POINCARÉ'S THEOREM FOR FUCHSIAN GROUPS

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Abstract. We present a proof of Poincaré's Theorem on the existence of a Fuchsian group for any signature (g; m_1, ..., m_r), where g, m_1, ..., m_r provide an admissable solution to the formula describing the hyperbolic area of the group's quotient space. Along the way we elucidate relevant concepts in hyperbolic geometry and the theory of Fuchsian groups.

Contents

1. Introduction

Our goal is to prove, as well as provide appropriate context for, the following theorem.

Theorem 1.1 (Poincaré's Theorem). *If $g \geq 0$, $r \geq 0$, $m_i \geq 2$ $(1 \leq i \leq r)$ are integers and if*

$$(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \geq 0$$

then there exists a Fuchsian group with signature (g; m_1, ..., m_r).

We first examine the relevant hyperbolic geometry, followed by an introduction to Fuchsian groups. Our chief reference is Svetlana Katok's *Fuchsian Groups* [1], in which the main definitions and results of this paper may be found; I have strived to maintain Katok's notation throughout, to facilitate cross-referencing with her work. Basic knowledge of algebra and topology will be assumed (e.g. terms such as "genus" and "quotient group").

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2. Hyperbolic Geometry and the Group $PSL(2,\mathbb{R})$

The set $\mathcal{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ is called the *hyperbolic (half) plane*. Notice that $\mathcal{H}$ does not contain the real line. If we wish to consider these points as well, we will find it fruitful to append a point labeled ∞ to the "end" of the real line, such that the real line together with ∞ is homeomorphic to the circle – the ends of this line are "joined" at infinity. The set $\tilde{\mathcal{H}} = \mathcal{H} \cup \{\infty\}$ is the *Euclidean closure* of $\mathcal{H}$, since it is the closure in $\mathbb{C} \cup \{\infty\}$ of $\mathcal{H}$ with the standard Euclidean metric.

The set of fractional linear transformations from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$ specified by $\{z \mapsto (az+b)/(cz+d) | a,b,c,d \in \mathbb{R},\ ad-bc = 1\}$, together with the operation of function composition, form a group referred to as $PSL(2,\mathbb{R})$. Each of these transformations is associated with two distinct 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

both of whose determinants are 1. This gives us a ready definition for the *trace* of an element of $PSL(2,\mathbb{R})$, namely the absolute value of the trace of either of its corresponding matrices. The group of all 2×2 real matrices is called $SL(2,\mathbb{R})$, and we immediately see that $PSL(2,\mathbb{R})$ is isomorphic to $SL(2,\mathbb{R})/\{\pm 1_2\}$, where 1_2 is the 2×2 identity matrix.

We can discriminate the elements of $PSL(2,\mathbb{R})$ into three different types based on the values of their traces. Those with trace < 2 are the *elliptic* elements, those with trace $= 2$ are *parabolic*, and those with trace > 2 are *hyperbolic*. This geometric terminology stems from the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$: A matrix in $SL(2,\mathbb{R})$ is hyperbolic (resp. elliptic) if and only if its invariant curve in $\mathbb{R}^2$ is a hyperbola (resp. ellipse), and the parabolic transformations are thought of as "intermediate" between the hyperbolic and elliptic transformations.

Finally, we expand our consideration to the group $PS^*L(2,\mathbb{R})$ comprised of the fractional linear transformations from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto (az+b)/(cz+d)$, $ad - bc = \pm 1$; $PSL(2,\mathbb{R})$ is a subgroup of $PS^*L(2,\mathbb{R})$ of index 2.

We now consider two representations of plane hyperbolic geometry, both embedded in the complex plane. The set $\mathcal{H}$ together with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y} \tag{2.1}$$

form our first representation, the half-plane model.

The function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{zi+1}{z+i} \tag{2.2}$$

is a bijective map that sends $\mathcal{H}$ to $\mathcal{U} = \{z \in \mathbb{C} \big| |z| < 1\}$, a (Euclidean) circle of radius 1 centered at $z = 0$. In a sense, f shrinks the "infinitely large circle" $\mathcal{H}$ down to a circle of radius 1 whose boundary Σ is the set $\{z \in \mathbb{C} \big| |z| = 1\}$, referred to as the *principal circle*. $\mathcal{U}$ together with its principal circle Σ is $\tilde{\mathcal{U}}$, the Euclidean closure of $\mathcal{U}$.

To obtain the appropriate metric ds to use in this model, we first examine the metric on $\mathcal{H}$ and how hyperbolic distances are transformed by the function f. For a path γ through $\mathcal{H}$ parametrized by $t \in [0,1]$, its *hyperbolic length* $h(\gamma)$ is given by

$$h(\gamma) = \int_0^1 \frac{ds}{dt}dt = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} \tag{2.3}$$

and the *hyperbolic distance* $\rho(z,w)$ between points $z, w \in \mathcal{H}$ is taken to be the infimum of $h(\gamma)$ over all γ joining z and w. For $z, w \in \mathcal{U}$, the hyperbolic distance $\rho^*(z,w)$ between them is thus equal to $\rho(f^{-1}z, f^{-1}w)$.

A brief calculation confirms that for $z = x + iy \in \mathcal{H}$,

$$\frac{2|df(z)/dz|}{1 - |f(z)|^2} = \frac{1}{\operatorname{Im}(z)},$$

which in turn equals

$$\frac{1}{y} = \frac{\sqrt{dx^2 + dy^2}}{y\sqrt{dx^2 + dy^2}} = \frac{ds}{dz}$$

so

$$ds = \frac{2|df(z)|}{1 - |f(z)|^2} = \frac{2|dz^*|}{1 - |z^*|^2} \tag{2.4}$$

for $z^* \in \mathcal{U}$, which is the metric we were looking for. This model of hyperbolic geometry will be referred to as the *unit disk* model.

Returning to our consideration of $PS^*L(2,\mathbb{R})$ and its subgroup $PSL(2,\mathbb{R})$, we now look at its action on $\mathcal{H}$. As it turns out, $PS^*L(2,\mathbb{R})$ is isomorphic to the group $\operatorname{Isom}(\mathcal{H})$ of hyperbolic isometries of $\mathcal{H}$ [Katok, Thm. 1.3.1]. For the remainder of this paper, we will convern ourselves only with the subgroup $PSL(2,\mathbb{R})$.

Theorem 2.5. *$PSL(2,\mathbb{R})$ is the group of orientation-preserving isometries of $\mathcal{H}$.*

Proof. An isometry is orientation-preserving if it is composed only of rotations and translations, and not reflections, in the hyperbolic plane. A reflection would be some rotations and translations combined with the operation $z \mapsto -\bar{z}$, but the latter operation does not belong to $PSL(2,\mathbb{R})$ - therefore, $PSL(2,\mathbb{R})$ contains only orientation-preserving isometries. Combining the above operation with any elements from $PSL(2,\mathbb{R})$ results in a transformation whose determinant is -1, and recalling that $PSL(2,\mathbb{R})$ is of index 2 in $PS^*L(2,\mathbb{R})$, we see that the demarcation between elements whose determinant is plus or minus one is precisely the same as that between transformations which are orientation-preserving or -reversing, and that the orientation-reversing isometries are those with negative determinant. $\square$

One can similarly show that the orientation-preserving isometries of $\mathcal{U}$ are the fractional linear transformations from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto (az + \bar{c})/(cz + \bar{a})$ where $a, c \in \mathbb{C}$, $a\bar{a} - c\bar{c} = 1$, which together form their own group isomorphic to $PSL(2,\mathbb{R})$.

Remark 2.6. Parabolic and hyperbolic transformations have one and two fixed points, respectively, which are located in $\mathbb{R} \cup \{\infty\}$. Elliptic transformations have one fixed point, in $\mathcal{H}$.

We wrap up our introduction to hyperbolic geometry with a discussion of the hyperbolic version of the polygon. First we investigate the hyperbolic equivalent of a straight line: A *geodesic* in hyperbolic space is the shortest curve (with respect to the appropriate metric) joining two points.

Theorem 2.7. *The geodesics in $\mathcal{H}$ are segments of either straight lines perpendicular to the real line, or of semicircles with their ends touching the real line whose tangents at the points of intersection are perpendicular to it.*

Proof. First consider two points $z_1 = ia, z_2 = ib \in \mathcal{H}$, with $b > a$. For any path $\gamma(t) = (x(t), y(t))$ joining z_1 and z_2,

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)} dt \geq \int_0^1 \frac{|dy/dt|}{y(t)} dt \\ &\geq \int_0^1 \frac{dy/dt}{y(t)} dt = \int_a^b \frac{dy}{y} = \ln \frac{b}{a}, \end{aligned}$$

meaning $\ln(b/a) \leq \rho(z_1, z_2)$. Meanwhile, the particular path $\zeta(u(t), v(t))$ where $u(t) = 0$ and $v(t) = a + t/(b - a)$, i.e. the segment of the imaginary axis joining z_1 and z_2, has hyperbolic length

$$\begin{aligned} h(\zeta) &= \int_0^1 \frac{\sqrt{(du/dt)^2 + (dv/dt)^2}}{v} dt = \int_0^1 \frac{dv/dt}{v(t)} dt \\ &= \int_a^b \frac{dv}{v} = \ln \frac{b}{a} \end{aligned}$$

so $\ln(b/a) \geq \rho(z_1, z_2)$, and hence $\ln(b/a)$, the hyperbolic length of the straight vertical line joining z_1 and z_2, is also the hyperbolic distance between them.

Any straight line can be mapped to the imaginary axis by the transformation $z \to z + b$ for some $b \in \mathbb{R}$, and all such transformations are elements of $PSL(2, \mathbb{R})$ and hence isometries. One can also show that, for a semicircle as described above which meets the real line at α and β, the transformation $z \to (\gamma z - (\alpha\gamma + 1))/(z - \alpha)$ for $\gamma = (\beta - \alpha)^{-1}$ has determinant 1 and maps the semicircle to the imaginary axis. Thus the hyperbolic length of the line or semicircle segment joining two points in $\mathcal{H}$ is the hyperbolic distance between their images under the isometry that sends them to the imaginary axis, and so is the hyperbolic distance between the points themselves. □

In this paper we will refer to both the above line segments (resp. arcs) as well as the full lines (resp. semicircles) of which they are a part as "geodesics," distinguishing the former with the term "geodesic segments." If two geodesics intersect in $\mathcal{H}$, their point of intersection is a *vertex*. If when extended to $\mathbb{R} \cup \{\infty\}$ they intersect at a point in $\mathbb{R} \cup \{\infty\}$, the point of intersection is a *vertex at infinity*. An *(hyperbolic) angle* between two geodesics meeting at a given vertex, in the half-plane model, is simply the corresponding Euclidean angle between the tangent lines to the two geodesics at the vertex; we see that the angle between two geodesics at a vertex at infinity is zero.

A *hyperbolic polygon* is a region of $\mathcal{H}$ bounded by a union of geodesics, and a polygon is said to be *convex* if the geodesic segment joining any two points in the polygon is itself in the polygon. A hyperbolic n-gon, specifically, is a hyperbolic polygon bounded by n geodesics. The *hyperbolic area* $\mu(A)$ of any $A \subseteq \mathcal{H}$ is the two-dimensional Riemannian volume form associated with the hyperbolic metric on $\mathcal{H}$,

$$\mu(A) = \int_A \frac{dxdy}{y^2} \tag{2.8}$$

Hence, $\mu^*(A^*)$ of $A^* \subseteq \mathcal{U}$ is $\mu(f^{-1}(A^*))$. Clearly, hyperbolic area is invariant under hyperbolic isometries.

A rather surprising result of hyperbolic geometry is that we can find the area of a hyperbolic triangle simply by knowing its angles. For a hyperbolic triangle A with angles α, β, γ, the Gauss-Bonnet formula tells us that the area $\mu(A)$ is given by

$$\mu(A) = \pi - \alpha - \beta - \gamma \tag{2.9}$$

Notice then that the sum of the angles of any hyperbolic triangle must be less than π – we can in fact construct a hyperbolic triangle with any nonnegative angles α, β, γ such that $\alpha+\beta+\gamma < \pi$ [Katok, Exercise 4.7]. Meanwhile, dividing any hyperbolic polygon into triangles lets us use the Gauss-Bonnet formula to derive the total area of the polygon: We find that for an n-gon Π with angles α_1, ..., α_n,

$$\mu(\Pi) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i \tag{2.10}$$

Remark 2.11. Hyperbolic circles in $\mathcal{H}$ are exactly the same as Euclidean circles, though with different centers. Thus, the Euclidean and hyperbolic metrics induce the same topology on $\mathcal{H}$.

The above definitions regarding hyperbolic polygons were developed for the half-plane model specifically, but they transfer to the unit circle model in the obvious way. Hyperbolic angles between geodesics intersecting at a given vertex are once again the angles between the tangent lines to the geodesics at the vertex. Representations of hyperbolic geometry such as the half-plane and unit circle models, in which the hyperbolic angles between geodesics are the same as the Euclidean angles between their drawings in the model, are known as *conformal* representations. Even though angles are thus always "preserved," other geometric properties are not; for instance, hyperbolic geodesics are not always Euclidean geodesics, and hyperbolic lengths are not always Euclidean lengths. Other models exist which preserve various combinations of these other kinds of properties, but we will remain focused on the two conformal models developed above.

3. Fuchsian Groups

Since every element of $PSL(2,\mathbb{R})$ is characterized by four real numbers, we can topologize $PSL(2,\mathbb{R})$ based on the topology of the corresponding subset of $\mathbb{R}^4$. More specifically, since *two* points in $\mathbb{R}^4$ are associated with each $T \in PSL(2,\mathbb{R})$ – (a,b,c,d) and $(-a,-b,-c,-d)$ for some $a,b,c,d \in \mathbb{R}$ – let us consider the set $X = \{(a,b,c,d) \in \mathbb{R}^4 | ad - bc = 1\}$ and the function $\delta(a,b,c,d) = (-a,-b,-c,-d)$.

The function δ together with the identity form Δ, a cyclic group of order 2 acting on X, and $PSL(2,\mathbb{R})$ corresponds with X/Δ, the *quotient space* of X with respect to Δ (the space formed by identifying points in the same Δ-cycle). We induce a topology on $PSL(2,\mathbb{R})$ based on the topology of X/Δ, and we induce topologies on subgroups of $PSL(2,\mathbb{R})$ based on the topologies of the subgroups considered as subspaces of $PSL(2,\mathbb{R})$. A *discrete* subgroup of $PSL(2,\mathbb{R})$, known as a *Fuchsian group*, is one whose induced topology is a discrete topology. In other words, a Fuchsian group is a discrete group of orientation-preserving isometries of the hyperbolic plane.

We can equivalently define a Fuchsian group in terms of its action on $\mathcal{H}$, and to do that we first define the notion of a properly discontinuous group action. There are several equivalent definitions of this type of action in the literature, many of which are listed in Katok, but the two that will serve our ends most directly are the following: A group G of homeomorphisms of a metric space X acts *properly discontinuously* if and only if for any compact set $K \subset X$, $T(K) \cap K \neq \emptyset$ for only finitely many $T \in G$, if and only if for any compact set $K \subset X$ and point $x \in X$, $\cup_{T \in G}[T(x) \cap K]$ is a finite set.

Lemma 3.1. *Let Γ be a subgroup of $PSL(2, \mathbb{R})$ acting properly discontinuously on $\mathcal{H}$, and $p \in \mathcal{H}$ be fixed by some element of Γ. Then there is a neighborhood W of p such that no other point of W is fixed by an element of Γ other than the identity.*

Proof. If $T(p) = p$ for some $T \in \Gamma$ other than the identity, suppose that in any neighborhood of p there are fixed points of transformations of T. Hence there exists a sequence of points $\{p_n\} \subset \mathcal{H}$ such that $p_n \to p$ and $T_n(p_n) = p_n$ for some $T_n \in \Gamma$, meaning that for any $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that for any $n > N_1$, $\rho(p_n, p) < \epsilon$.

The closed hyperbolic disc $\overline{B_{2\epsilon}(p)}$ of radius $2\epsilon > 0$ centered at p is compact, recalling the Remark at the end of section 3. Since Γ acts properly discontinuously, $\{T \in \Gamma | T(p) \in B_{2\epsilon}(p)\}$ is a finite set, so there exists some $N_2 \in \mathbb{N}$ such that for any $n > N_2$, $\rho(T_n(p), p) > 2\epsilon$. Thus for $N = \max(N_1, N_2)$, for any $n > N$ it is true that both $\rho(p_n, p) < \epsilon$ and $\rho(T_n(p), p) > 2\epsilon$.

By the triangle inequality,

$$\begin{aligned} \rho(T_n(p), p) &\leq \rho(T_n(p), T_n(p_n)) + \rho(T_n(p_n), p) \\ &= \rho(T_n(p), T_n(p_n)) + \rho(p_n, p) \end{aligned}$$

and since elements of $PSL(2, \mathbb{R})$ are isometries,

$$\rho(T_n(p), T_n(p_n)) = \rho(T^{-1}(T_n(p)), T^{-1}(T_n(p_n))) = \rho(p, p_n).$$

Hence

$$\begin{aligned} 2\epsilon < \rho(T_n(p), p) &\leq \rho(T_n(p), T_n(p_n)) + \rho(T_n(p_n), p) \\ &= \rho(p, p_n) + \rho(p_n, p) < 2\epsilon, \end{aligned}$$

a contradiction. □

Theorem 3.2. *A subgroup Γ of $PSL(2, \mathbb{R})$ is a Fuchsian group if and only if it acts properly discontinuously on $\mathcal{H}$.*

Proof. For Γ a Fuchsian group, let $z \in \mathcal{H}$ and let K be a compact subset of $\mathcal{H}$. It is a fact that for any such point and set, the set $E = \{T \in PSL(2, \mathbb{R}) | T(z_0) \in K\}$ is compact [Katok, Lemma 2.2.4]. A space is discrete iff every point has a neighborhood containing no other points, i.e. there are no accumulation points. A subset of a compact set can only have infinitely elements if one of its points is an accumulation point with respect to that subset, and so the intersection of a discrete set with a compact set contains only finitely many points. Hence $E \cap \Gamma$, the intersection of a compact set with a discrete set, is a finite set, meaning $\{T \in \Gamma | T(z) \in K\} = E \cap K$ is finite, meaning in turn that Γ acts properly discontinuously on $\mathcal{H}$.

A group Γ is discrete if and only if for any sequence $\{T_k\} \subset \Gamma$ such that $T_k \to T \in \Gamma$, there exists some $N \in \mathbb{N}$ such that $T_i = T_j$ for any $i, j > N$, if and only if, fixing any $T \in \Gamma$, for any sequence $\{S_k\} \subset \Gamma$ such that $S_k \to Id$ (noting that each S_k equals $T_k T^{-1}$ for some $T_k \in \Gamma$) there exists some $N \in \mathbb{N}$ such that $S_i = S_j$ for any $i, j > N$.

Now let Γ be a subset of $PSL(2, \mathbb{R})$ that acts properly discontinuously on $\mathcal{H}$. We know from the preceding lemma any subgroup of $PSL(2, \mathbb{R})$ acting properly discontinuously on $\mathcal{H}$ has at least one point $s \in \mathcal{H}$ not fixed by any non-identity element in the subgroup; assign such an s to the present group Γ. Since Γ is not discrete, there must be some sequence of distinct transformations T_k such that $T_k \to Id$, meaning $\{T_k(s)\}$ is a sequence of points distinct from s that approaches s. Hence for any closed hyperbolic disc $B_\epsilon(s)$ centered at s, $T(s) \cap B_\epsilon(s) \neq \emptyset$ for infinitely many $T \in \Gamma$, meaning there exists a compact set $B_\epsilon(s)$ such that $[T(s) \cap B_\epsilon(s)] \subseteq [T(B_\epsilon(s)) \cap B_\epsilon(s)] \neq \emptyset$ for infinitely many $T \in \Gamma$, and so Γ does not act properly discontinuously, a contradiction. □

4. Fundamental and Dirichlet Regions

One of the key features of any Fuchsian group is its capacity to tile the hyperbolic plane when given various tile shapes, where each tile is "set down" once and only once in a given location by an element of the group. To make this notion more precise, let G be a group of homeomorphisms acting properly discontinuously on a metric space X, and let $F \subset X$ be the closure of a nonempty open set F^o. We say F is a *fundamental region* for G if $\cup_{T \in G} T(F) = X$ and $F^o \cap T(F^o) = \emptyset$ for all $T \in G - \{\text{Id}\}$. The set $\partial F = F - F^o$ is the *boundary* of F, and the family $\{T(F) | T \in G\}$ is the *tessellation* of X.

By Lemma 3.1, for any Fuchsian group we can find a point $p \in \mathcal{H}$ that is not fixed by any element of the group. Thus let Γ be a Fuchsian group and $p \in \mathcal{H}$ be one such point. The *Dirichlet region* for Γ centered at p is the set

$$D_p = \{z \in \mathcal{H} | \rho(z, p) \leq \rho(T(z), p) \text{ for all } T \in \Gamma\} \tag{4.1}$$

For a given $T_1 \in \Gamma$, the sets $H_p(T_1) = \{z \in \mathcal{H} | \rho(z, p) \leq \rho(z, T_1(p))\}$ are hyperbolic half-planes, split off from the rest of $\mathcal{H}$ by the perpendicular bisector of the geodesic segment joining p and $T_1(p)$, and $\cap_{T \in \Gamma} H_p(T) = D_p(\Gamma)$. Since each set $H_p(T_1)$ is closed, $D_p(\Gamma)$ is closed, and therefore compact iff it has finite hyperbolic area and no vertices at infinity. Since the shortest path connecting two points which are closer to p than $T(p)$ for all $T \in \Gamma$ must clearly consist of points which are themselves closer to p than $T(p)$ for all $T \in \Gamma$, $D_p(\Gamma)$ is convex, and hence connected.

Theorem 4.2. *Let Γ be a Fuchsian group and $p \in \mathcal{H}$. If p is not fixed by any element of Γ other than the identity, then $D_p(\Gamma)$ is a fundamental region for Γ.*

Proof. Every point in $\mathcal{H}$ is acted on by Γ, so every point in $\mathcal{H}$ is a member of a Γ-orbit. Hence, the union of all Γ-orbits is $\mathcal{H}$, and if $D_p(\Gamma)$ contains at least one point from every Γ-orbit, then $\cup_{T \in \Gamma} T(D_p(\Gamma)) = \mathcal{H}$. Furthermore, if the interior of $D_p(\Gamma)$ contains no more than one point from each Γ-orbit, then $D_p(\Gamma)^o \cap T(D_p(\Gamma)^o) = \emptyset$ for all $T \in \Gamma - \{\text{Id}\}$.

Let $z \in \mathcal{H}$ and let Γz be its Γ-orbit. Since Γz is discrete, there exists a point $z_0 = T_0(z) \in \Gamma z$ such that $\rho(T_0(z), p) \leq \rho(T(z), p)$ for all $T \in \Gamma$, and so

$$\rho(z_0, p) \leq \rho(ST_0(z), p) = \rho(S(z_0), p)$$

for all $S \in \Gamma$, meaning $z_0 \in D_p(\Gamma)$. Thus, $D_p(\Gamma)$ contains at least one point from every Γ-orbit.

Let $L_p(T)$ denote the perpendicular bisector of the geodesic segment joining p and $T(p)$ for some $T \in \Gamma - \{\text{Id}\}$, so

$$L_p(T) = \{z \in \mathcal{H} | \rho(p, z) = \rho(z, T(p))\}.$$

Any such $L_p(T)$ must therefore lie either outside of or on the boundary of $D_p(\Gamma)$. If $\rho(z, p) = \rho(T(z), p)$ for some $T \in \Gamma - \{\text{Id}\}$, then

$$\rho(z, p) = \rho(T^{-1}(T(z)), T^{-1}(p)) = \rho(z, T^{-1}(p))$$

and so $z \in L_p(T^{-1})$, meaning z is not in the interior of $D_p(T)$. Hence the interior of $D_p(\Gamma)$ cannot contain any z such that $T(z)$ is also in the interior, i.e. the interior of $D_p(\Gamma)$ contains at most one point from every Γ-orbit. □

Let Γ be a Fuchsian group. Two points $u, v \in \mathcal{H}$ are called *congruent* if they belong to the same Γ-orbit. Congruency is an equivalence relation, and the equivalence classes are called *congruence classes*. Since the interior of any Dirichlet region contains at most one point from every Γ-orbit, two points in the same Dirichlet region for Γ can only be congruent if they lie on its boundary. In particular, a set of congruent vertices is called a *cycle*. Recall now the division of elements of $PSL(2, \mathbb{R})$ into three *types* – elliptic, parabolic, and hyperbolic.

Remark 4.3. For $S, T \in PSL(2, \mathbb{R})$, S and TST^{-1} both have the same type.

If u is fixed by $S \in \Gamma$, then $T(u)$ for any $T \in \Gamma$ is fixed by TST^{-1}, so all the vertices in a given cycle are fixed by isometries of the same type. When the type is elliptic, we say the cycle is an *elliptic cycle* and the vertices are *elliptic vertices*.

The locally finite action of Fuchsian groups gives rise to several "finite" properties, and some which will shortly be employed are collected in the following Remark.

Remark 4.4. Any elliptic transformation has a finite order. For any Dirichlet region, there are finitely many vertices in a congruent cycle. The stabilizer in Γ of a point in $\mathcal{H}$ is a maximal finite cyclic subgroup of Γ, and every maximal finite cyclic subgroup of Γ is a stabilizer of one and only one point in $\mathcal{H}$.

Choosing some fundamental region F for Γ, notice that every fixed point of an elliptic transformation lies on the boundary of $T(F)$ for some $T \in \Gamma$. For an elliptic transformation of order $k \geq 3$ fixing u on the boundary of F, u must be a vertex whose angle is at most $2\pi/k$. If the order is 2, the fixed point can lie in the middle of a side of F, and the transformation then switches the two segments of the side; we will find it useful to extend our definition of a vertex to include such fixed points, and the angle at such a vertex is π.

Theorem 4.5. *Let F be a Dirichlet region for Γ, and let θ_1, θ_2, ..., θ_t be the internal angles of all the vertices in some congruence class. Let m be the order of the stabilizer in Γ of one of these vertices. Then $\theta_1 + \theta_2 + ... + \theta_t = 2\pi/m$.*

Proof. Let v_1, v_2, ..., v_t be the vertices corresponding to the angles θ_1, θ_2, ..., θ_t, respectively. Since vertices in a given cycle are each fixed by conjugate elements of Γ, the stabilizers of two points in a congruent set are conjugate subgroups of Γ, so the stabilizers of each of these vertices have the same order. Let

$$H = \{\text{Id}, S, S^2, ..., S^{m-1}\}$$

be the stabilizer of v_1 in Γ. For every $0 \leq r \leq m-1$, $S^r(F)$ has a vertex at v_1 whose angle is θ_1 – each rotates F by some amount about v_1. If $T_k(v_k) = v_1$ for some $T_k \in \Gamma$, $1 \leq k \leq t$, then the set of elements which map v_k to v_1 is given by

$$HT_k = \{S^j T_k | 0 \leq j \leq m-1\},$$

a set which has m elements, and each $S^j T_k(F)$ has v_1 as a vertex with an angle of θ_k.

Thus for a given S^r we have determined t regions of the form $S^r T_k(F)$ with v_1 as a vertex, and so we have distinguished a total of mt regions with that vertex. Each of these regions is in fact distinct: If $S^r T_k(F) = S^q T_l(F)$, then $S^r T_k = S^q T_l$, and hence $r = q$ and $k = l$. If $S^r T_k(F) \neq S^q T_l(F)$, then the two regions in question cannot overlap except at their boundaries, since F is a fundamental region for Γ. The sum of the interior angles at v_1 of all these regions is $m(\theta_1 + ... + \theta_t)$.

Meanwhile, for any region $A(F)$ for $A \in \Gamma$ with v_1 as a vertex, $A^{-1}(v_1) \in F$, so v_1 is congruent with $A^{-1}(v_1)$. Two points in a fundamental region can only be congruent if they are on the boundary, and vertices specifically must be congruent with vertices, so $A^{-1}(v_1) = v_i$ for $1 \leq i \leq t$. Rewriting this as $A(v_i) = v_1$, we see that $A \in HT_i$, meaning $A = S^j T_i$ for some $0 \leq m-1$. Every region with v_1 as a vertex is therefore one of the mt distinct regions we have already considered, so the sum of the angles at v_1 of all these regions must be 2π. Thus,

$$m(\theta_1 + ... + \theta_t) = 2\pi$$

□

Remark 4.6. This theorem is also true for the case where the vertices are not fixed points, i.e. $m = 1$.

Theorem 4.7. *Every vertex of a Dirichlet region has a neighborhood containing no other vertices. Hence, a compact Dirichlet region has a finite number of vertices.*

Proof. Let $F = D_p(\Gamma)$ where p is not fixed by any $T \in \Gamma - \{\text{Id}\}$, and let v be some vertex of F not at infinity.

Suppose that there exists an infinite sequence of distinct vertices $v_i \in F$ such that $v_i \to v$. For any point x on a side of a Dirichlet region, including a vertex not at infinity, there exists a transformation $T_x \in \Gamma$ such that $\rho(p, x) = \rho(T_x(p), x)$, and hence $\rho(p, x) = \rho(p, T_x^{-1}(x))$. Thus, for any v_i in the sequence there exists $T_i \in \Gamma$ such that $\rho(p, v_i) = \rho(p, T_i(v_i))$.

$$\begin{aligned}
\rho(v, T_i(v_i)) &\leq \rho(v, v_i) + \rho(v_i, p) + \rho(p, T_i(v_i)) \text{ by successive applications of the triangle inequality} \\
&= \rho(v, v_i) + 2\rho(v_i, p) \\
&\leq \rho(v, v_i) + 2\rho(v_i, v) + 2\rho(v, p) \\
&= 3\rho(v, v_i) + 2\rho(v, p)
\end{aligned}$$

and thus for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i > N$,

$$\rho(v, T_i(v_i)) < 2\rho(v, p) + \epsilon$$

Therefore, setting $\epsilon > 0$, $T_i(v_i) \in \overline{B_\epsilon(v)}$ for all $i > N$, so there exists a compact set $\overline{B_\epsilon(v)} \subset \mathcal{H}$ such that $T(F) \cap \overline{B_\epsilon(v)} \neq \emptyset$ for infinitely many $T \in \Gamma$, contradicting the fact that Γ acts properly discontinuously on $\mathcal{H}$. □

5. Quotient Spaces of Fuchsian Groups

Let Γ be a Fuchsian group. We induce an area function on the quotient space $\mathcal{H}/\Gamma$ based on the hyperbolic area function on $\mathcal{H}$.

Remark 5.1. If F_1 and F_2 are fundamental regions for a Fuchsian group Γ, $\mu(F_1) < \infty$, and $\mu(\partial F_1) = \mu(\partial F_2) = 0$, then $\mu(F_1) = \mu(F_2)$. Thus, the hyperbolic area of a fundamental region (with a nice boundary) for a given Fuchsian group, as long as one such region has finite hyperbolic area, is a well-defined function of the group.

The points of $\mathcal{H}/\Gamma$ are Γ-orbits, and two points in a given fundamental region are only in the same Γ-orbit if they lie on the boundary, so it is clear that, if $\mu(\mathcal{H}/\Gamma) < \infty$, $\mu(\mathcal{H}/\Gamma) = \mu(F)$ for any fundamental region F for Γ. In fact, if F is a Dirichlet region, $\mathcal{H}/\Gamma$ is homeomorphic to F/Γ [Beardon, Thm 9.2.4, as stated in Katok, pg. 75], and $\mathcal{H}/\Gamma$ is compact if and only if all of its Dirichlet regions are [Katok, Corollary 4.2.3].

In the parlance of manifolds, Γ is the *fundamental group* of the quotient space $\mathcal{H}/\Gamma$. Elliptic cycles of vertices in a given fundamental region give rise to *marked points* in $\mathcal{H}/\Gamma$, and vertices at infinity produce *cusps*. The quotient space assumes the structure of an *orbifold*, which for our purposes means an oriented surface of genus g with some number of marked points.

Let us now compare Fuchsian groups to discrete groups of orientation-preserving Euclidean isometries. The quotient spaces of the latter are always homeomorphic to the torus [Katok, pg. 76], meaning that their fundamental group is always isomorphic to $\mathbb{Z} \times \mathbb{Z}$. However, no Fuchsian group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (see below), indicating that there are nontrivial differences between the actions of these two types of groups. Searching for the topologies that quotient spaces of Fuchsian groups may have leads us to our next set of theorems.

Now assume that Γ has a compact Dirichlet region F. We saw in the proof of Theorem 4.5 that the stabilizers of any vertex in an elliptic cycle all have the same order, so we define the *order of the stabilizer of an elliptic cycle* to be the order of the stabilizer of one of its vertices. By Theorem 4.7, F has finitely many vertices and hence finitely many elliptic cycles, with stabilizers of orders m_1, ..., m_r. The quotient space $\mathcal{H}/\Gamma$ has genus g, and we say that Γ has *signature* $(g; m_1, ..., m_r)$.

Theorem 5.2. *Let Γ have signature $(g; m_1, ..., m_r)$. Then*

$$\mu(\mathcal{H}/\Gamma) = 2\pi \left[(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \right] \tag{5.3}$$

Proof. A Dirichlet region F of Γ has r elliptic cycles of vertices, and by Theorem 4.5 the sum of the angles at all the elliptic vertices is $\sum_{i=1}^{r} 2\pi/m_i$. If there are s other cycles of vertices, then they are not elliptic cycles and hence not stabilized by elliptic transformations, and hence their stabilizers would have to be either the identity or parabolic and/or hyperbolic transformations. The latter types of transformations only stabilize points in $\mathbb{R} \cup \{\infty\}$, so the order of the stabilizer of each of these vertices is 1, and hence (in light of the Remark at the end of Theorem 4.5) the sum of the angles at all these vertices is $2\pi s$. Thus, the sum of all the interior angles of F is

$$2\pi \left[\Big(\sum_{i=1}^{r} \frac{1}{m_i} \Big) + s \right]$$

In addition to identifying vertices, Γ also identifies various sides of F with each other, and so if there are n such sets of identified sides, $\mathcal{H}/\Gamma$ has $(r+s)$ vertices, n edges, and 1 face. By the Euler formula,

$$2-2g=(r+s)-n+1$$

Meanwhile, by Equation (2.10),

$$\mu(F)=(n-2)\pi-2\pi\left[\Big(\sum_{i=1}^{r}\frac{1}{m_i}\Big)+s\right]$$

Incorporating the Euler formula and rearranging terms, we get

$$\mu(F)=2\pi\left[(2g-2)+\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)\right]$$

and finally we recall that $\mu(F)=\mu(\mathcal{H}/\Gamma)$. □

5.1. **Proof of Poincaré's Theorem.** Equation (5.3) provides a restriction on the possible signatures of Fuchsian groups, since the numbers $g, m_1, ..., m_r$ need to be such that $\mu(\mathcal{H}/\Gamma)>0$. Surprisingly, this is the only restriction on which Fuchsian groups are allowed to exist, which is the result stated in Poincaré's Theorem. The following proof of the theorem constructs a fundamental polygon for a Fuchsian group of a given signature, in the manner of Jones and Singerman [3] (as employed by Katok).

Proof of Theorem 1.1. We will be constructing many geodesics and labeling them in various ways, so it may help to grab a pencil and paper and actually draw a specific example of what we are constructing as we go along. Choose specific numbers for g, r, $m_1, ..., m_r$, checking that they satisfy the equation in the theorem statement.

Working in the unit disc model, draw $(4g+r)$ radii from the center of $\mathcal{U}$ spaced at equal angles (so the angle between any two adjacent radii is $2\pi/(4g+r)$), and label the points a Euclidean distance t from the center on each radius, for some constant $0<t<1$. Start at one of these points and draw the geodesic connecting it to one of the two nearest labeled points, then continue in the same direction drawing these geodesics all the way around until we get a regular hyperbolic $(4g+r)$-gon, $M(t)$. Assign to the first r sides of $M(t)$ the numbers 1 to r sequentially. Construct isosceles hyperbolic triangles on each of them, with the $M(t)$ side as the base and the vertex opposite this side on the outside of $M(t)$, and so that the angle between the equal sides is $2\pi/m_j$, where j is the $M(t)$ side's number. If $m_i=2$, the vertex "opposite the base" will actually be in the middle of the base, and the triangle will be degenerate.

The union of these triangles with $M(t)$ forms a new polygon $N(t)$ with $4g+2r$ sides. Label the sides that aren't part of the above isosceles triangles $\alpha_1, \beta_1^{'}, \alpha_1^{'}, \beta_1$, ..., $\alpha_g, \beta_g^{'}, \alpha_g^{'}, \beta_g$, and label the sides that are in the above triangles $\xi_1, \xi_1^{'}, ..., \xi_r, \xi_r^{'}$. For each quartet $\alpha_k, \beta_k^{'}, \alpha_k^{'}, \beta_k$, assign an orientation to each of their corresponding sides – draw an arrow – which points towards the vertex between $\beta_k^{'}$ and $\alpha_k^{'}$. For each pair $\xi_k, \xi_k^{'}$, orient the corresponding sides so that they point toward the vertex between them.

As $t \to 0$, so does $\mu(N(t))$. Meanwhile, by the Gauss-Bonnet formula for an n-sided polygon, and remembering that the interior angle of any vertex at infinity is zero, we see that as $t \to 1$,

$$\mu(N(t)) \to (4g + 2r - 2)\pi - \sum_{i=1}^{r} 2\pi/m_i$$
$$= 2\pi[(2g - 1) + \sum_{i=1}^{r} 1 - \frac{1}{m_i}]$$

The hyperbolic area $\mu(N(t))$ is a continuous function of t, so there must accordingly be some $0 < t_0 < 1$ such that

$$\mu(N(t_0)) = 2\pi[(2g - 2) + \sum_{i=1}^{r} 1 - \frac{1}{m_i}],$$

which is 2π below the upper limit but still above zero by virtue of the restrictions on g, r, and the m_i's.

By construction, α_i has the same hyperbolic length as $\alpha_i^{'}$, β_j has the same length as $\beta_j^{'}$, and ξ_k has the same length as $\xi_k^{'}$ for all $1 \leq i, j \leq g$, $1 \leq k \leq r$. At the end of the proof of Thm. 2.7 we saw that for any geodesic in $\mathcal{H}$ there exists an orientation-preserving isometry of $\mathcal{H}$ that maps it to the imaginary axis, which means that for each pair of geodesics in $\mathcal{U}$ there exists an orientation-preserving isometry of $\mathcal{U}$ that maps one to the other. Thus there exist orientation-preserving hyperbolic isometries A_i, B_j, X_k on $\mathcal{U}$ for which

$$A_i(\alpha_i^{'}) = \alpha_i, B_j(\beta_j^{'}) = \beta_j, X_k(\xi_k^{'}) = \xi_k$$

where i, j, k are as above. These transformations divide the boundary of $N(t_0)$ into pairs of congruent sides. Let Γ be the group generated by

$$\{A_i, B_j, X_k | 1 \leq i, j \leq g, 1 \leq k \leq r\}.$$

It can be shown that $N(t_0)$ tessellates $\mathcal{U}$ via the transformations in Γ, and hence $N(t_0)$ is a fundamental region for Γ. A compact, hence closed and bounded, region in $\mathcal{U}$ intersects with only finitely many Γ-images of $N(t_0)$, and so we see that Γ acts properly discontinuously on $\mathcal{U}$. The corresponding group of orientation-preserving isometries of $\mathcal{H}$ is thus a Fuchsian group.

Since the side β_g has an orientation, it has a beginning and an end marked by the vertices that bound it. Label its beginning vertex v_1. It is congruent via the transformation B_g^{-1} to the beginning vertex of $\beta_g^{'}$, which we label v_2. This vertex is also the end vertex of α_g, and is congruent via the transformation A_g^{-1} to the end vertex of $\alpha_g^{'}$, which we label v_3. This vertex is also the end vertex of $\beta_g^{'}$, and is congruent via the transformation B_g to the end vertex of β_g, which we label v_4. Finally, this vertex is also the beginning vertex of $\alpha_g^{'}$ and is congruent via the transformation A_g to the beginning vertex of α_g, which is also the beginning vertex of β_{g-1}.

Following this pattern all the way around the polygon, each α-β quartet has all of its members labeled, and so we see that all the vertices of the original polygon $M(t_0)$ are in the same congruence class under combinations of the various A and B transformations. Meanwhile, the other r vertices (the tips of the isosceles triangles), which we label w_1, ..., w_r, form r separate congruence classes (under the X

transformations) composed of one vertex each. Thus there are $r+1$ distinct congruence classes of vertices of $N(t_0)$, meaning $\mathcal{U}/\Gamma$ has $r+1$ vertices. Since $N(t_0)$ has $4g+2r$ sides and these are paired off under Γ-congruency, the quotient space $\mathcal{U}/\Gamma$ has $2g+r$ sides. It also has 1 simply-connected face, so by the Euler formula the genus h of $\mathcal{U}/\Gamma$ satisfies

$$2-2h=(r+1)-(2g+r)+1=2-2g$$

and so its genus does in fact equal g.

The only vertices of $N(t_0)$ which are stabilized by elements of Γ are w_1, ..., w_r, and each of these belongs to a congruence class containing only itself, so there are r elliptic cycles, namely the sets containing one of these vertices. The interior angle at the vertex w_i is $2\pi/m_i$, so the stabilizer of w_i and hence of the elliptic cycle containing w_i is of order m_i. Thus, we have constructed a Fuchsian group with signature $(g; m_1, ..., m_r)$. □

In particular, we see that for any integer $g>1$ there exists a Fuchsian group for which $\mathcal{H}/\Gamma$ has genus g. Meanwhile, there cannot exist a Fuchsian group of genus 1 with no marked points, proving that no quotient space of a Fuchsian group is homeomorphic to the torus. The same can be said with regards to the sphere, plane, and punctured plane, but all other orientable surfaces are quotient spaces of Fuchsian groups [Katok, pg. 76].

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