THE CONTINUUM HYPOTHESIS AND ITS RELATION TO THE LUSIN SET

CLIVE CHANG

ABSTRACT. In this paper, we prove that the Continuum Hypothesis is equivalent to the existence of a subset of $\mathbb{R}$ called a Lusin set and the property that every subset of $\mathbb{R}$ with cardinality $< c$ is of first category. Additionally, we note an interesting consequence of the measure of a Lusin set, specifically that it has measure zero. We introduce the concepts of ordinals and cardinals, as well as discuss some basic point-set topology.

CONTENTS

1. INTRODUCTION

Throughout much of the early and middle twentieth century, the Continuum Hypothesis (CH) served as one of the premier problems in the foundations of mathematical set theory, attracting the attention of countless famous mathematicians, most notably, Gödel, Cantor, Cohen, and Hilbert. The conjecture first advanced by Cantor in 1877, makes a claim about the relationship between the cardinality of the continuum ($\mathbb{R}$) and the cardinality of the natural numbers ($\mathbb{N}$), in relation to infinite set hierarchy. Though ultimately proven undecidable by Cohen in 1963 (under ZF : Zermelo-Frankel Set Theory), the hypothesis implies many interesting

Date: AUGUST 25, 2011.

consequences. Additionally, CH has led to a more universal statement, called the generalized continuum hypothesis (GCH), which establishes a connection between all cardinalities of higher Cantor (transfinite) sets and the sets themselves or their ordinal numbers.

In this paper, we discuss an equivalent statement of CH and note its correspondence with a unique topological set, called the Lusin set.

2. Ordinals

Definition 2.1. A set S is called *well-ordered* if every non-empty subset of S has a least element under this ordering.

Definition 2.2. A *monotone function* is a function that preserves order.

Definition 2.3. Two sets X and Y have the same *order type* if there exists a bijection, F from X to Y, such that F^{-1} and F are both monotone functions.

Definition 2.4. An *ordinal number*, or *ordinal*, is the order type of a well-ordered set.

Under the definitions above, every well-ordered set can then be assigned a unique ordinal number (i.e. is order isomorphic to a particular ordinal). It is in this sense that we can now view ordinals as extensions of the natural numbers. We associate the 0 ordinal with the empty set, 1 with the set containing the empty set, 2 with the set containing 0 and 1 and etc.

3. Cardinals and Countability

We now move on to discussing the notion of cardinals and their direct relation to ordinals.

In an imprecise sense, the cardinality of a set is a measure of the size of the set. We can define this notion more formally as follows.

Definition 3.1. Two sets X and Y are called *equinumerous* if there exists a bijection F from X to Y.

Definition 3.2. The *cardinality* of a set A, $|A|$, is defined to be its equivalence class under equinumerosity.

We can adjust our current definition of the cardinality of a set A, the set of all sets that have a bijection with A, by assigning to the set a representative set that is equinumerous to A. Conventionally, this set is the smallest ordinal number equinumerous to A.

As well, we can define an order on cardinals. For cardinals A and B, $A < B$ if there exists an injective map from A to B, but there does not exist an injection from B to A.

Definition 3.3. A *countable set* is a set that has the same cardinality as $\mathbb{N}$ or a subset of $\mathbb{N}$.

4. The Continuum Hypothesis and Aleph Numbers

Before we state the Continuum Hypothesis, we introduce the aleph number notation for ordinals. $\aleph_0$ is assigned to the ordinal ω, or the set of natural numbers. $\aleph_1$ is assigned to the first uncountable ordinal ω_1 or the first ordinal that cannot be put in bijection with a subset of ω. In general, the aleph numbers are a sequence of numbers that denote the cardinalities of infinite sets. We only treat the above two in this paper.

The Continuum Hypothesis states the following: there is no set whose cardinality is strictly between that of the integers and that of the real numbers. Symbolically, this is equivalent to saying, $|\mathbb{R}| = \aleph_1$. Namely, there are no subsets of $\mathbb{R}$ that have cardinality between the natural numbers and the continuum.

5. The Topology on $\mathbb{R}$

We now provide a quick reminder of open, closed, and dense sets in $\mathbb{R}$. Recall that the open intervals form a basis for the topology on $\mathbb{R}$. Any open set in $\mathbb{R}$ can be expressed as a countable union of open intervals. Moreover, a set is closed if and only if it contains all its limit points.

Definition 5.1. A subset A of $\mathbb{R}$ is called *dense* (in $\mathbb{R}$) if any point x in $\mathbb{R}$ belongs to A or is a limit point of A.

Definition 5.2. A subset A of $\mathbb{R}$ is called *nowhere dense* (in $\mathbb{R}$) if there is no neighborhood in $\mathbb{R}$ on which A is dense.

Two key examples of nowhere dense sets in $\mathbb{R}$ are the rational numbers of the form $1/N$, where $N \in \mathbb{N}$, and the Cantor Set.

6. Meagre (First Category) and F_σ Sets

Definition 6.1. A *meagre set* or a *set of first category* is a countable union of nowhere dense sets.

Definition 6.2. An F_σ *set* is a countable union of closed sets.

We will now introduce a lemma that will be crucial to a later part of the proof.

Lemma 6.3. *There exists a function F from the set of all first category sets to a subset S of first-category F_σ sets, which has cardinality equal to the continuum.*

Proof. We begin by considering an arbitrary first category set, X. By definition, $X = S_1 \cup S_2 \cup S_3 \cup \ldots$, where $S_1, S_2, S_3, \ldots$ are nowhere dense sets. Consider the closure $\overline{S_i}$ of each set S_i. Each set $\overline{S_i}$ is nowhere dense (and closed). Lastly, we define our function F, as follows, $F(X) = \overline{S_1} \cup \overline{S_2} \cup \overline{S_3} \cup \ldots$ It is clear that each $F(X)$ is a first category F_σ set. Call F's image, S. We now wish to find the cardinality of S.

The important fact we use here is that every closed nowhere dense set is the boundary of an open set. Every open set is the countable union of open intervals. Thus, this implies that every closed nowhere dense set is the countable union of the boundary points of open intervals. From here, we observe that each open interval is determined by two points, $a, b \in \mathbb{R}$, making the cardinality of the set of all open intervals, $|\mathbb{R}| \cdot |\mathbb{R}|$. So after applying cardinal multiplication, we obtain $|\mathbb{R}| \cdot |\mathbb{R}| = |\mathbb{R}|$ cardinality number of open intervals.

Thus, the cardinality of the set of all closed nowhere dense sets is at most $(|\mathbb{R}| + |\{\emptyset\}|)^{\mathbb{N}} = |\mathbb{R}|^{\mathbb{N}}$, as for each closed nowhere dense set we can choose a cardinality continuum number of boundary points of open intervals or the empty set to compose the countable union. Applying cardinal exponentiation, we obtain, $|\mathbb{R}|^{\mathbb{N}} = |\mathbb{R}|$. There are $|\mathbb{R}|$ cardinality number of closed nowhere dense sets, as each singleton set is a closed nowhere dense set and there are cardinality continuum number of singletons in $\mathbb{R}$.

Similarly, the cardinality of S is again at most, $(|\mathbb{R}| + |\{\emptyset\}|)^{\mathbb{N}} = |\mathbb{R}|^{\mathbb{N}} = |\mathbb{R}|$, as each $F(X) \in S$ is a countable union of closed nowhere dense sets. As above, the singletons serve as $|\mathbb{R}|$ cardinality number of countable closed nowhere dense sets, so S has cardinality continuum.

□

7. The existence of a Lusin set, assuming CH

Definition 7.1. A *Lusin set* is a subset $A \subset \mathbb{R}$ of cardinality continuum such that A intersects every set of first category in a countable set.

Theorem 7.2. *If CH holds, then there exists a Lusin set and every subset of* $\mathbb{R}$ *with cardinality* $< c = |\mathbb{R}|$ *is of first category.*

Proof. Suppose that the Continuum Hypothesis holds. Now consider an arbitrary set of first category. It is clear that such a set is contained in a first category F_σ set, namely the union of the closures of each nowhere dense set that composes it. By Lemma 6.3, there are continuum many such first category F_σ sets. By the Continuum Hypothesis, we can enumerate our set of first category F_σ sets as $\{A_\alpha : \alpha < \omega_1\}$. We define our proposed Lusin set A, as follows. Let $A = \{x_\alpha : \alpha < \omega_1\}$, where for each α, the x_α is any element chosen from the set $x_\alpha \in \mathbb{R} \setminus (\bigcup\{A_\beta : \beta < \alpha\})$. By our construction, A has cardinality continuum (as each x_α is distinct). Also, we can note that the intersection, $A \cap A_\alpha$ is contained in the countable set, $\{x_\beta : \beta \leq \alpha\}$. This is true, as for any $x_\beta \in A$, where $\beta > \alpha$, $x_\beta \notin A_\alpha$, since by definition, $x_\beta \in \mathbb{R} \setminus (\bigcup\{A_\gamma : \gamma < \beta\})$. Thus, A is a Lusin set.

Additionally, if we assume the Continuum Hypothesis, every subset of $\mathbb{R}$ of cardinality $< c$ is countable. Taking the countable union of every singleton (nowhere dense) set in such a subset, we obtain a first category set. And so, we are done.

□

8. The Lebesque Measure on $\mathbb{R}$

Recall that there is a standard measure defined on $\mathbb{R}$, known as the Lebesgue measure, and denoted by μ. We will not construct it here, but we will recall some essential properties and definitions. The construction is discussed in detail in Rudin's Real and Complex Analysis. For the remainder of the paper, we will only consider the Lebesgue measure on $\mathbb{R}$.

We now state a few fundamental properties of the Lebesque measure. Firstly, the Lebesque measure on $\mathbb{R}$ is non-negative: for any measurable set $X \subset \mathbb{R}$, $\mu(X) \geq 0$. It is also countably additive. That is, the sum of a countable number of pairwise disjoint, measurable sets $X_1, X_2, X_3, \ldots$ is the sum of their measures. All open subsets (and hence closed subsets) of $\mathbb{R}$ are measurable under the Lebesgue measure. Moreover, any open interval (a, b) has measure equal to $b - a$. Lastly, a subset A of $\mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, there exists a countable collection of open intervals $\{I_n\}$ such that $A \subset \bigcup_n \{I_n\}$ and $\sum_n \mu(I_n) \leq \epsilon$.

9. Additional Property of the Lusin set

Lemma 9.1. *Every countable subset of* $\mathbb{R}$ *has measure zero.*

Proof. Let $X \subset \mathbb{R}$ be a countable set with elements $\{x_n : n \in \mathbb{N}\}$. Consider the open set U_ϵ, which is the union of open intervals of length $\epsilon/2^n$ around x_n, over all n. Then the measure of U_ϵ is at most $\sum \epsilon/2^i = \epsilon$. Since $X \subset U_\epsilon$, the measure of X is at most ϵ. Since $\epsilon > 0$ is arbitrary, X has zero measure.

□

Corollary 9.2. *Every Lusin set is of measure zero.*

Proof. We will exhibit a decomposition of $\mathbb{R}$ as a disjoint union $\mathbb{R} = X \cup Y$, where X is of first category and Y is of measure zero. If A is a Lusin set, then A is the disjoint union $A = (A \setminus X) \cup (A \setminus Y)$. Since X is of first category, $A \setminus X$ will be countable and hence will have measure zero by the previous lemma. Since Y has measure zero, $A \setminus Y$ will also have measure zero. This will show that $\mu(A) = \mu(A \setminus X) + \mu(A \setminus Y) = 0$.

We now define Y as follows. For every $N \in \mathbb{N}$, we can construct the open set V_N to be the union of open intervals of length $1/2^N$ centered around every rational. Let Y equal the intersection of these coverings. By the previous lemma, the set of rationals has measure zero. As in the proof of the previous lemma, construct the set U_ϵ, which is the union of countably many intervals, each of length $1/2^n$ around the nth rational. It is easy to see that the intersection $Y = \bigcap_{N\in\mathbb{N}} V_N$ is contained in U_ϵ, so $\mu(Y) \leq \mu(U_\epsilon) = \epsilon$. Since this is true for every $\epsilon > 0$, we see that $\mu(Y) = 0$.

We will now show that $X = \mathbb{R} \setminus Y$ is of first category. To do this, we prove that for every $N \in \mathbb{N}$, the set $\mathbb{R} \setminus V_N$ is nowhere dense. Since X is the union of all the sets $\mathbb{R} \setminus V_N$ (by the extended DeMorgan's Law), we will be able to conclude that X is of first category.

Assume for contradiction that for some $N \in \mathbb{N}$, $\mathbb{R} \setminus V_N$ is not nowhere dense. That is, on some open interval (a, b) of $\mathbb{R}$, we assume that $\mathbb{R} \setminus V_N$ is dense. Consider, however, any rational, $r \in (a, b)$. This point is not within $\mathbb{R} \setminus V_N$, as V_N covers all rationals. As well, this point is not a limit point of $\mathbb{R} \setminus V_N$, as we can simply consider the chosen open interval around r in V_N and that interval shares no points in common with $\mathbb{R} \setminus V_N$. Thus, this contradicts our assumption that there exists an open interval on which $\mathbb{R} \setminus V_N$ is dense, and so for every $N \in \mathbb{N}$, the set $\mathbb{R} \setminus V_N$ is nowhere dense. Therefore, we now have proven that X is a set of first category.

By the reasoning above, every Lusin set is of measure zero.

□

10. Lemma: CH is equivalent to $\mathbb{R}$ being representable as an increasing chain of countable sets

Definition 10.1. An increasing chain of sets $\{S_i\}$ is a collection of sets indexed on an ordered set I such that $i < j$ implies $S_i \subset S_j$.

Lemma 10.2. *CH holds if and only if $\mathbb{R}$ is the union of an increasing chain of countable sets.*

Proof. Assume that the Continuum Hypothesis holds. Then, $\mathbb{R}$ may be expressed as $\mathbb{R} = \{r_\alpha : \alpha < \omega_1\}$ where r_α is a real number. Define $A_\alpha = \{r_\beta : \beta < \alpha\}$. Thus, $\mathbb{R} = \bigcup\{A_\alpha : \alpha < \omega_1\}$, and $\mathbb{R}$ is representable as the union of an increasing chain of countable sets.

Assume that $\mathbb{R}$ is representable as the union of an increasing chain of countable sets. That is, $\mathbb{R} = \{A_i : i \in I\}$, where I is some ordered index set such that $A_i \subset A_j$

if $i < j$ and each A_i is a countable set. Consider now a subset X of $\mathbb{R}$, of cardinality $\aleph_1$. By our assumption, for every $x \in X$, $x \in A_{i(x)}$ for some $i(x) \in I$. Now if there existed some $j \in I$ such that for every $x \in X$, $i(x) < j$, then $X \subset A_j$, by the definition of an increasing chain. This is a contradiction as X is uncountable and A_j is countable. Thus, for every $j \in I$, there exists an $i(x) > j$, for some $x \in X$. So, $\mathbb{R} = \{A_i : i \in I\} \subseteq \{A_{i(x)} : i(x) \in X\}$. Thus, by cardinal multiplication, the cardinality of $\mathbb{R}$ is at most $\aleph_1\aleph_0 = \aleph_1$. $\mathbb{R}$ is an uncountable set, so $|\mathbb{R}| = \aleph_1$ and the Continuum Hypothesis holds.

□

11. The Converse: Proof of CH from the existence of a Lusin set and a property of $\mathbb{R}$

Theorem 11.1. *If there exists a Lusin set and every subset of $\mathbb{R}$ of cardinality $<$ c is of first category, then the Continuum Hypothesis holds.*

Proof. Assume that there exists a Lusin set, A, and every subset of $\mathbb{R}$ of cardinality $< c$ is of first category. Let us now enumerate the reals as $\{r_\alpha : \alpha < c\}$. Consider now A_α as defined as $A \cap \{r_\beta : \beta \leq \alpha\}$. By the definition of a Lusin set and our assumption, each A_α is countable. Thus, we can now observe that $A = \bigcup\{A_\alpha : \alpha < c\}$. To prove this, consider an arbitrary element $a \in A$. Then, $a \in \mathbb{R}$, so $a = r_\alpha$ for some α. Thus, $a \in A_\alpha$ and $a \in \bigcup\{A_\alpha : \alpha < c\}$. Consider now an arbitrary element, $x \in \bigcup\{A_\alpha : \alpha < c\}$. By the definition of union, $x \in A_\alpha$ for some non-empty A_α, as A is non-empty. Thus, by the definition of A_α, $x \in A \cap \{r_\beta : \beta \leq \alpha\}$, and $x \in A$.

Applying the proof of Lemma 10.2, we see that A has cardinality at most $\aleph_1$. By our assumption, A has cardinality continuum. The continuum is uncountable, so $|A| = \aleph_1 = c$. Thus, the Continuum Hypothesis holds.

□

12. Closing Comments

Acknowledgments. It is a pleasure to thank my mentor, Carsen Berger, for his tireless help and digilent time, discussing with and aiding me in this project. Also, I would like to thank my graduate student mentor, Asilata Bapat, for her very helpful edits and timely guidance.

References

[1] Peter Komjath and Vilmos Totik. Problems and Theorems in Classical Set Theory. Springer Science+Business Media, LLC. 2006.