THE CLASSICAL RISK MODEL WITH EXPONENTIALLY DISTRIBUTED CLAIMS

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Abstract. This paper will analyze an insurer's susceptibility to losing all of its capital given an incoming cashflow gained from premiums and an outgoing cashflow caused by claims. First, we discuss the model and give a detailed justification for the distributions of outgoing cashflows. Then, we solve the the model to obtain a closed form solution for the probability of insolvency.

Contents

1. Introduction

Ruin theory, a subset of actuarial science, is the study of an insurance company's vulnerability to losing all of its capital due to a large number of claims in a short period of time, a particularly large set of claims, or, more likely, a combination of the two. This topic is interesting not only in theory, but also in practice as it gives insurers an idea of how to price their premiums in order to secure the company's survival. Ruin theory arose around 1900 through the Swedish actuary Filip Lundberg whose ideas were developed and spread by Harald Cramér. He introduced the topic by assuming claims arrived according to a Poisson process and based his model from there, which became known as the "classical risk model." Since his introduction, much work has been done to better model risk through complications of the model such as including interest, operating costs, and generalizing Lundberg's assumptions. The biggest advancement to ruin theory after Lundberg and Cramér was achieved by Powers in 1995 and Gerber and Shiu in 1998 who created the expected penalty discount function, which will not be discussed in this paper. Since this advancement, numerous papers have been written generalizing and tweaking the model with different assumptions.

This paper will use a very intuitive, basic model to explain an insurer's total capital, denoted $M(t)$ at a given time t. We let the insurer's initial capital, i.e. at

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$t = 0$, be denoted by $u \geq 0$, we assume that premiums are recovered at a continuous, constant rate $c > 0$, and we also assume that the insurer has lost money at time t equal to the total amount claimed up until that time, denoted $S(t)$. We can express the insurer's available capital at time t as his starting capital plus the money the company has accumulated from premiums minus the total amount claimed:

$$M(t) = u + ct - S(t) \tag{1.1}$$

Furthermore, it is clear that $S(t)$, the total amount claimed at time t, can be written as the sum of X_i, $i = 1, \ldots, n$, where each X_i is the size a particular customer claims up until time t and $N(t)$ is the number of customers making claims by time t. Therefore

$$M(t) = u + ct - \sum_{i=1}^{N(t)} X_i \tag{1.2}$$

In this paper, we are interested in the chance $M(t) < 0$ for any $t > 0$, the event in which the insurer has negative capital. In this case we say the insurer is "ruined" even though he is not ruined in a practical sense of bankruptcy, but rather that the insurer needs to have more money if he wishes to pay off the claims.

In Section 2, the details of the model are discussed. In particular, the assumptions are stated and then it is shown how the model follows from these assumptions. In Section 3, it is shown why these assumptions are logical and their potential weaknesses. In Section 4, a general solution for the probability of ruin is derived. In Section 5, a particular claim size distribution is assumed, and we use the derivation of Section 4 to find a closed form expression of an insurance company's probability of ruin.

2. The Model

Let $S(t)$ be the total amount claimed by time t, so that $S(t)$ is dependent on two random variables, the number of claims by time t, denoted $N(t)$, and the claim sizes, denoted $X_1, X_2, \ldots$. As stated before,

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

In this section we derive an appropriate distribution for $N(t)$ and hypothesize an approximate distribution for the X_is. We first make the following justifiable assumptions about the claims.

(1) The X_is are independent.
(2) Each X_i occurs with some probability p with $0 < p < 1$.

Condition (1) simply states that no client has an influence on a different client. Condition (2) states that whether or not there is some claim amount by a given client occurs with the same probability p for each client. These X_i will all take different values, but they are each independent and have the same chance of occurring. The effect of changing assumption (2) will be considered later in the paper.

Given these assumptions, we can derive a distribution for $N(t)$. Recall that given n independently distributed events $X_1, X_2, ..., X_n$, each with probability p of happening, the number $N < n$ of total occurrences has a binomial distribution. This means that for all k such that $0 < k < n$, $\mathbb{P}\{N = k\} = \binom{n}{k}(p)^k(1-p)^{n-k}$. Insurance companies tend to have a large number of clients n, each with a small probability p of making a claim. Unfortunately, the binomial theorem is difficult to work with for large n and small p as one ends up with large factorials and very small exponents. Therefore, we use the following proposition, frequently known as the law of rare events.

Proposition 2.1. *If the random variable X follows a binomial distribution with parameter $p = \frac{\lambda}{n}$, where n is the number of trials of X, then as n tends to infinity, X follows a Poisson distribution with mean λ.*

Proof. We compute the following limit:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\{X_n = k\} &= \lim_{n\to\infty} \frac{n!}{(n-k)!k!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k} \\
&= (1)\frac{\lambda^k}{k!}e^{-\lambda} \\
&= \frac{\lambda^k e^{-\lambda}}{k!},
\end{aligned}$$

which is exactly the Poisson distribution. □

This proposition allows for the conclusion that the number of claims to occur in each unit of time follows a Poisson process with parameter $\lambda = np$. By scaling, the number of occurrences in any time interval of length t follows a Poisson distribution with parameter λt. Another random variable of interest is the time between claims, which leads to the following proposition.

Proposition 2.2. *Suppose the number of events X to occur in any time interval of length t follows a Poisson distribution with mean λt, and the number of events to have occurred at time 0 is 0. Suppose further that each event occurs independently of every other event. Then the times between claims are independent identically distributed exponential random variables with parameter $\frac{1}{\lambda}$.*

Proof. First let's show the distribution for the waiting time of the first arrival is exponential and then conclude by showing the waiting time for each arrival follows the same distribution. Let T_k denote the time until the kth arrival. For $k = 1$,

$$\mathbb{P}\{T_1 \le t\} = 1 - \mathbb{P}\{T_1 > t\} = 1 - \mathbb{P}\{N(t) = 0\} = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t},$$

which is exactly the cumulative distribution function for an exponential random variable. In general,

$$\begin{aligned}\mathbb{P}\{T_k - T_{k-1} \le t | T_{k-1} = s\} &= 1 - \mathbb{P}\{T_k - T_{k-1} > t | T_{k-1} = s\} \\ &= 1 - \mathbb{P}\{T_k > t + T_{k-1} | T_{k-1} = s\} \\ &= 1 - \mathbb{P}\{T_k > t + s | T_{k-1} = s\} \\ &= 1 - \mathbb{P}\{\text{no events in } (s, s+t) | T_k = s\} \\ &= 1 - \mathbb{P}\{\text{no events in } (s, s+t)\} \\ &= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t},\end{aligned}$$

where the last line holds by the assumption that number of events X to occur in any time interval of length t follows a Poisson distribution with mean λt. □

Definition 2.3. If the number of occurrences of a random variable X by time t satisfy the assumptions of Proposition 2.2 then we say X follows a Poisson process.

Consider our assumptions, and write p for the probability of a claim happening and n for the number of clients. Then the number of claims by time t follows a Poisson distribution with parameter λt, where λ is approximated by np, and the time between claims follows an exponential distribution with parameter $\frac{1}{\lambda}$. The distribution of claim sizes X_i comes next in the analysis. In order to assign a logical distribution to the claim sizes, a few assumptions about the claim sizes are necessary:

(1) Insuring a client is profitable for the insurer.
(2) The X_is are identically distributed.
(3) There exist constants C and α such that $\mathbb{P}\{X_i > x\} \le Ce^{-\alpha x}$, $i = 1, 2, \ldots$ This condition is called the "small claims condition."

Assumption (1) is only used in deriving the density of the probability of ruin given in Section 5 of the paper. Assumption (2) is analyzed in Section 3, and it is used in both Section 4 and 5 to derive the probability of ruin. Assumption (3), the "small claims condition," simply says that claim sizes decrease to zero at least exponentially fast. This implies that most claims are relatively small, and then larger sized claims are exponentially unlikely. This condition is used in Section 4 to derive a general formula for the probability or ruin. Furthermore, this condition motivates using $\mathbb{P}\{X_i > x\} = Ce^{-kx}$ for the distribution of the claim sizes. In fact, in Section 5 we take $C = 1$, implying each X_i is explained by the exponential distribution, and then we solve for a closed form expression of $\psi(u)$, the probability of ruin starting with initial capital u.

3. Justifying the Assumptions of the Model

In order to conclude that the claim arrival followed a Poisson process, it was necessary to assume the following

(1) Claim arrivals are independent.
(2) Each X_i occurs with some probability p with $0 < p < 1$.

Furthermore, to proceed into sections (4) and (5) it is necessary to assume the following about the claim sizes;

(1) Insuring a client is profitable for the insurer.

(2) The X_is are identically distributed.
(3) There exist constants C and α such that $\mathbb{P}\{X_i > x\} \leq Ce^{-\alpha x}$, $i = 1, 2, \ldots$

In this section we will analyze the effects of eliminating the assumption that each X_i occurs with some probability p with $0 < p < 1$, and the assumption that the X_is are identically distributed. First, we proceed with the following lemma.

Lemma 3.1. *Suppose $S = X_1 + X_2 + \ldots + X_N$, where $X_1, X_2, \ldots$ are independent identically distributed random variables, N is a random variable taking positive integer values, and N is independent of X_i for all i, then*

(1) $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X_1]$
(2) $\text{var}[S] = \mathbb{E}[N]\text{var}[X_1]+\text{var}[N]\mathbb{E}^2[X_1]$

Proof. To prove (1), we condition on the value of N

$$\begin{aligned}
\mathbb{E}[S] &= \mathbb{E}[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{\infty} \mathbb{E}[\sum_{j=1}^{i} X_j]\mathbb{P}[N = i] \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i} \mathbb{E}[X_j]\mathbb{P}[N = i] = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \mathbb{E}[X_1]\mathbb{P}[N = i] \\
&= \sum_{i=1}^{\infty} i\mathbb{E}[X_1]\mathbb{P}[N = i] = \mathbb{E}[X_1]\sum_{i=1}^{\infty} i\mathbb{P}[N = i] \\
&= \mathbb{E}[X_1]\mathbb{E}[N]
\end{aligned}$$

To prove (2), we again condition on the value of N

$$\begin{aligned}
\text{var}[S] &= \text{var}[\sum_{i=1}^{N} X_i] \\
&= \mathbb{E}[(\sum_{i=1}^{N} X_i)^2] - \mathbb{E}^2[\sum_{i=1}^{N} X_i] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[(\sum_{j=1}^{i} X_j)^2]\mathbb{P}[N = i] - \mathbb{E}^2[X_1]\mathbb{E}^2[N] \\
&= \sum_{i=1}^{\infty} (\sum_{j=1}^{i} \mathbb{E}[X_j^2]\mathbb{P}[N = i] + \sum_{k \neq j}^{i} \mathbb{E}[X_j X_k]\mathbb{P}[N = i]) - \mathbb{E}^2[X_1]\mathbb{E}^2[N] \\
&= \sum_{i=1}^{\infty} (i\mathbb{E}[X_1^2]\mathbb{P}[N = i] + (i)(i-1)\mathbb{E}^2(X_1)\mathbb{P}[N = i] - \mathbb{E}^2[X_1]\mathbb{E}^2[N] \\
&= \mathbb{E}[N]\mathbb{E}[X_1^2] + \mathbb{E}[N^2]\mathbb{E}^2[X_1] - \mathbb{E}[N]\mathbb{E}^2[X_1] - \mathbb{E}^2[X_1]\mathbb{E}^2[N] \\
&= \mathbb{E}[N](\mathbb{E}[X_1^2] - \mathbb{E}^2[X_1]) + (\mathbb{E}[N^2] - \mathbb{E}^2[N])\mathbb{E}^2[X] \\
&= \mathbb{E}[N]\text{var}[X_1] + \text{var}[N]\mathbb{E}^2[X_1]
\end{aligned}$$ □

With this lemma in hand it is now possible to analyze the justifiability of assuming all claims have the same probability of arriving in a given time interval and that each claim has the same size distribution. Consider the following generalization of the model, where each claim has a different chance of occurring or not occurring, and each claim has a different distribution for its claim size.

Proposition 3.2. *Let n be the number of clients, X_i be independent random variables denoting the total amount claimed by the i^{th} client, and S be the total amount claimed in a year so that $S = \sum_{i=1}^{n} X_i$. Suppose that*

$$X_i = \begin{cases} 0 & : \text{with probability } 1 - p_i \\ Y_i & : \text{with probability } p_i \end{cases}$$

where $0 < p_i < 1$ and Y_i has distribution function f_i, mean μ_i, and variance σ_i^2. Then the expected value and variance of S, is as follows

$$\mathbb{E}[S] = \sum_{i=1}^{n} p_i \mu_i$$
$$\text{var}[S] = \sum_{i=1}^{n} \left((\sigma_i^2 + \mu_i^2) p_i - p_i^2 \mu_i^2 \right)$$

Proof. To show that $\mathbb{E}[S] = \sum_{i=1}^{n} p_i \mu_i$ simply note that

$$\mathbb{E}[S] = \mathbb{E}[\sum_{i=1}^{n} S_i] = \sum_{i=1}^{n} \mathbb{E}[S_i] = \sum_{i=1}^{n} (p_i \mu_i + (1 - p_i)0) = \sum_{i=1}^{n} p_i \mu_i$$

To show that $\text{var}(S) = \sum_{i=1}^{n} \left((\sigma_i^2 + \mu_i^2) p_i - p_i^2 \mu_i^2 \right)$ write,

$$\begin{aligned} \text{var}[S] &= \text{var}[X_1 + \ldots + X_n] \\ &= \sum_{i=1}^{n} \text{var}[X_i] + 2 \sum_{i<j}^{n} \text{cov}[X_i, X_j] \\ &= \sum_{i=1}^{n} \text{var}[X_i] + 0 \\ &= \sum_{i=1}^{n} (\mathbb{E}[X_i^2] - \mathbb{E}^2[X_i]) \end{aligned}$$

To find $\mathbb{E}[X_i^2]$ define a new indicator random variable Z_i as follows:

$$Z_i = \begin{cases} 0 : X_i \neq Y_i \\ 1 : X_i = Y_i \end{cases}$$

We now have

$$\mathbb{E}[X_i^2] = \mathbb{E}[Z_i^2 Y_i^2] = \mathbb{E}[Z_i^2]\mathbb{E}[Y_i^2] = p_i(\text{var}[Y_i] + \mathbb{E}^2[Y_i]) = (\sigma_i^2 + \mu_i^2) p_i$$

Thus we have the following relationship as desired;

$$\text{var}[S] = \sum_{i=1}^{n} (\mathbb{E}[X_i^2] - \mathbb{E}^2[X_i]) = \sum_{i=1}^{n} \left((\sigma_i^2 + \mu_i^2) p_i - p_i^2 \mu_i^2 \right)$$

$\square$

Above is the case where our assumptions are generally discarded, the claims have different size densities and different chances of occurring. Let's consider how this compares to the case where we derived a Poisson arrival time, based off the assumption that all claims have the same chance of occurring. Also, we will consider the effect of assuming all the claims have the same distribution.

Proposition 3.3. *Consider our model $S = \sum_{i=1}^{N(t)} X_i$, where $N(t)$ follows the Poisson distribution with mean $\lambda t = npt$. Let $f(x)$ be the density of each X_i, let μ be the mean of the X_is, and let σ^2 be the variance, then for a time span of one year, i.e. $t = 1$, the mean and variance of S is as follows;*

$$\mathbb{E}[S] = np\mu$$
$$\text{var}[S] = np(\sigma^2 + \mu^2)$$

Proof. Apply lemma 3.1 to obtain the following:

$$\begin{aligned}\mathbb{E}[S] &= \mathbb{E}[N]\mathbb{E}[X_1] = np\mu \\ \text{var}[S] &= \mathbb{E}[N]\text{var}[X_1] + \text{var}[N]\mathbb{E}^2[X_1] \\ &= np\sigma^2 + np\mu^2 = np(\sigma^2 + \mu^2)\end{aligned}$$ □

Comparing Proposition 3.2 with Proposition 3.3, it is clear that not much has changed. When finding the mean, the only difference is that in Proposition 3.2 we add up each individual claim instead of multiplying by the number of claims as we do in Proposition 3.3. When finding the variance, we have the same summation versus multiplication difference as well as an additional $p_i^2\mu_i^2$ term in Proposition 3.2. Fortunately, by the small claims condition, each p_i is very small for any μ_i big enough to make a significant contribution to the model, which makes this additional p_i^2 term so small it is almost irrelevant to the calculation. Thus, by taking our value of p and X in Proposition 3.3 to be the average of all the p_is and X_is respectively in Proposition 3.2, we would get a very similar result. This implies that it is possible to pick a value for p and a distribution for X which would make our mean and variances to Proposition 3.3 and Proposition 3.2 very similar, which demonstrates that making these assumptions is something we can do without drastically changing the result.

4. Probability of Ruin with Unknown Claim Distribution

Definition 4.1. The probability of ruin $\psi(u)$ is given by

$$\psi(u) = \mathbb{P}\{M(t) < 0 | M(0) = u\}.$$

The survival probability $\phi(u)$ is given by

$$\phi(u) = 1 - \psi(u).$$

Given our assumptions about $N(t)$ and the conclusion that $N(t)$ followed a Poisson distribution, it is possible to solve for $\phi(u)$. In order to find $\phi(u)$ one must know that $\phi(\infty) = 1$, and thus arises the following lemma.

Lemma 4.2. *We have*

$$\lim_{u\to\infty} \phi(u) = 1.$$

The proof of this lemma is outside the level of this paper; however, we give some intuition for the lemma. First, consider the model as a random walk with jump size equivilant to the claim size X and exponentially distributed waiting times. Furthermore, the random walk has a linear drift c. By the law of iterated logarithm, this random walk will only decrease in the limit at a rate of $\sqrt{t}$ and increase at a linear rate t. Therefore, as we take the random walk out to infinite time, we never get back to zero. Although this lemma may appear confusing, it simply states that as one's starting captial tends to infinity, the chance of survival goes to one.

Remark 4.3. Clearly if

$$\lim_{u\to\infty} \phi(u) = 1 \text{ then } \lim_{u\to\infty} \psi(u) = 0.$$

This simply states that as one's starting capital goes to infinity, the probability of being ruined tends to zero.

The following lemma is included to smooth out the computation involved in the proof of the theorem of this section.

Lemma 4.4. *Given $\frac{d}{dx}F(x) = f(x)$ then the following equality holds;*

$$\begin{aligned}\int_0^t \int_0^u \phi(u-x)f(x)\,dx\,du = &-\phi(0)\int_0^t (1-F(u))\,du + \int_0^t \phi(u)\,du \\ &- \int_0^t (1-F(x))(\phi(t-x)-\phi(0))\,dx\end{aligned}$$

Proof. Note that $-d(1-F(X)) = f(x)dx$ and integrate by parts to obtain the following;

$$\begin{aligned}\int_0^u \phi(u-x)f(x)\,dx &= -\int_0^u \phi(u-x)d(1-F(x)) \\ &= -\phi(u-x)(1-F(x))|_0^u - \int_0^u \phi'(u-x)(1-F(x))\,dx \\ &= -\phi(0)(1-F(u)) + \phi(u)(1-F(0)) \\ &- \int_0^u \phi'(u-x)(1-F(x))\,dx\end{aligned}$$

Now integrate with respect to u from 0 to t to obtain

$$\begin{aligned}\int_0^t \int_0^u \phi(u-x)f(x)\,dx\,du &= \int_0^t [-\phi(0)(1-F(u)) + \phi(u)(1-F(0)) \\ &- \int_0^u \phi'(u-x)(1-F(x))\,dx]\,du \\ &= -\phi(0)\int_0^t (1-F(u))\,du + \int_0^t \phi(u)(1-F(0))\,du \\ &- \int_0^t \int_0^u \phi'(u-x)(1-F(x))\,dx\,du \\ &= -\phi(0)\int_0^t (1-F(u))\,du \\ &- \int_0^t (1-F(x))(\phi(t-x)-\phi(0))\,dx\end{aligned}$$

□

Theorem 4.5. *Consider the model $M(t) = u + ct - \sum_{i=1}^{N(t)} X_i$, where $N(t)$ follows the Poisson distribution. Let $f(x)$ be the density of each X_i, let $F(x)$ be the cumulative distribution function, and let μ be the mean. Then the probability of survival $\phi(u)$ is given by the following equation;*

$$\phi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^t \phi(t-x)(1-F(x))\,dx \tag{4.6}$$

Proof. We use the independence of events X_i from each other and from $N(t)$ along with Proposition 2.2 to compute as follows:

$$\phi(u) = \int_{s=0}^{\infty}\int_{x=0}^{\infty} \mathbb{P}[M(t) \geq 0 | X_1 = x, T_1 = s]\mathbb{P}[X_1 = x]\mathbb{P}[T_1 = s]\, dx\, ds$$

$$= \int_{s=0}^{\infty}\int_{x=0}^{u+cs} \phi(u + cs - x) f(x) \lambda e^{-\lambda s}\, dx\, ds$$

let $z = u + cs$

$$= \int_{z=u}^{\infty}\int_{x=0}^{z} \frac{\lambda}{c} e^{\frac{z-u}{c}} \phi(z - x) f(x)\, dx\, dz.$$

Take the derivative with respect to u,

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u - x) f(x)\, dx. \tag{4.7}$$

Integrate (4.7) from 0 to t and apply Lemma 4.2 to obtain

$$\phi(t) = \phi(0) + \frac{\lambda}{c}\int_0^t \phi(u)\, du - \frac{\lambda}{c}\int_0^t\int_0^u \phi(u - x) f(x)\, dx\, du$$

$$= \phi(0) + \frac{\lambda}{c}\int_0^t \phi(u)\, du + \frac{\lambda}{c}\phi(0)\int_0^t (1 - F(u))\, du - \frac{\lambda}{c}\int_0^t \phi(u)\, du$$

$$+ \frac{\lambda}{c}\int_0^t \phi(t - x)(1 - F(x))\, dx - \frac{\lambda}{c}\phi(0)\int_0^t (1 - F(x))\, dx$$

$$= \phi(0) + \frac{\lambda}{c}\int_0^t \phi(t - x)(1 - F(x))\, dx.$$

To find $\phi(0)$, let t tend to infinity, giving

$$\phi(\infty) = \phi(0) + \frac{\lambda}{c}\phi(\infty)\int_0^{\infty} (1 - F(x))\, dx$$

$$= \phi(0) + \frac{\lambda\mu}{c}\phi(\infty).$$

We are able to take the limit as t goes to infinity in this manner because our function is bounded, demonstrated below:

$$\int_0^t \phi(t - x)(1 - F(x))\, dx = \int_0^t \phi(t - x)(1 - F(x)) I_{[0,1]}\, dx$$

$$\leq \int_0^t (1 - F(x))\, dx$$

$$\leq \int_0^t Ce^{-\alpha x}\, dx \leq \infty$$

where the last line holds by the small claims condition. So $1 = \phi(0) + \frac{\lambda\mu}{c}$ which implies $\phi(0) = 1 - \frac{\lambda\mu}{c}$. Thus, the desired conclusion follows

$$\phi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^t \phi(t - x)(1 - F(x))\, dx. \qquad \square$$

Corollary 4.8. *We have*

$$\frac{c}{\lambda}\psi'(u) = \psi(u) - \int_0^u f(u - x)\psi(x)\, dx - \int_u^{\infty} f(x)\, dx \tag{4.9}$$

Proof. We know $\psi(u) = 1 - \phi(u)$ which implies $\psi'(u) = -\phi'(u)$. Therefore, by plugging $-\psi'(u)$ into (4.7) and multiplying through by $-\frac{c}{\lambda}$:

$$\begin{aligned}\frac{c}{\lambda}\psi'(u) &= -\phi(u) + \int_0^u \phi(u-x)f(x)\,dx \\ &= \psi(u) - 1 + \int_0^u (1-\psi(u-x))f(x)\,dx \\ &= \psi(u) - \int_0^u \psi(u-x)f(x)\,dx + \int_0^u f(x)\,dx - 1 \\ &= \psi(u) - \int_0^u \psi(x)f(u-x)\,dx - \int_u^\infty f(x)\,dx.\end{aligned}$$

In the last step, we use the fact that $\int_0^u f(x)\,dx + \int_u^\infty f(x)\,dx = 1$ since $f(x)$ is a probability distribution. This implies that $\int_0^u f(x)\,dx - 1 = -\int_u^\infty f(x)\,dx$ as used in the proof. ☐

Definition 4.10. The conditions and assumptions in Theorem 4.5 form what is known as the classical risk model.

5. Probability of Ruin with Exponential Claims

From the assumption that insuring a client is profitable for the insurer, we get

$$0 < \mathbb{E}[u + ct - \sum_{i=1}^{N(t)} X_i] = \mathbb{E}[u] + \mathbb{E}[ct] - \mathbb{E}[\sum_{i=1}^{N(t)} X_i] = u + ct - \lambda t\mu.$$

This equation implies $c > \lambda\mu - \frac{u}{t}$, but since this holds for all t, we obtain $c > \lambda\mu$ so there exists a positive ρ such that $c(1+\rho) = \lambda\mu$. In this case, the value of ρ is known as the premium loading factor. Ideally the insurance company wishes to maximize ρ, but obviously competition gets in the way.

In the following theorem we use the exponential distribution to explain the claim sizes, motivated by the small claims condition.

Theorem 5.1. *Assuming the classical risk model with claim size density $f(x) = \alpha e^{-\alpha x}$ for $x > 0$, ρ the premium loading factor, and μ the mean of $f(x)$, then*

$$\psi(u) = \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u} \tag{5.2}$$

Proof. First, (4.9) is used to show that

$$\psi''(u) + \left(\alpha - \frac{\lambda}{c}\right)\psi'(u) = 0 \tag{5.3}$$

and then (5.3) will be solved to find $\psi(u)$. In order to show (5.3), note that by taking the derivative with respect to u of $\psi'(u)$

$$\psi''(u) = \frac{\lambda}{c}\psi'(u) - \left(\frac{\lambda}{c}\int_0^u \psi(x)f(u-x)\,dx - \int_u^\infty f(x)\,dx\right)'$$

which implies that (5.3) holds if and only if

$$
\begin{aligned}
\frac{c\alpha}{\lambda}\psi'(u) &= \left(\int_0^u \psi(x) f(u-x)\,dx - \int_u^\infty f(x)\,dx\right)' \\
&= \left(\int_0^u \alpha e^{-\alpha(u-x)}\psi(x)\,dx - \int_u^\infty \alpha e^{-\alpha x}\,dx\right)' \\
&= \left(\alpha e^{-\alpha u}\int_0^u e^{\alpha x}\psi(x)dx\right)' - \alpha e^{-\alpha u} \\
&= -\alpha^2 e^{-\alpha u}\int_0^u e^{\alpha x}\psi(x)dx + \alpha e^{-\alpha u}\left(\int_0^u e^{\alpha x}\psi(x)dx\right)' - \alpha e^{-\alpha u} \\
&= -\alpha^2 e^{-\alpha u}\int_0^u e^{\alpha x}\psi(x)dx + \alpha e^{-\alpha u}\left(\int_0^u 0\,dx + e^{\alpha u}\psi(u)\right) - \alpha e^{-\alpha u} \\
&= -\alpha^2 e^{-\alpha u}\int_0^u e^{\alpha x}\psi(x)dx + \alpha\psi(u) - \alpha e^{-\alpha u} \\
&= \alpha\left(\psi(u) - \int_0^u \alpha e^{-\alpha(u-x)}\psi(x)\,dx - \int_u^\infty \alpha e^{-\alpha u}\right) \\
&= \alpha\left(\psi(u) - \int_0^u f(u-x)\psi(x)\,dx - \int_u^\infty f(x)\,dx\right) \\
&= \frac{c\alpha}{\lambda}\psi'(u)
\end{aligned}
$$

where the last two steps come from (4.9). To find $\psi(u)$, we are only left to solve (5.3). The solution will be in the form

$$\psi(u) = k_1 e^{\gamma_1 u} + k_2 e^{\gamma_2 u}$$

where γ_1 and γ_2 solve for r

$$r^2 + \left(\alpha - \frac{\lambda}{c}\right) r = 0 \tag{5.4}$$

which implies $\gamma_1 = 0$ and $\gamma_2 = -\alpha + \frac{\lambda}{c}$. So

$$\psi(u) = k_1 + k_2 e^{(-\alpha+\frac{\lambda}{c})u} \tag{5.5}$$

Letting u tend to infinity we get $\psi(\infty) = k_1 + k_2 e^{(-\alpha+\frac{\lambda}{c})\infty}$. By the profitability assumption we know $c > \lambda\mu$ which is equivalent to saying $-\alpha + \frac{\lambda}{c} < 0$. This implies $0 = \psi(\infty) = k_1 + 0$ which implies $k_1 = 0$. To solve for k_2 use the fact that $\psi(0) = \frac{\lambda}{\alpha c}$ which implies that $k_2 = \frac{\lambda}{\alpha c}$. So our solution for $\psi(u)$ is as follows

$$\psi(u) = \frac{\lambda}{\alpha c} e^{(-\alpha+\frac{\lambda}{c})u} \tag{5.6}$$

Now use the fact that $\alpha = \frac{1}{\mu}$ and that $c = (1+\rho)\lambda\mu$ and substitute into (5.6) to achieve

$$\psi(u) = \frac{1}{1+\rho} e^{-\frac{\rho}{(1+\rho)\mu}u} \tag{5.7}$$

□

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