INTRODUCTION TO REPRESENTATION THEORY AND CHARACTERS

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Abstract. In this paper, we will first build up a background for representation theory. We will then discuss some interesting topics in representation theory, including the basic properties of a representation, G-invariant subspaces, characters, and regular representations.

Contents

1. Groups

We begin with a short discussion of groups, which will be important for the rest of the paper.

Definition 1.1. A group is a set G with an associative binary operation for which there exists an identity element with respect to which every element of G has its own inverse. The order of a group is the number of elements contained in the set G.

Example 1.2. An example of a group would be the collection of all invertible $n \times n$ matrices with matrix multiplication as the operation. Each matrix has an inverse, and the identity matrix is the identity of the group. This is the general linear group or $GL_n(F)$, where F is the field from which the entries of the matrices are taken.

Example 1.3. Another example would be the set $\{ x^m \mid m \in \mathbb{Z}\}$ under the relation $x^n = 1$ with multiplication as the operation and 1 as the identity element. This group is called the cyclic group of order n. The element x is known as the generator of this group.

Theorem 1.4. *Cancellation Law : Let f,g,h be elements of a group G. If $fg = fh$, then $g = h$ and if $gf = hf$, then $g = h$.*

Proof. Assume $fg = fh$. Let f^{-1} be the inverse of f. Multiply both sides of $fg = fh$ byf^{-1} on the left. This yields

$$g = f^{-1}fg = f^{-1}fh = h.$$

The other case is similar. □

Definition 1.5. A subset S of a group G is called a subgroup if it has the following properties:

(i) *Closure*: If $a \in S$ and $b \in S$, then $ab \in S$.
(ii) *Identity*: $e_G \in S$, where e_G is the identity of G.
(iii) *Inverses*: If $a \in S$, then $a^{-1} \in S$.

Note that since all elements of S are also elements of G, all elements of S satisfy the properties of G.

Definition 1.6. Two elements g and g' of a group G are called conjugate if $g' = aga^{-1}$ for some a in G. A conjugacy class is a subset of the group such that all elements of the conjugacy class are conjugates of each other.

Definition 1.7. Let G and H be groups. A group homomorphism is a mapping $\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Proposition 1.8. *Let e_G be the entity element of a group G. Then a group homomorphism$\phi : G \to H$ satisfies $\phi(e_G) = e_H$ and $\phi(a^{-1}) = \phi(a)^{-1}$.*

Proof. Since $e_G = e_G e_G$ and by the definition of a group homomorphism,

$$\phi(e_G)e_H = \phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G).$$

Then by the cancellation law, we have $e_H = \phi(e_G)$.

Next, for all a in G,

$$\phi(a)\phi(a^{-1}) = \phi(aa^{-1}) = \phi(e_G) = e_H.$$

Also, $\phi(a)\phi(a)^{-1} = e_H$. Therefore, $\phi(a)\phi(a^{-1}) = \phi(a)\phi(a)^{-1}$, which implies $\phi(a^{-1}) = \phi(a)^{-1}$. □

2. The Hermitian Product

Definition 2.1. For any two vectors $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ in the vector space $\mathbb{C}^n$, the standard hermitian product is a map $\langle , \rangle$: $V \times V \to \mathbb{C}$ defined by the formula

$$\langle X, Y\rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + ... + \bar{x}_n y_n,$$

where $\bar{a}$ is the complex conjugate of a. The general hermitian form is defined by the following properties:

(i) *Linearity in the second variable*:

$$\langle X, cY\rangle = c\langle X, Y\rangle \text{ and } \langle X, Y_1 + Y_2\rangle = \langle X, Y_1\rangle + \langle X, Y_2\rangle$$

(ii) *Conjugate symmetry*:

$$\langle X, Y\rangle = \bar{c}\langle Y, X\rangle e$$

The standard hermitian product is a hermitian form.

Definition 2.2. Two vectors v and w are orthogonal if $\langle v, w\rangle = 0$; alternatively, this relation can be written as $v \perp w$. If two subspaces W and U are such that

$$\forall\ w \in W \text{ and } u \in U,\ \langle w, u\rangle = 0,$$

then W and U are orthogonal; this relation can also be written as $W \perp U$. Also, the term $W^{\perp}$ refers to the set of all vectors orthogonal to W.
In addition, an orthonormal basis is a basis $B = (v_1, v_2, ..., V_n)$ such that

$$\langle v_i, v_i\rangle = 1 \text{ and } \langle v_i, v_j\rangle = 0 \text{ if } i \neq j$$

Theorem 2.3. *If W is a subspace of a vector space V and $U = W^{\perp}$, then the vector space V is a direct sum of W and U or $V = W \oplus U$.*

We will only prove this theorem for the case when $\dim W = 1$, and omit the rest of the proof.

Proof. We first need to prove that W and U only share the zero vector. This is already clear, since they can only share the zero vector because $U = W^{\perp}$.
Next we need to prove that for any x in V, x can be written as a sum of elements of W and U. In other words, we want $x = aw + u$ for some scalar a, w in W and u in U. It is enough to prove $x - aw$ is in U for some scalar a. This is true if and only if $\langle x - aw, w\rangle = 0$. Solving for a using the properties of the hermitian product, we have $a = \frac{\langle x,w\rangle}{\langle w,w\rangle}$. This solution will ensure that $x - aw$ is in U. □

Definition 2.4. A linear operator T is said to be unitary if

$$\forall\ v, w,\ \langle v, w\rangle = \langle T(v), T(w)\rangle$$

Definition 2.5. A complex matrix P is unitary if $\bar{P}^t P = I$, where $\bar{P}$ is the matrix with entries that are the complex conjugates of the entires of P and $\bar{P}^t$ is the transpose of that matrix. The set of all such matrices together with the operation matrix multiplication forms the unitary group U_n.

3. Representations

Definition 3.1. A matrix representation R of a group G is a group homomorphism $R : G \to GL_n(F)$, where F is a field. For any g in G, R_g will denote the image of g.

If we do not consider a specific basis, then we can talk of the concept of a representation of a group on a finite-dimensional vector space V. The group of invertible linear operators on V is denoted by $GL(V)$, with composition of functions as the binary opeartion.

In addition, if we do choose a specific basis B, then there will be an isomorphism between this group and the group of invertible matrices $GL(V) \to GL_n(F)$, which maps the linear operator to the matrix of the linear operator with respect to the basis B.

We will define representation of a group G on V as a group homomorphism $\rho : G \to GL(V)$. The dimension of the representation ρ is defined to be the dimension of V. Similarly to matrix representations, we will use ρ_g to denote the image of the representation.

Consider a representation ρ. If we also choose a basis B, then for any g in the group, the matrix R_g of ρ_g gives a matrix representation of the group. In addition, for each vector v_i in the basis B, $\rho_g(v_i)$ is equal to the linear combination of the vectors of B where the coefficients are the i^{th} column of R_g, which is a $n \times n$ matrix. This is true for every vector in B, so we can write this relationship as $\rho_g(B) = BR_g$.

Example 3.2. Let G_8 be the cyclic group of order 8 with x as its generator. Let $Rot(\theta)$ be rotation by θ. Then we can define a representation $\rho : G_8 \to GL(\mathbb{R}^2)$

$$x^n \mapsto Rot(\frac{2n\pi}{8}).$$

Note that if $n = k8$, where k is an integer, then the image will be the identity; this means ρ is a well-defined function.

Definition 3.3. Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ of a group G are called isomorphic if there exists an isomorphism $T : V \to V'$ between the two vector spaces V and V' such that

$$\rho'_g(T(v)) = T(\rho_g(v)),\ \forall\ v \in V \text{and } g \in G.$$

We can also talk of isomorphism classes, which are just collections of representations that are isomorphic to each other.

Corollary 3.4. *If two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are isomorphic such that $T : V \to V'$ is an isomorphism, and if $B = \{v_1, v_2, ..., v_n\}$ and $B' = \{T(v_1), T(v_2), T(v_3), ...T(v_n)\}$ are corresponding bases for V and V', then the corresponding matrix representations R with respect to B and R' with respect to B' will have the same image for each member of the group.*

Proof. We want to show that for all g in G, $R_g = R'_g$. Let $B = \{v_1, v_2, ..., v_n\}$ be a basis for V and let B' be the basis for V' such that $B' = \{T(v_1), T(v_2), T(v_3), ...T(v_n)\}$. We know by the definition of isomorphic representations that there exists an isomorphism $T : V \to V'$ such that

$$T(\rho_g(v_i)) = \rho'_g(T(v_i)). \tag{3.5}$$

Let R_g and R'_g be matrix representations of ρ and ρ' with respect to bases B and B'. Then we know that $\rho_g(v_i) = BR_g$ and $\rho'_g(T(v_i)) = B'R'_g$, as above. Plugging this into Equation 3.5, we have $T(BR_g) = B'R'_g$. Note that the two ordered list of vectors $T(B)$ and B' are the same, so $T(B)R_g = T(B)R'_g$. If we let $R_g = (a_i^j)$ and $R'_g = (b_i^j)$ so the i^{th} columns R_g and R'_g are $(a_i^j)_{1/lej/len}$ and $(b_i^j)_{1/lej/len}$ respectively, then we have for each i, $\sum_j T(v_j)(a_i^j) = \sum_j T(v_i)(b_i^j)$. Since $T(v_i)$ is a basis,$a_i^j = b_i^j$, which implies that $R_g = R'_g$. □

4. Irreducible Representations

Definition 4.1. Let ρ be a representation of a group G on a vector space V, and let W be a subspace of V. Then W is G-invariant if

$$\rho_g(w) \in W, \forall\ w \in W \text{and } g \in G.$$

Example 4.2. If we let ρ be the trivial representation which maps every element of a group G to the identity linear operator, then any subspace W of the vector space V will be G-invariant:

$$\forall\ w \in W \text{and } \forall\ g \in G,\ \rho_g(w) = w, \text{therefore } \rho_g(w) \in W.$$

Definition 4.3. If a reprepresentation ρ of a group G on a vector space V has no proper G-invariant subspaces, then it is an irreducible representation. In addition, if $V = W_1 \oplus W_2$, where W_1 and W_2 are G-invariant subspaces, then ρ is the direct sum of its restrictions to the subspaces. If we let ρ_i be the representation ρ restricted to W_i, then we can write $\rho = \rho_1 \oplus \rho_2$.

Example 4.4. Let the representation from Example 3.2 be ρ_1 and let the representation from Example 4.2 be ρ_2. Let $Rot(\theta)$ be rotation by θ. The representation $\rho = \rho_1 \oplus \rho_2$ is a representation of G_8 over the vector space $\mathbb{R}^3$:

$$\rho : G_8 \to GL(\mathbb{R}^3) \text{ and } \rho : x^n \mapsto Rot(\frac{2n\pi}{8}) \text{ around } z - axis.$$

. In this case, the G-invariant subspaces are

$$W_1 = \{w = (a, b, 0) \mid w \in \mathbb{R}^3 \ and\ a, b, \in \mathbb{R}\}$$

$$W_2 = \{v = (0, 0, c) \mid v \in \mathbb{R}^3 \ and\ c \in \mathbb{R}\}$$

Now take B to be the standard basis of $\mathbb{R}^3$ and let $B_1 = \{(1, 0, 0), (0, 1, 0)\}$ and $B_2 = \{(0, 0, 1)\}$. If we let $R_{1,g}$, $R_{2,g}$, and $R_{3,g}$ be the matrix representation of an element g in G with respect to B, B_1, B_2, respectively, then the matrix representation of the generator x of G will be

$$R_{1,x} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that this matrix is made up of blocks of the form

$$\begin{array}{c|c} R_{2,x} & 0 \\ \hline 0 & R_{3,x} \end{array}$$

In fact, all matrices of representations that are direct sums of other representations will have this block form. This is because the direct sum is made up of representations that are restricted to the subspaces.

Definition 4.5. A representation of a group is called unitary if all its images are unitary linear operators on a vector space that satisfies the conditions of the hermitian product.

Theorem 4.6. *Let ρ be a unitary representation on a vector space V. Let W be a G-invariant subspace. Then $W^\perp$ is also G-invariant, and ρ is a direct sum of its restrictions to W and $W^\perp$.*

Proof. Let v be an element of $W^\perp$ then for all w in W, $v \perp w$. Since all ρ_g are unitary, they preserve orthogonality. This means for all g in G, $\rho_g(v) \perp \rho_g(w)$. Let w be an element of W. $\rho_g(w)$ is an invertible linear operator so there exists w' in W such that $\rho(w') = w$. Also, $v \perp w'$, which implies that $\rho_g(v) \perp w$. Therefore all such $\rho_g(v)$ are in $W^\perp$. This means $W^\perp$ is G-invariant. Thus, ρ is a direct sum of its restrictions to W and $W^\perp$ because $V = W \oplus W^\perp$, which was prove in Theorem 2.3. □

Corollary 4.7. *Every unitary representation on a vector space V is a direct sum of irreducible representations.*

Proof. We proceed by induction on $\dim(V)$. If V does not contain any G-invariant subspaces, then it is already irreducible and we are done. This is the case if $\dim(V) = 1$. If V contains a non-trivial G-invariant subspace W, then the statement is true by the last theorem and induction on W and $W^{\perp}$. □

We will need the following lemma for the final result of this section. However, we will not prove this lemma in the paper.

Lemma 4.8. *Any finite subgroup of $GL(V)$ is conjugate to a subgroup of the group of all unitary linear operators.*

Corollary 4.9 (**Maschke's Theorem**). *Every representation ρ of a finite group G is a direct sum of irreducible representations.*

Proof. Since the image of ρ is a subgroup of $GL(V)$, ρ is conjugate to a unitary representation by the above lemma. Therefore, by Corollary 4.7, this statement is true. □

5. Characters

Definition 5.1. The character χ of a representation ρ is a mapping $\chi : G \to \mathbb{C}$. It is defined by the formula $\chi(g) = Tr(\rho_g)$, where Tr is the trace function. This is well defined since the trace of a linear operator is well defined. The dimension of a character is equal to the dimension of the vector space that the representation is defined on.

The character can be calculated through matrix representations of a group. For an element g in a group, the trace of its matrix representation will be the character of g. Note that the trace of any matrix is the sum of its eigenvalues. For any diagonalized matrix, it is the sum of the entries on the diagonal.

Theorem 5.2. *Let χ be the character of a representation ρ of a finite group G on a vector space V. The following are true:*

(i) *The character of the identity element of the group is the dimension of the character.*
(ii) *For $a, g \in G$, $\chi(g) = \chi(aga^{-1})$. This means that the character is constant on each conjugacy class.*
(iii) *If χ' is the character of another representation ρ', then the character of the direct sum of ρ and ρ' is $\chi + \chi'$.*

Proof. (i) Let e_G be the identity element of G and let I be the identity matrix. Since the identity element of G will be mapped to the identity matrix through the matrix representation, we have that $\chi(e_G) = tr\ I$. The trace of the identity matrix is the dimension of the vector space, or $tr\ I = \dim V$.
(ii) Since matrix representation is a group homomorphism, we have that for g, a in G, $R_{aga^{-1}} = R_a R_g {R_a}^{-1}$. Then, by the properties of the trace function, we can conclude

$$tr\ (R_a R_g {R_a}^{-1}) = tr\ (R_g {R_a}^{-1} R_a) = tr\ (R_g)$$

(iii)The matrix of the representation $\rho \oplus \rho'$ will have a block form, as in Example 4.4, so

$$tr\ (\rho \oplus \rho') = tr\ R + tr\ R' = \chi + \chi'$$

This means the character of this direct sum will be the sum of the individual characters. □

Characters actually have alot to do with the hermitian product. In order to relate characters to the hermitian product, we first have to write characters in vector form. Each member of a group will get an entry in the character vector, and the members of each conjugacy class are adjacent entries in the vector. Suppose that the G has members $g_1, g_2, g_3, ..., g_n$, then the hermitian product between two characters χ and χ' is calculated by the formula

$$\langle \chi, \chi' \rangle = \frac{1}{n} \sum_{i=1}^{n} \overline{\chi(g_i)} \chi'(g_i)$$

Note that the $\frac{1}{n}$ is used to normalize the product; this is the standard way to calculate the hermitian product of characters.

The following theorem is very important to the study of characters. However, we will only prove Equation 5.4 in a later section of the paper. The rest of the theorem will not be proved in this paper.

Theorem 5.3. *Let G be a group of order n, let $\rho_1, \rho_2, \rho_3, ...$ be representatives of the distinct isomorphism classes of irreducible representations of G, and let χ_i be the character of ρ_i.*

(i) $\langle \chi_i, \chi_j \rangle = 0$ if $i \neq j$, and $\langle \chi_i, \chi_j \rangle = 1$ if $i = j$.

(ii) There are finitely many isomorphism classes of irreducible representations, the same number as the number of conjugacy classes in the group.

(iii) Let d_i be the dimension of the irreducible representations within the isomorphism class of ρ_i, and let r be the number of irreducible representations. Then d_i divides n, and

$$n = d_1^2 + ... + d_r^2. \tag{5.4}$$

With this powerful theorem, we can prove many useful corollaries. Before that, we need to define class functions.

Definition 5.5. A complex value $\varphi : G \to \mathbb{C}$ that is constant on each conjugacy class of the group G is called a class function. The collection of all such functions L is a vector space over $\mathbb{C}$.

Corollary 5.6. *The irreducible characters of a group G form an orthonormal basis of L.*

Proof. This corollary easily follows from Theorem 5.3. The irreducible characters must be linearly independent by part (i) of Theorem 5.3. They must span the vector space by part (iii) of Theorem 5.3, there are r vectors in the collection of irreducible characters and the dimension of L must also be r by part (ii) of Theorem 5.3. □

With this corollary, we can write the character of ρ as a linear combination of the characters of the irreducible representations by part (iii) of Theorem 5.2. In addition, this gives us the power to write any representation ρ as a linear combination of the irreducible representations of the group.

Corollary 5.7. *Let $\chi_1, ..., \chi_r$ be the irreducible characters of a finite group G, and let χ be some character. Then $\chi = n_1\chi_1 + ... + n_r\chi_r$, where $n_i = \langle \chi, \chi_i \rangle$.*

Proof. We already know that we can write any character as a linear combination of the irreducible characters. We just need to prove that $n_i = \langle \chi, \chi_i \rangle$. We are given that $\chi = n_1\chi_1 + ... + n_r\chi_r$, which means

$$\langle \chi, \chi_i \rangle = \langle n_1\chi_1, \chi_i \rangle + ... + \langle n_i\chi_i, \chi_i \rangle + ... + \langle n_r\chi_r, \chi_i \rangle.$$

Since every term will be 0 except for

$$\langle n_i\chi_i, \chi_i \rangle = n_i\langle \chi_i, \chi_i \rangle = n_i,$$

we have what we needed. □

Corollary 5.8. *It two representations ρ, ρ' have the same character, they are isomorphic.*

Proof. Let χ be the character of ρ and let χ' be the character of ρ'. We know that the representations can be written as linear combinations of the irreducible representations, and the same is true for their characters and the irreducible characters. This means that

$$\text{If } \rho = n_1\rho_1 \oplus ... \oplus n_r\chi_r \text{ and } \rho' = n_i'\chi_i \oplus ... \oplus n_r'\chi_r$$

$$\text{then } \chi = n_1\chi_1 + ... + n_r\chi_r \text{ and } \chi' = n_1'\chi_1 + ... + n_r'\chi_r.$$

However, we also know the irreducible characters are linearly independent, which means any vector in the space of class functions L can only be written in one way using the vectors in the basis. This implies that if $\chi = \chi'$, then $n_i = n_i'$ for each i, so $\rho = \rho'$ □

Corollary 5.9. *A character χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof. If $\chi = n_1\chi_1 + ... + n_r\chi_r$, then $\langle \chi, \chi \rangle = {n_1}^2 + ... + {n_r}^2$, which will yield 1 if and only if a single n_i is 1 while the rest are 0. □

Example 5.10. As an example, we will find all the irreducible representations and characters of the cyclic group $G = \{e, x, x^2, x^3\}$ of order 4.

In order to determine how many such irreducible representations there are, we will first need to find the number of conjugacy classes. We want to find all the conjugates of an arbitrary element x^n of the group. Let us call the conjugate g, then we want all g such that $g = x^m x^n x^{-m}$ for arbitrary x^m. However, since the operation of this group is commutative, we can conclude that $g = x^n$, which means that each element only has itself as a conjugate. This implies that there are 4 conjugacy classes in this group; then by part (ii) of Theorem 5.3, there must be 4 distinct irreducible representations are their characters.

Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the irreducible representations, then by Equation 5.4, we know that $4 = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2$, where d_i is the dimension of ρ_i. The only solution to this equation is $4 = 1^2 + 1^2 + 1^2 + 1^2$, which implies that all 4 irreducible representations are 1-dimensional. This means we can think of the characters as functions that map the elements of this group to $\mathbb{C}$.

We will let ρ_1 be the identity representation, which maps every element of G to the identity element of $\mathbb{C}$, which is just 1. This means $\chi_1 = (1, 1, 1, 1)$.

The next representation ρ_2 will be rotation around the complex plane by $\frac{\pi}{2}$ counter-clockwise. The identity element will obviously still map to 1. This time,

$$\chi_2(x) = i \quad \chi_2(x^2) = -1 \quad \chi_3(x^3) = -i.$$

It is easy to show that this is a homomorphism, since rotation and the group operation are both commutative. In addition, $\langle \chi_2, \chi_2 \rangle = 1$, which means this is an irreducible representation, and $\chi_2 = (1, i, -1, -i)$.

Similarly, ρ_3 will be rotation around the complex plane by $\frac{\pi}{2}$ clockwise. Then $\chi_3 = (1, -i, -1, i)$.

Lastly, ρ_4 will be rotation around the complex plane by π. The direction of rotation does not matter in this case, because both directions will produce the same result. Then $\chi_4 = (1, -1, 1, -1)$. Note that $\langle \chi_4, \chi_4 \rangle = 1$, so this representation is irreducible.

We now have all 4 distinct irreducible representations with different characters. We can organize all of this information in a character table, with each row headed by an irreducible character and each column headed by a representative of a conjugacy class:

	e	x	x^2	x^3
χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_3	1	$-i$	-1	i
χ_4	1	-1	1	-1

6. Regular Representation

Some groups have a natural action on a set. This means each member of the group G can act on a set $S = \{s_1, s_2, s_3, ..., s_n\}$ by permuting its elements. Moreover, we require that there exists a homomorphism between G and $Sym(S)$, the collection of all symmetric actions on S. Then each g in G is mapped to σ_g, a permutation action. We can define a vector space $V(S)$ as a vector space over the field $\mathbb{C}$ with basis S. A vector v in this space will be in the form $v = \sum_i^n a_i s_i$, where a_i is a complex number. Then the permutation $\sigma_g : S \to S$ is an invertible linear operator on $V(S)$ permuting on the basis S. This gives a representation of G on $V(S)$.

With respect to basis S, the matrix representation R_g will have the same columns as the identity matrix, except the columns will be shuffled around, depending on how g permutes the elements of S. For example, in a 3-dimensional vector space, where g switches the first two elements of S, R_g will be

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the character of g is 1, since only the third column has a non-zero entry on the diagonal. In fact, $\chi(g)$ is the number of elements of S not moved by g, since only those corresponding columns of R_g will have 1 on the diagonal, while the rest of the diagonal is 0.

Definition 6.1. If we let G be the set that G permutes and let the action be left multiplication, the representation of G on $V(G)$ is called the regular representation ρ^{reg}.

Proposition 6.2. *The character of ρ^{reg} is given by $\chi^{reg}(e) = n$ and $\chi^{reg}(g) = 0$, where e is the identity of G, n is the order of G, and g is an arbitrary non-identity element of G.*

Proof. The first equation is clear by part (i) of Theorem 5.2; the dimension of ρ^{reg} must be the order of the group since the basis of the vector space that the images of the representation are acting on is made up of the elements of G. The second formula follows from the fact that $\chi(g)$ is the number of elements of G not moved by g. Since left multiplication by g will necessarily move anything it acts on unless $g = e$, the second formula is true. □

Because of these equations, we know that if ρ is a representation of a group with character χ, then $\langle \chi^{reg}, \chi \rangle = \dim \rho$, where χ^{reg} is the character of the regular representation.

Theorem 6.3. *Let $\rho_1, ..., \rho_r$ be the irreducible representations of a group G and let $\chi_1, ..., \chi_r$ be their characters. Then $\chi^{reg} = d_1\chi_1 + ... + d_r\chi_r$ and thus $\rho^{reg} = d_1\rho_1 \oplus ... \oplus d_r\rho_r$, where d_i is the dimension of ρ_i.*

Proof. We know that $\langle \chi^{reg}, \chi_i \rangle = d_i$. Since the χ^{reg} is a linear combination of the irreducible characters, $\langle \chi^{reg}, \chi_i \rangle$ must also be equal to the coefficient of χ_i in the linear combination. Therefore, $\chi^{reg} = d_1\chi_1 + ... + d_r\chi_r$. Next, by Corollary 5.8, we can conclude that $\rho^{reg} = d_1\rho_1 \oplus ... \oplus d_r\rho_r$. □

Note that because $\rho^{reg} = d_1\rho_1 \oplus ... \oplus d_r\rho_r$, we can also relate the dimension of ρ^{reg} with the dimensions of the irreducible representations. Namely, if we let d_{reg} be the dimension of ρ^{reg}, then $d_{reg} = {d_1}^2 + ... + {d_r}^2$. However, $d_{reg} = n$, where n is the order of the group. Therefore we can conclude that $n = {d_1}^2 + ... + {d_r}^2$, which is Equation 5.4.

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