any loop $f : I \to X$, we can use the compactness of I to subdivide the interval into finitely many disjoint segments, each of which maps into some $U \in \mathscr{O}$. Considering the union of these subsets, we see $\mathrm{im}(f) \subset U_{\mathscr{L}}$ for some $U_{\mathscr{L}} \in \mathscr{O}\mathscr{F}$. Utilizing this decomposition, define $\tilde{\eta} : \pi_1(X,x) \to G$ by $\tilde{\eta}[f] = \eta_{U_{\mathscr{L}}}[f]$. This morphism makes the colimit diagram commute, and therefore we just need to verify it is well-defined.

To check this, suppose $f \sim g$ and let $h : I \times I \to X$ be the required basepoint-preserving homotopy. Using the compactness of $I \times I$ we can subdivide the unit square into finitely many subrectangles, each of which maps into some $U \in \mathscr{O}$. In particular, we can choose this subdivision such that it refines the subdivisions of $I \times \{0\}$ and $I \times \{1\}$ showing $\mathrm{im}(f) \subset U_{\mathscr{L}}$ and $\mathrm{im}(g) \subset U_{\mathscr{G}}$ respectively. The resulting subdivision of $I \times I$ shows $\mathrm{im}(h) \subset U_{\mathscr{T}}$ for some $\mathscr{T} \in \mathscr{F}$. Clearly $U_{\mathscr{L}}, U_{\mathscr{G}} \subset U_{\mathscr{T}}$, and therefore we have inclusion morphisms $d_1 : U_{\mathscr{L}} \to U_{\mathscr{T}}$ and $d_2 : U_{\mathscr{G}} \to U_{\mathscr{T}}$. Since η is a natural transformation, the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U_{\mathscr{L}}, x) \arrow[r, "\pi_1(d_1)"] \arrow[dr, "\eta_{U_{\mathscr{L}}}"'] & \pi_1(U_{\mathscr{T}}, x) \arrow[d, "\eta_{U_{\mathscr{T}}}"] & \pi_1(U_{\mathscr{G}}, x) \arrow[l, "\pi_1(d_2)"'] \arrow[dl, "\eta_{U_{\mathscr{G}}}"] \\
& G. &
\end{tikzcd}$$

This gives us the equalities $\eta_{U_{\mathscr{L}}}[f] = \eta_{U_{\mathscr{T}}}[f] = \eta_{U_{\mathscr{T}}}[g] = \eta_{U_{\mathscr{G}}}[g]$, verifying that $\tilde{\eta}$ is well-defined.

STEP 3: $\operatorname{colim}_{U\in\mathscr{O}} \pi_1(U,x) \cong \pi_1(X,x)$.

If a natural transformation indexed on $\mathscr{O}$ extends uniquely to one indexed on $\mathscr{O}\mathscr{F}$ and if a natural transformation indexed on $\mathscr{O}\mathscr{F}$ restricts uniquely to one indexed on $\mathscr{O}$, then the diagrams have isomorphic colimits. By Step 2, the statement about restrictions is obvious; and therefore, verifying that a natural transformation indexed on $\mathscr{O}$ has a unique extension completes the proof.

By Step 1, we know $\operatorname{colim}_{U\in\mathscr{L}} \pi_1(U,x) \cong \pi_1(U_{\mathscr{L}},x)$. This allows us to extend a natural transformation $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$ to a natural transformation $\hat{\eta} : \pi_1|_{\mathscr{O}\mathscr{F}}$ as follows:

- For each $\mathscr{L} \in \mathscr{F}$ we obtain a morphism $\hat{\eta}_{U_{\mathscr{L}}} : \pi_1(U_{\mathscr{L}},x) \to G$ by the colimit property of $\pi_1(U_{\mathscr{L}},x)$. We claim the $\hat{\eta}_{U_{\mathscr{L}}}$ determine a natural transformation $\hat{\eta} : \pi_1|_{\mathscr{O}\mathscr{F}} \to \underline{G}$.
- Suppose $U_{\mathscr{L}} \subset U_{\mathscr{G}}$ and define $\overline{\mathscr{L}\cup\mathscr{G}}$ to be the finite intersections of elements of $\mathscr{L}$ and $\mathscr{G}$. Then $U_{\mathscr{G}} = U_{\overline{\mathscr{L}\cup\mathscr{G}}}$. Since the map $\hat{\eta}_{U_{\mathscr{G}}}$ is unique by the colimit property of $\pi_1(U_{\mathscr{G}},x)$, this forces the equality $\hat{\eta}_{U_{\mathscr{G}}} = \hat{\eta}_{U_{\mathscr{L}\cup\mathscr{G}}}$. Therefore, we only need to consider cases where $\mathscr{L} \subset \mathscr{G}$.
- Suppose $\mathscr{L} \subset \mathscr{G}$ and note that the natural transformation given by the composite $\mathscr{L} \to \pi_1(U_{\mathscr{G}},x) \to \underline{G}$ is equal to $\eta|_{\pi_1|_{\mathscr{L}}}$. Therefore, these induce the same map $\pi_1(U_{\mathscr{L}},x) \to G$. Letting $d : U_{\mathscr{L}} \to U_{\mathscr{G}}$ be the inclusion, this forces the following diagram to commute:

$$\begin{tikzcd}
\pi_1(U,x) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & \pi_1(U_{\mathscr{L}},x) \arrow[dl, "\pi_1(d)"'] \arrow[ddl, bend left, "\hat{\eta}_{U_{\mathscr{L}}}"] \\
& \pi_1(U_{\mathscr{G}},x) \arrow[d, "\hat{\eta}_{U_{\mathscr{G}}}"] & \\
& G. &
\end{tikzcd}$$

Commutativity gives us the relation $\hat{\eta}_{U_{\mathscr{G}}} \circ \pi_1(d) = \hat{\eta}_{U_{\mathscr{L}}}$, verifying that $\hat{\eta} : \pi_1|_{\mathscr{O}\mathscr{F}} \to \underline{G}$ is a natural transformation. Uniqueness follows from the observation that the $\hat{\eta}_{U_{\mathscr{L}}}$ were uniquely determined by the η_U.

Since we have unique extensions and restrictions, we know diagrams indexed on $\mathscr{O}$ and diagrams indexed on $\mathscr{O}\mathscr{F}$ have isomorphic colimits. □

Corollary. *Let $X = U\cup V$ where U, V and $U\cap V$ are path-connected neighborhoods of the basepoint of X and $\pi_1(V) = 0$. Then $\pi_1(U) \to \pi_1(X)$ is a surjection whose kernel is the smallest normal subgroup that contains the image of $\pi_1(U\cap V)$.*

Proof. Let N be the kernel described above and consider the following commutative diagram:

$$\begin{tikzcd}
& \pi_1(U) \arrow[dr] \arrow[drr] & & \\
\pi_1(U\cap V) \arrow[ur] \arrow[dr] & & \pi_1(X) \arrow[r, "\xi"] & \pi_1(U)/N. \\
& \pi_1(V) = 0 \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

The van Kampen Theorem tells us that $\pi_1(X)$ is the pushout of the diagram above, guaranteeing the existence ξ. By a quick inspection, we also see that $\pi_1(U)/N$ is the pushout of the homomorphisms $\pi_1(U) \longleftarrow \pi_1(U \cap V) \longrightarrow \pi_1(V)$. Therefore, ξ is an isomorphism, completing the proof. $\square$

5. Application to cell complexes

In this section, we will introduce the basic theory of cell complexes to illustrate some applications of the van Kampen Theorem. However, before we treat cell complexes, we will develop some further background needed for the applications. Therefore, we will begin by introducing additional homotopy theory and then computing the fundamental group for some key spaces.

More homotopy theory

Proposition 5.1. *Suppose X is a path-connected space and $x, y \in X$. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. Choose a path $a : x \to y$ and define a homomorphism $\gamma[a] : \pi_1(X, x) \to \pi_1(X, y)$ by setting $\gamma[a][f] = [a \cdot f \cdot a^{-1}]$ for $[f] \in \pi_1(X, x)$. To check that this is well-defined, suppose $g \in [f]$ and let $h : f \simeq g$ be a basepoint-preserving homotopy. The map $k = a \cdot h \cdot a^{-1}$ shows $a \cdot f \cdot a^{-1} \sim a \cdot g \cdot a^{-1}$, as needed.

Given a path $b : y \to z$, note that $\gamma[b \cdot a][f] = [(b \cdot a) \cdot f \cdot (b \cdot a)^{-1}] = [b \cdot (a \cdot f \cdot a^{-1}) \cdot b^{-1}] = (\gamma[b] \circ \gamma[a])[f]$ for each $[f] \in \pi_1(X, x)$. This means $\gamma[a]$ is an isomorphism with inverse $\gamma[a^{-1}]$, establishing the desired result. $\square$

Remark. For path-connected spaces, this result shows that given any two basepoints, the resulting fundamental groups are isomorphic. Therefore, so long as we are only concerned with isomorphism classes, this result justifies dropping the basepoint notation.

Proposition 5.2. *Suppose $p, q : X \to Y$ are continuous maps of topological spaces and $h : p \simeq q$ is a homotopy. Let $a : p(x) \to q(x)$ be the path specified by $a(t) = h(x, t)$. Then the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x) \arrow[rr, "\pi_1(q)"] \arrow[dr, "\pi_1(p)"'] & & \pi_1(Y, q(x)). \\
& \pi_1(Y, p(x)) \arrow[ur, "\gamma[a]"'] &
\end{tikzcd}$$

Proof. Let $f : I \to X$ be a loop at x. To make the diagram commute, we must show that $q \circ f \sim a \cdot (p \circ f) \cdot a^{-1}$. A quick argument shows this is equivalent to proving $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f) \sim c_{p(x)}$. Defining $j : I \times I \to Y$ by $j(s, t) = h(f(s), t)$ and examining the behavior of this map on the boundary of $I \times I$, we obtain the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "q \circ f"] & \cdot \\
\cdot \arrow[u, "a"] \arrow[r, "p \circ f"'] & \cdot \arrow[u, "a"']
\end{tikzcd}$$

Beginning at the bottom left corner and travelling counterclockwise around the diagram, we trace out the path $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$.

Now, define $r_t : I \to I \times I$ such that each quarter interval maps linearly onto an edge of the subsquare $[0,t] \times [0,t]$. Setting $k(s,t) = j(r_t(s))$ proves $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f) \sim c_{p(x)}$ as needed. □

Definition 5.3. A continuous map $f : X \to Y$ is called a homotopy equivalence if there exists a continuous map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We say two spaces X and Y are homotopy equivalent if there is a homotopy equivalence $f : X \to Y$. The space X is said to be contractible if it is homotopy equivalent to a point.

Proposition 5.4. *Suppose $f : X \to Y$ is a homotopy equivalence and $x \in X$. Then $\pi_1(X,x) \cong \pi_1(Y, f(x))$.*

Proof. Let $g : Y \to X$ be an inverse homotopy equivalences. Considering the compositions $g \circ f$ and $f \circ g$, the functoriality of π_1 yields:

$$\pi_1(X,x) \xrightarrow{\pi_1(f)} \pi_1(Y, f(x)) \xrightarrow{\pi_1(g)} \pi_1(X, (g \circ f)(x))$$

$$\pi_1(Y,y) \xrightarrow{\pi_1(g)} \pi_1(X, g(y)) \xrightarrow{\pi_1(f)} \pi_1(Y, (f \circ g)(y))$$

Because f and g are homotopy equivalences, we have homotopies $h : g \circ f \simeq \mathrm{id}_X$ and $k : f \circ g \simeq \mathrm{id}_Y$. Applying the previous proposition, we obtain paths $\alpha : x \to (g \circ f)(x)$ and $\beta : y \to (f \circ g)(y)$ such that $\pi_1(g) \circ \pi_1(f) = \gamma[\alpha] \circ \pi_1(\mathrm{id}_X) = \gamma[\alpha]$ and $\pi_1(f) \circ \pi_1(g) = \gamma[\beta] \circ \pi_1(\mathrm{id}_Y) = \gamma[\beta]$. Since $\gamma[\alpha]$ is an isomorphism, the map $\pi_1(g)$ is injective and $\pi_1(f)$ is surjective. Similarly, since $\gamma[\beta]$ is an isomorphism, the map $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective. This completes the proof. □

Corollary. *Suppose X is a contractible space and $x \in X$. Then $\pi_1(X,x) = 0$.*

Some computational preliminaries

Lemma 5.5. $\pi_1(\mathbb{R}^n, 0) = 0$

Proof. The map $h(s,t) = (1-t)s$ is a contraction homotopy, proving $\mathbb{R}^n$ is contractible. □

Theorem 5.6. $\pi_1(S^1, 1) \cong \mathbb{Z}$

Proof. Define the loop $f_n : I \to S^1$ by $f_n(s) = e^{2\pi i n s}$. Since $[f_n][f_m] = [f_{m+n}]$, we have a homomorphism $i : \mathbb{Z} \to \pi_1(S^1, 1)$ defined by $i(n) = [f_n]$. To complete the proof we will show that i is an isomorphism.

Define $p : \mathbb{R} \to S^1$ by $p(s) = e^{2\pi i s}$. We claim that for every path $f : I \to S^1$ with $f(0) = 1$, there is a unique path $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $f = p \circ \tilde{f}$. To see this, consider a cover of S^1 by small connected neighborhoods $U \subset S^1$. Using the compactness of I, subdivide the interval into finitely many disjoint segments, each of which f carries into some U. Proceeding subinterval by subinterval now, we can construct the desired lifting by noting that the lifting on each subinterval is uniquely determined by its lifting on its initial point.

Using this lifting, define the map $j : \pi_1(S^1, 1) \to \mathbb{Z}$ by $j[f] = \tilde{f}(1)$. To check that this is well-defined, note that $(p \circ \tilde{f})(1) = 1$ implies $\tilde{f} \in \mathbb{Z}$. Moreover, suppose $g \in [f]$ and let $h : f \simeq g$ be a basepoint-preserving homotopy. Arguing as we did before, we can construct a unique lift $\tilde{h} : I \times I \to \mathbb{R}$ such that $p \circ \tilde{h} = h$. Since $h(1,t) = p(\tilde{h}(1,t))$ specifies a constant path, we know $\tilde{h}(1,t)$ is a constant path. This tells us $\tilde{f}(1) = \tilde{h}(1,0) = \tilde{h}(1,1) = \tilde{g}(1)$, verifying that j is well-defined.

Since $j[f_n] = n$, we have $j \circ i = \mathrm{id}_{\mathbb{Z}}$, establishing that j is onto. If we can show j is one-to-one, this will complete the proof. Therefore, suppose $j[f] = j[g]$. This means $\tilde{f}(1) = \tilde{g}(1)$, so $[\tilde{g}^{-1} \cdot \tilde{f}] = [c_0]$ by the lemma introduced above. Utilizing the functoriality of π_1, we see $\pi_1(p)[\tilde{g}^{-1} \cdot \tilde{f}] = [g^{-1}][f] = [c_1]$, so $[f] = [g]$. This shows i and j are inverse isomorphisms. □

Theorem 5.7. *$\pi_1(S^n) = 0$ for $n \geq 2$.*

Proof. Let $n, s \in S^n$ be antipodal points. Stereographic projection yields a homeomorphism between $S^n \setminus \{n\}$ and R^n. Therefore $\pi_1(S^n \setminus \{n\}) = 0$ and $\pi_1(S^n \setminus \{s\}) = 0$. Set $U = S^n \setminus \{n\}$ and $V = S^n \setminus \{s\}$. Applying the corollary to Theorem 4.2, it follows that $\pi_1(S^n) \cong \pi_1(S^n \setminus \{n\})/N = 0$. □

Proposition 5.8. *Suppose $\bigvee_{i \in I} V_i$ is a wedge sum of contractible spaces. Then $\bigvee_{i \in I} V_i$ is contractible.*

Proof. Let x_0 be the point at which we identify the V_i. Let $f_i : V_i \to \{x_0\}$ and $g_i : \{x_0\} \to V_i$ be homotopy equivalences. Therefore, for each $i \in I$, we have homotopies $h_i : g_i \circ f_i \simeq \mathrm{id}_{V_i}$ and $k_i : f_i \circ g_i \simeq \mathrm{id}_{x_0}$.

Define $f : \bigvee_{i \in I} V_i \to \{x_0\}$ by setting $f(x) = f_i(x)$ for $x \in V_i$. Choose $g : \{x_0\} \to \bigvee_{i \in I} V_i$ to be the inclusion. Setting $h(s,t) = h_i(s,t)$ for $s \in V_i$, we see $g \circ f \simeq \mathrm{id}_{\bigvee_{i \in I} V_i}$. Defining $k(s,t) = k_i(s,t)$ for $s \in V_i$, we obtain the relation $f \circ g \simeq \mathrm{id}_{x_0}$, completing the proof. □

Proposition 5.9. *Let X be any path-connected space and suppose V is path-connected and contractible. Then $\pi_1(X \vee V) \cong \pi_1(X)$*

Proof. Let x_0 be the point at which we identify X and V. Since V is contractible, this gives us homotopy equivalences $f_1 : V \to \{x_0\}$ and $g_1 : \{x_0\} \to V$. Let $h_1 : g_1 \circ f_1 \simeq \mathrm{id}_V$ and $k_1 : f_1 \circ g_1 \simeq \mathrm{id}_{\{x_0\}}$ be the homotopies obtained from these maps.

Define $f : X \vee V \to X$ such that $f|_X = \mathrm{id}_X$ and $f|_V = f_1$. Choose $g : X \to X \vee V$ to be the inclusion. Define $h : g \circ f \simeq \mathrm{id}_{X \vee V}$ by $h(x,t) = (g \circ f)(x)$ for $x \in X$ and $h(x,t) = h_1(x,t)$ for $x \in V$. Likewise, define $k : f \circ g \simeq \mathrm{id}_X$ by $k(x,t) = (f \circ g)(x)$. These maps show X and $X \vee V$ are homotopy equivalent. Applying Proposition 5.4 and utilizing the path-connectedness of X and $X \vee V$, the proposition follows. □

Theorem 5.10. *Let $X = \bigvee_{i \in I} X_i$ be the wedge sum of path-connected based spaces X_i, each of which contains a contractible neighborhood V_i of its basepoint. Then $\pi_1(X) = *_{i \in I} \pi_1(X_i)$.*

Proof. Let $U_i = X_i \bigvee_{j \neq i} V_j$ and apply the van Kampen Theorem with $\mathscr{O}$ taken as the U_i and their finite intersections. We claim $*_{i \in I} \pi_1(X_i) \cong \mathrm{colim}_{U \in \mathscr{O}} \pi_1(U)$. To see this, consider $U \in \mathscr{O}$. If U is the intersection of two or more U_i, then U is a wedge sum of contractible spaces. Applying Proposition 5.8 and our corollary to Proposition 5.4, this tells us $\pi_1(U) = 0$. On the other hand, if $U = U_i$ for some $i \in I$, Proposition 5.9 tell us $\pi_1(U) \cong \pi_1(X_i)$. This tells us only the U_i make a contribution to the colimit. Applying Theorem 2.17, the proof follows immediately. □

Corollary. *The fundamental groups of a wedge sum of circles is a free group with one generator for each circle.*

Cell complexes

Definition 5.11. A CW complex is a space X which we form inductively as follows:

- Let X_0 be a discrete set which we call the 0-skeleton of X.
- Given an index set I, form the n-skeleton X^n from X^{n-1} by attaching disks D^n via maps $j_i : S^{n-1} \to D^{n-1}$. That is $X^n = (X^{n-1} \coprod_{i \in I} D_i^n)/\sim$ where we identify $x \sim j_i(x)$ for $x \in S_i^{n-1}$.
- Each X^n is an n-dimensional CW complex. Letting $X = \bigcup_{n \in \mathbb{N}} X^n$ with the weak topology, we obtain an infinite dimensional CW complex.

We call each mapping $\Phi_i : D_i^n \to X$ an n-cell. We can characterize the topology on X as the coarsest topology for which each map Φ_i is homeomorphic on $\mathrm{int}(D_i^n)$. We write $e_i^n = \Phi_i(\mathrm{int}(D_i^n))$.

Theorem 5.12. *Attach 2-cells to a path-connected space X via maps $j_i : S^1 \to X$, producing the space Y. Each j_i determines a loop at $j_i(1)$. Given a basepoint $x \in X$ and paths $\gamma_i : j_i(1) \to x$, the map $\gamma_i \cdot j_i \cdot \gamma_i^{-1}$ is a loop at x. Let $N \subset \pi_1(X, x)$ be the normal subgroup generated by these loops. Then the inclusion $X \to Y$ induces a surjection $\pi_1(X, x) \to \pi_1(Y, x)$ whose kernel is N.*

Proof. Step 1: The result holds when the number of 2-cells is finite.

Suppose first that we add a single 2-cell via a map $j : S^1 \to X$. This induces a map $\Phi : D^2 \to Y$. Choose $y = \Phi(0)$ and select a basepoint $d_0 \in \mathrm{int}(D^2 \setminus \{0\})$. Setting $U = Y \setminus \{y\}$ and $V = e^2$, we obtain a cover of Y by path-connected open subsets. The subspace $U \cap V = e^2 \setminus \{y\}$ is clearly path-connected and homotopy equivalent to S^1. Applying the van Kampen Theorem to $\{U, V, U \cap V\}$ produces the following commutative diagram:

$$\begin{tikzcd}
 & \pi_1(U, \Phi(d_0)) \arrow[dr] \arrow[drr] & & \\
\mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1(Y, \Phi(d_0)) \arrow[r] & \pi_1(U, \Phi(d_0))/N, \\
 & \pi_1(V, \Phi(d_0)) = 0 \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

where N is the smallest normal subgroup containing the image of the homomorphism $\mathbb{Z} \to \pi_1(U, \Phi(d_0))$. In particular, this tells us $N = \langle [j] \rangle$. Since U and X are homotopy equivalent spaces, we obtain the following relations

$$\pi_1(U, \Phi(d_0))/\langle [j] \rangle \cong \pi_1(U, x)/\langle [\gamma \cdot j \cdot \gamma^{-1}] \rangle \cong \pi_1(X, x)/\langle [\gamma \cdot j \cdot \gamma^{-1}] \rangle,$$

establishing the case for a single 2-cell. Proceeding by induction, we obtain the result when the number of 2-cells is finite.

Step 2: The result holds when the number of 2-cells is arbitrary.

Suppose we attach an arbitrary number of 2-cells via maps $j_i : S^1 \to X$. Define $y_i = \Phi_i(0)$ and set $U_i = Y \setminus \bigcup_{j \neq i} \{y_j\}$. A simple argument shows U_i is homotopy equivalent to $X \cup e_i^2$. Apply the van Kampen Theorem with $\mathscr{O}$ taken as the U_i and their finite intersections. We claim that $\pi_1(X, x)/N \cong \mathrm{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x)$. To see this, define the morphism of diagrams $\iota : \pi_1|_{\mathscr{O}} \to \underline{\pi_1(X, x)/N}$ as follows:

- Define $\iota_X : \pi_1(X,x) \to \pi_1(X,x)/N$ such that $\iota_X|_{\pi_1(X,x)/N}$ is the identity homomorphism and $\ker(\iota_X) = N$.
- Suppose $d_i : U_i \to X$ is the inclusion. Step 1 shows $\pi_1(U_i,x) = \pi_1(X,x)/\langle[\gamma_i \cdot j_i \cdot \gamma_i^{-1}]\rangle$. Applying this result again, we see that $\pi_1(d_i)|_{\pi_1(U_i,x)}$ is the identity homomorphism and $\ker(\pi_1(d_i)) = \langle[\gamma_i \cdot j_i \cdot \gamma_i^{-1}]\rangle \subset N$. This shows defining the map $\iota_{U_i} : \pi_1(U_i,x) \to \pi_1(X,x)/N$ by requiring $\iota_X = \iota_{U_i} \circ \pi_1(d_i)$ is well-defined.

Now suppose $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$ is a morphism of diagrams. In particular, this means the following diagram commutes:

$$\begin{tikzcd} \pi_1(U_i,x) \arrow[dr, "\eta_{U_i}"'] & \pi_1(X,x) \arrow[l, "\pi_1(d_i)"'] \arrow[d, "\eta_X"] \\ & G. \end{tikzcd}$$

This implies $\ker(\eta_X) \supset N$. Therefore, η_X factors through the quotient map $\pi_1(X,x) \to \pi_1(X,x)/N$, producing the unique map $\tilde{\eta} : \pi_1(X,x)/N \to G$. This morphism clearly makes the colimit diagram commute and the van Kampen Theorem then yields the relations

$$\pi_1(Y,x) \cong \operatorname{colim}_{U_i \in \mathscr{O}} \pi_1(U_i,x) \cong \pi_1(X,x)/N,$$

completing the proof. □

Theorem 5.13. *Attach n-cells ($n \geq 3$) to a path-connected space X via maps $j_i : S^{n-1} \to X$, producing the space Y. Choose a basepoint $x \in X$. Then $\pi_1(X,x) \cong \pi_1(Y,x)$.*

Proof. Step 1: The result holds when the number of n-cells is finite.

Suppose first we attach a single n-cell via the map $j : S^{n-1} \to X$. This induces a map $\Phi : D^n \to Y$. Choose $y = \Phi(0)$ and select a basepoint $d_0 \in \operatorname{int}(D^n \setminus \{0\})$. Set $U = Y \setminus \{y\}$ and $V = e^n$. A simple argument shows V is contractible and U is homotopy equivalent to X. Likewise, the subspace $U \cap V = e^n \setminus \{y\}$ is path-connected and homotopy equivalent to S^{n-1}. Applying the van Kampen Theorem to the cover $\{U, V, U \cap V\}$ gives us the relation $\pi_1(Y, \Phi(d_0)) \cong \pi_1(U, \Phi(d_0))/N$ where N is the smallest normal subgroup containing the image of the homomorphism $\pi_1(U \cap V, \Phi(d_0)) \to \pi_1(U, \Phi(d_0))$. Applying Theorem 5.7, we see $\pi_1(U \cap V, \Phi(d_0)) \cong \pi_1(S^{n-1}) = 0$, showing that $N = 0$. Therefore

$$\pi_1(Y,x) \cong \pi_1(Y, \Phi(d_0)) \cong \pi_1(U, \Phi(d_0)) \cong \pi_1(U,x) \cong \pi_1(X,x),$$

as desired.

Step 2: The result holds when the number of n-cells is arbitrary.

Now suppose we attach arbitrarily many n-cells via maps $j_i : S^{n-1} \to X$. Define $y_i = \Phi_i(0)$ and set $U_i = Y \setminus \bigcup_{j \neq i} \{y_j\}$. A quick argument will show U_i is homotopy equivalent to $X \cup e_i^n$. Apply the van Kampen Theorem with $\mathscr{O}$ taken as the U_i and their finite intersections. We claim $\pi_1(X,x) \cong \operatorname{colim}_{U_i \in \mathscr{O}} \pi_1(U_i,x)$. To see this, consider $U \in \mathscr{O}$. If U is the intersection of two or more of the U_i, then $U = X$. If $U = U_i$ then $\pi_1(U_i,x) \cong \pi_1(X,x)$ by the step above. Therefore, we can define $\iota : \pi_1|_{\mathscr{O}} \to \underline{\pi_1(X,x)}$ by choosing isomorphisms $\iota_U : \pi_1(U,x) \to \pi_1(X,x)$ for each $U \in \mathscr{O}$. Given a morphism of diagrams $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$, define $\tilde{\eta} : \pi_1(X,x) \to G$ by

$\tilde{\eta} = \eta_X$. This choice of morphisms makes the colimit diagram commute. The van Kampen Theorem yields the relations

$$\pi_1(X, x) \cong \operatorname{colim}_{U_i \in \mathscr{O}} \pi_1(U_i, x) \cong \pi_1(Y, x),$$

completing the proof. □

Remark. Theorem 5.10 shows that 1-cells produce generators, Theorem 5.12 shows that 2-cells produce relations, and Theorem 5.13 shows that higher order cells do nothing. Therefore, the fundamental group of a CW complex is determined entirely by its 2-skeleton.

Applications

Definition 5.14. Consider a regular $4g$-gon whose edges we label sequentially as $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$. Give these edges orientation as follows: Each edge with a "-1" superscript is oriented clockwise, and the edges without the superscript are oriented counter-clockwise. An orientable surface of genus g is the quotient space of this $4g$-gon under the oriented identification $a_i \sim a_i^{-1}$.

Now consider a $2h$-gon whose edges we label sequentially as $a_1, a_1, \ldots, a_h, a_h$. Assign each edge a counter-clockwise orientation. A non-orientable surface of genus h is the quotient space of this $2h$-gon under the oriented identification $a_i \sim a_i$.

Theorem 5.15 (Classification of surfaces)**.** *Every compact connected surface is homeomorphic to one and only one of the following: The 2-sphere S^2, an orientable surface of genus g, or a non-orientable surface of genus h.*

Proof. A proof of the classification of compact surfaces can be found in many standard texts on topology and is therefore omitted (See [3] or [5] for a detailed discussion of the theorem including its proof). □

Theorem 5.16. *Suppose X is an orientable surface of genus g. Then $\pi_1(X) \cong \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle$.*

Proof. We can easily check that X has the following cell-decomposition: A 0-skeleton consisting of a single point, a 1-skeleton consisting of the wedge sum of $2g$ circles, and a 2-skeleton consisting of a single 2-cell glued along the loop $[a_1, b_1] \cdots [a_g, b_g]$. Applying Theorem 5.10 and Theorem 5.12, we immediately see $\pi_1(X) \cong \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle$, as desired. □

Theorem 5.17. *Suppose X is a non-orientable surface of genus h. Then $\pi_1(X) \cong \langle a_1, \ldots, a_h | a_1^2 \cdots a_h^2 \rangle$.*

Proof. We can easily check that X has the following cell-decomposition: A 0-skeleton consisting of a single point, a 1-skeleton consisting of the wedge sum of h circles, and a 2-skeleton consisting of a single 2-cell glued along the loop $a_1^2 \cdots a_h^2$. Applying Theorem 5.10 and Theorem 5.12, we immediately see $\pi_1(X) \cong \langle a_1, \ldots, a_h | a_1^2 \cdots a_h^2 \rangle$, as desired. □

Remark. Theorem 5.6, Theorem 5.16, and Theorem 5.17 compute the fundamental group for every surface listed under our classification theorem. Since homeomorphism is a homotopy equivalence, this means we have computed the fundamental group for every compact connected surface.

Theorem 5.18. *Suppose G is a group. Then there exists a space X such that $\pi_1(X) \cong G$.*

Proof. Choose a presentation $G = \langle S|R\rangle$ and construct X by letting its 0-skeleton consist of a single point, its 1-skeleton consist of a wedge sum of circles (one for each generator), and its 2-skeleton consist of 2-cells glued along the words specified in R. Applying Theorem 5.10 and Theorem 5.12 shows $\pi_1(X) \cong \langle S|R\rangle = G$, as desired. □

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