PERSISTENT HOMOLOGY OF FINITE TOPOLOGICAL SPACES

HANEY MAXWELL

Abstract. We introduce homology and finite topological spaces. From the basis of that introduction, persistent homology is applied to finite spaces. We prove an equivalence between persistent homology and normal homology in the context of finite topological spaces and introduce an extended pseudometric on finite topological spaces, using the results of Minian.

Contents

1. Introduction

One of the most interesting aspects of finite topological spaces is their tendency to make dramatically different mathematical concepts equivalent. In this discussion we will explore an equivalence between persistent homology and normal homology, and will make use of that equivalence to introduce a notion of distance between finite topological spaces. This notion of distance will incorporate the concepts of persistent homology, in that it will depend not only on the homology of the spaces, but also on transient features in their associated filtrations.

Most discussions of either homology or finite topological spaces expect the reader to be familiar with basic algebraic topology and category theory. This discussion does not; we will begin with an in depth introduction to homology, and will go through the relevant basic properties of finite topological spaces in detail. It is

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intended that this paper be accessible to any reader with some knowledge of linear algebra, point-set topology, and basic algebra. Should the reader be lacking in any of those areas, it would be quite effective to have a basic text for each on hand, and look up as they appear.

2. Homology

Homology is one of many topological invariants, but has the useful property of being easily computable. In essence, homology detects holes in a space, and the higher and lower dimensional equivalents. The first three homology groups detect the connected components, tunnels, and voids of a space.

2.1. Simplicial Homology.

The most simple environment for homology is the simplicial complex. The basic idea of a simplicial complex is that of gluing together points, lines, triangles, tetrahedra, and the higher dimensional equivalents. This concept has a very simple abstraction, the abstract simplicial complex.

Definitions 2.1 (Abstract Simplicial Complex)**.** Let V be a set, called the vertex set. An *abstract simplicial complex* Δ is a collection of subsets of V such that for all $\sigma \in \Delta$ any subsets of σ are also elements of Δ. The elements of Δ are called *simplicies*.

Let Δ be an abstract simplicial complex and let σ be one of its simplicies; a simplex σ is an *n-simplex* , where n is a natural number, if it has cardinality $n+1$. A simplex σ' is a *face* of σ if $\sigma' \subseteq \sigma$.

An abstract simplicial complex can generate a simplicial complex in euclidean space. First, define an injective function f from V to a sufficiently high dimensional euclidean space, such that $\mathrm{Im}(f)$ is linearly independent. The geometric realization of the abstract simplicial complex Δ is then a subset of the union of the spans of the simplicies.

Definition 2.2. (Geometric Realization) Let $f\colon V \to \mathbb{R}^N$ be an injection, such that for all elements v of V $f(v)$ is an element of the standard basis. The *geometric realization* of an abstract simplicial complex Δ is as follows.

$$|\Delta| = \bigcup_{\sigma \in \Delta} \left\{ \sum_{i=1}^{|\sigma|} a_i f(\sigma_i) : \sum_{i=1}^{|\sigma|} a_i = 1 \text{ and } (\forall i)(a_i \geq 0) \right\}$$

Where $\{\sigma_i\}$ is equal to Δ.

Definition 2.3. (Abstract Standard Simplex) The *abstract standard n-simplex* is defined to be $\tilde{\Delta}^n = \mathcal{P}(V)$, such that $|V| = n+1$.

Definition 2.4. (Standard Simplex) The *standard n-simplex* Δ^n is the geometric realization of the abstract standard n-simplex. A subset of Δ^n is called a *face* of Δ^n if it is the geometric realization of a face of $\tilde{\Delta}^n$.

From the definition of a standard simplex, we can define a simplicial complex.

Definition 2.5. (Simplicial Complex) A *simplicial complex* Δ is a collection of standard n-simplicies in a euclidean space such that every face of a simplex in Δ is in Δ, and the intersection of any two simplicies in Δ is in Δ.

Example 2.6. With a vertex set V such that V has three elements, and $\Delta = \mathcal{P}(V)$, the geometric realization is a triangle. Were V to have four elements, and $\Delta = \mathcal{P}(V)$, the geometric realization would be a tetrahedron.

However, a simplicial complex is not quite enough for homology, which requires an ordering on a simplicial complex.

Definition 2.7 (Ordered Simplicial Complex)**.** An *ordered simplicial complex* is a simplicial complex, along with a partial ordering of the verticies, which restricts to a total ordering on each simplex.

Definition 2.8 (Partially Ordered Set)**.** Let X be a set and $\leq$ be a relation. $(X, \leq)$ is a *partially ordered set* , also called a *poset*, if $\leq$ is reflexive, transitive, and antisymmetric, that is, if the following are satisfied for all $x, y, z \in X$.

(1) $x \leq x$
(2) $x \leq y$ and $y \leq z \Rightarrow x \leq z$
(3) $x \leq y$ and $y \leq x \Rightarrow x = y$

The reader should note that there is no requirement that the relation exist between any two elements of X. If there is such a relation between any two elements, then it is a totally ordered set.

Definition 2.9 (Totally Ordered Set)**.** Let $(X, \leq)$ be a partially ordered set. $(X, \leq)$ is a *totally ordered set* if for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

This elucidates the definition of an ordered simplicial complex; an ordered simplicial complex is simply a simplicial complex where the verticies of each simplex are totally ordered in a manner consistent with the orderings of the verticies of all of the other simplicies.

Definition 2.10. (Chain Complex) A *chain complex* is a sequence of abelian groups C_n with $n \in \mathbb{N}$ and homomorphisms $\partial_n \colon C_n \to C_{n-1}$, where ∂_0 maps from C_0 to 0, such that for all $n, \partial_n \circ \partial_{n+1} = 0$.

This definition sounds exceedingly abstract. However, there is a simple way of generating a chain complex from a simplicial complex, stated here using the terminology from Hatcher [1]. Let Δ_n be the free abelian group with the n-simplicies of an ordered simplicial complex Δ as its basis, and consider the map ∂_n from the basis of Δ_n, where $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \setminus \{\sigma_i\})$. Since there is a total ordering on each simplex, σ_i can be defined to be the i-th vertex in the ordering. Because ∂_n is defined on the basis elements of Δ_n, it extends to a homomorphism $\partial_n \colon \Delta_n \to \Delta_{n-1}$.

Proposition 2.11. *The groups Δ_n and the homomorphisms ∂_n are a chain complex.*

Proof. Consider $\partial_n \circ \partial_{n+1}(\sigma)$, where σ is an $(n+1)$-simplex. Let c be a non-zero element of $\partial_n \circ \partial_{n+1}(\sigma)$. Let the simplex σ' be an element of the basis for Δ_{n-1} which has a non-zero coefficient in c. The simplex σ' must be equal to $\sigma \setminus \{\sigma_i, \sigma_j\}$. However, there are going to be two basis elements with equal non-zero coefficients of $\partial n + 1(\sigma)$ which will yield σ' under ∂_n, namely $\sigma \setminus \{\sigma_i\}$ and $\sigma \setminus \{\sigma_j\}$, and they will yield $\sigma \setminus \{\sigma_i, \sigma_j\}$ with opposite signs, as subtracting an element reverses the signs with which all elements less than it are added in the following homomorphism,

and i is not equal to j. This presents a contradiction, implying that there are no basis elements with non-zero coefficients for $\partial_n \circ \partial_{n+1}(\sigma)$, which is thus equal to 0. Because the set of $(n+1)$-simplicies is a basis for Δ_{n+1}, this concludes the proof. □

The definition of homology follows from the definition of a chain complex and the method of constructing one from a simplicial complex.

Definition 2.12 (Homology Groups)**.** Given a chain complex (C_n, ∂_n), define the *n=th homology group* H_n by the following equation.

$$H_n = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})}$$

The elements of the kernel of ∂_n are called *cycles*, and the elements of the image of image of ∂_{n+1} are called *boundaries*.

Proposition 2.13. *The homology groups of a chain complex are well defined.*

Proof. Since $\partial_n \circ \partial_{n+1} = 0$, $\mathrm{Im}(\partial_{n+1})$ must be a subset of $\mathrm{Ker}(\partial_n)$. □

Example 2.14. Consider the abstract simplicial complex with the vertex set $\{1, 2, 3\}$ and simplicies $\Delta = \mathcal{P}(V)$. In the first homology group, $\{1, 2\}$, $\{2, 3\}$, and $\{1, 3\}$, that is, the sides of the triangle, would be a cycle, but there is no tunnel in that complex, as can be seen by the fact that the chain $-\{1, 2\} - \{2, 3\} + \{1, 3\}$ is the image of $\{1, 2, 3\}$, the interior of the triangle. Therefore, that cycle is quotiented out and the first homology group is zero.

Because the homology groups are defined in terms of chain complexes, the elements of which can be easily computed from simplicial complexes, and the quotient of groups, which is also easily computable, it is possible to algorithmically compute the homology of finite simplicial complexes.

2.2. **Singular Homology.** Simplicial homology is only defined for simplicial complexes, so a more general homology theory is needed for other topological spaces. Singular homology fills that role.

Definition 2.15 (Singular Chain Complex)**.** Let X be a topological space. Let C_n be the free abelian group with a basis of the set of all continuous functions from the geometric realization of the standard n-simplex $|\Delta^n|$ to X. The homomorphisms ∂_n are defined in the same way as in simplicial homology: let σ be a continuous function from Δ^n to X, $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i$(the restriction of σ to $|\Delta^n \setminus \{\Delta_i^n\}|$). The *singular chain complex* of X is (C_n, ∂_n).

The singular chain complex defines the singular homology groups, as homology groups are defined for any chain complex.

Theorem 2.16. *For any simplicial complex, the singular homology groups are isomorphic to the simplicial homology groups.*

The proof for this is non-trivial and would require material outside the scope of this paper. Should the reader be interested, both Hatcher [1] and Munkres [3] have clear proofs of this theorem.

2.3. **Reduced Homology.** Unfortunately, the homology theory detailed above has a result that the homology groups of an n-sphere are zero except the n-th and the zeroth. It would seem logical that for every sphere S^m, H_n would be zero for all $n \neq m$. This motivates the formulation of reduced homology.

Definition 2.17 (Reduced Homology Groups)**.** Let (C_n, ∂_n) be a chain complex. Define $\epsilon\colon C_0 \to \mathbb{Z}$ with the equation $\epsilon\left(\sum_i a_i\sigma_i\right) = \sum_i a_i$, where each a_i is an integer. The *reduced homology groups* of (C_n, ∂_n), denoted $\tilde{H}_n$, are defined as equal to the homology groups H_n of (C_n, ∂_n) for $n \neq 0$, with the exception of $\tilde{H}_0$. The reduced homology group $\tilde{H}_0$ is equal to $\dfrac{\mathrm{Ker}(\epsilon)}{\mathrm{Im}(\partial_1)}$.

In essence, this decreases the rank of H_0 by one. From this point on, whenever this discussion references homology groups, it is referring to reduced homology groups.

3. Persistence

Persistent homology is a way of describing which topological features are most distinct in a filtration. This has many applications in the physical sciences, where one rarely gets an entire topological space as data, but instead often gets a sample of discrete points from some topological space. From such a sample, a simple way of determining the topology of the original space could be to consider $\cup_{x\in X} B_r(x)$, with the filtration defined to start at $r = 0$ and then to increase r, looking at what topological features persist the longes. Persistent homology gives a rigorous way of doing that, as well for any arbitrary space and any filtration.

3.1. **Persistent Homology Groups.**

Definition 3.1 (Filtration)**.** Let X be a topological space. A *filtration* of X is a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \subset X_{n+1}$ and $\bigcup_{n\in\mathbb{N}} X_n = X$.

Once a filtration is defined for a space, it is possible to consider the persistent homology groups. This paper will use the definition presented by Edelsbrunner et al. [4]. There are several other equivalent definitions, but this one has an useful similarity to the definition of an homology group.

Definition 3.2 (Persistent Homology Groups)**.** Let X be a topological space with a filtration $\{K^l\}$. The homomorphisms ∂_n^l are defined as the restrictions of ∂_n to the singular chain complex for K^l. The *p-persistent k-th homology group of X given the filtration K^l* is

$$H_n^{l,p}(X, K^l) = \frac{\mathrm{Ker}(\partial_n^l)}{\mathrm{Im}(\partial_{n+1}^{l+p}) \cap \mathrm{Ker}(\partial_n^l)}$$

In situation where the relevent filtration is clear, this will be abbreviated $H_n^{l,p}(X)$.

Over the course of the filtration, boundaries are added, and thus cycles which were non-trivial in homology at time l in the filtration may no longer be at time $l + p$. The p-persistent homology groups consist of the features which are still non-trivial at time $l + p$. It is worth noting that the zero-persistent homology groups of K^l are the same as the actual homology groups of K^l.

3.2. **Persistence Diagram.** In addition to the persistent homology groups, it would be useful to have a general indication of when features were born and died over the entire filtration. The persistence diagram gives such general indication. This discussion will use a version of the definition used by Morozov [6], modified to be consistent with the definition of persistent homology groups.

Definition 3.3 (Betti Numbers)**.** The *n-th Betti number* of a topological space X is denoted β_n and is equal to $\mathrm{Rank}(H_n)$ where H_n is the n-th homology group.

Definition 3.4 (Persistent Betti Numbers)**.** Let $\{K^l\}$ be a filtration of X. The *p-persistent n-th Betti number* of K^l is denoted $\beta_n^{l,p}$ and is equal to $\mathrm{Rank}(H_n^{l,p})$.

From the persistent Betti numbers comes a set of multiplicities $\mu_n^{i,j}, j > i$ with the following equation.

$$p = j - i, \mu_n^{i,j} = \beta_n^{i,p} - \beta_n^{i-1,p} - \beta_n^{i,p+1} + \beta_n^{i-1,p+1}$$

The multiplicity $\mu_n^{i,j}$ will then represent the number of features in the n-th homology group that are 'born' at i and 'die' at j. The n-th persistence diagram can be defined from these multiplicities.

Definition 3.5 (Persistence Diagram)**.** Let X be a topological space and $\{K^l\}$ be a filtration. The *n-th persistence diagram of X given the filtration* $\{K^l\}$, denoted $\mathrm{Dgm}_n(\{K^l\})$ is a subset of $\bar{\mathbb{R}}^2$, where $\bar{\mathbb{R}}^2 = (\mathbb{R} \cup \{\pm\infty\}) \times (\mathbb{R} \cup \{\pm\infty\})$, with each point (i, j) having a multiplicity of $\mu_n^{i,j}$ and all points on the diagonal (that is the set $\{(i, j) : i = j\}$) having infinite multiplicity.

There is one particularly important result about the persistence diagram, which is its stability under small changes in the filtration. However, any discussion of such stability requires a notion of distance.

Definition 3.6 (Bottleneck Distance)**.** Let $\mathrm{Dgm}_n(f)$ and $\mathrm{Dgm}_n(g)$ be persistence diagrams. *The bottleneck distance* , denoted $d_B(\mathrm{Dgm}_n(f), \mathrm{Dgm}_n(g))$ is the infimum over all bijections $h\colon \mathrm{Dgm}_n(f) \to \mathrm{Dgm}_n(g)$ of $\sup_i d(i, h(i))$.

Proposition 3.7. *Let* Dgm *denote the set of all persistence diagrams with a finite number of off diagonal points. The ordered pair* (Dgm, d_B) *is an extended metric space.*

Proof. Let $x, y, z \in \mathrm{Dgm}$. There are several properties which must be proven, the first of which is $d_B(x, y) = 0$ implies that $x = y$. Assume that $d_B(x, y) = 0$ and there exists a point $a \in x$ such that $a \notin y$. This implies that there is a sequence $\{b_n\}_{n\in\mathbb{N}} \subset y$ such that $\lim_{n\to\infty}(d_B(b_n, a)) = 0$. However, such a subset must contain infinite off-diagonal points, which presents a contradiction. Symmetry is inherent in the definition, as is non-negativity. The remaining property is the triangle inequality. Given $d_B(x, y)$ and $d_B(y, z)$ consider the infimum of $\sup_i d(i, h(i))$, where $h = a \circ b$, with a and b being bijections from y to z and from x to y respectively. As the standard metric on Euclidean space obeys the triangle inequality, and the relation $\leq$ is preserved under infimums, $\sup_i d(i, h(i) \leq \sup_j d(j, a(j)) + \sup_k d(k, h(k))$. $\square$

Any discussion of stability also requires a notion of distance between filtrations. Consider a filtration generated by a function from our space X to $\mathbb{R}$, with the filtration defined by the sublevel sets, that is, the subset X^r of X such that for all $x \in X^r$, the value $f(x) \leq r$. For functions, distance is defined by the l-infinity

norm, that is $||f-g||_\infty = \sup_x |f(x)-g(x)|$. With these concepts of distance, the stability result can be formalized.

Theorem 3.8 (Stability of the Persistence Diagram)**.** *Let X be a triangulable space, that is, a space to which there is a homeomorphism from a simplicial complex. Let f, g be continuous functions from X to $\mathbb{R}$ and let $\tilde{f}$ and $\tilde{g}$ be the filtrations they define, let f, g such that all the diagrams of the filtrations they define have finitely many off-diagonal points. Then $d_B(\mathrm{Dgm}_n(\tilde{f}) - \mathrm{Dgm}_n(\tilde{g})) \leq ||f-g||_\infty$.*

The proof of this is non-trivial, and its inclusion would preclude the discussion of finite spaces. If the reader is interested, a detailed proof is done by Cohen-Steiner et al. [5].

4. Finite Topological Spaces

Many of the applications of persistent homology have been to simplicial complexes, see Edelsbrunner et al. [4] for an example, but it can be applied to any topological space. In particular, finite spaces, that is, topological spaces with a finite number of points, offer an interesting application, due to the small number of possible filtrations of a particular space. However, before applying persistent homology to finite spaces, it is useful to elaborate on their basic properties. This discussion will only include those properties relevant to homology. For a more detailed account, May [2] gives a discussion of the homotopy properties of finite spaces.

4.1. Equivalence to Partially Ordered Sets.

The first important thing for finite topological spaces is to restrict ourselves to T_0 spaces, as the only T_1 space will be the discrete space. Non-T_0 spaces do not distinguish all their points, and thus are equivalent to a T_0 space via the Kolmogorov quotient.

Definition 4.1 (T_0)**.** Let X be a topological space, X is T_0 if for all $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x.

Definition 4.2 (T_1)**.** Let X be a topological space, X is T_1 if for all $x \in X, \{x\}$ is closed.

Definition 4.3 (Kolomogorov Quotient)**.** Let X be a topological space, and consider an equivalence relation $\sim$, where $x \sim y$ if there is neither an open set containing x but not y, nor an open set containing y but not x. The *Kolmogorov quotient* of X is the quotient space of X with $\sim$.

Clearly, the Kolmogorov quotient of any topological space X is T_0.

Definition 4.4 (Alexandrov Space)**.** Let X be a topological space, X is an *Alexandrov space* if the arbitrary intersection of open sets is open.

Proposition 4.5. *Every finite space is an Alexandrov Space.*

Proof. Let X be a finite topological space. As X is finite, the topology on X is also finite. Therefore, any arbitrary intersection of open sets is a finite intersection of open sets. □

Because finite spaces are Alexandrov spaces, there exists a unique minimal neighborhood for each point, that being the intersection of all open sets containing the point. This gives us a unique minimal basis. An equivalence between finite posets and finite topological spaces comes from the minimal basis. Consider the poset with elements the same as the finite space, and $x \leq y$ if and only if the minimal neighborhood of x is a subset of the minimal neighborhood of y. Any finite poset can also generate a finite space, by making the minimal neighborhood of any point equal to all the points it is greater than or equal to. Because of this, the reader should consider finite posets and finite topological spaces as isomorphic mathematical objects.

A poset X can also generate a finite simplicial complex, denoted $\mathcal{K}(X)$, where the verticies of $\mathcal{K}(X)$ are the elements of X, and the simplicies of $\mathcal{K}(X)$ are the totally ordered subsets of X. Similarly, a finite simplicial complex K can generate a finite poset $\mathcal{X}(K)$, where the simplicies of K are the elements of $\mathcal{X}(K)$, and the ordering is by inclusion. It is important to note that $\mathcal{X}(\mathcal{K}(X)) \neq X$.

Theorem 4.6. *Let X be a finite poset. The homology groups of $\mathcal{K}(X)$ are isomorphic to the homology groups of the finite topological space associated with X.*

The proof requires knowledge of homotopy, and thus will not be addressed in this paper. If the reader is interested, it is done by McCord [8].

4.2. **Persistence.** From here on in, consider finite spaces will be T_0, and homology groups will be reduced with integer coefficients unless indicated otherwise. In order to consider the persistent homology of finite topological spaces, it will be useful to have a few more definitions, for which this discussion will use those introduced by Minian [7].

Definitions 4.7 (Homogeneous Poset)**.** Let X be a poset. X is *homogeneous* if all maximal chains $x_0 < x_1 ... < x_n$ are of equal length, and the set X has *degree* n, where n is the length of all the maximal chains.

Definitions 4.8 (Graded Poset)**.** Let X be a poset, let $U_x^X = \{y \in X : y \leq x\}$ and let $F_x^X = \{y \in X : x \leq y\}$. Similarly, let $\hat{U}_x^X = \{y \in X : y < x\}$ and let $\hat{F}_x^X = \{y \in X : x < y\}$. The poset X is *graded* if for all $x \in X$, the set U_x^X is homogeneous. The *degree* of $x \in X$, denoted $\deg(x)$, is equal to the degree of U_x^X.

Definition 4.9 (Cellular Poset)**.** Let X be a graded poset. X is *cellular* if, for all x in X, the homology group $H_m(\hat{U}_x)$ is isomorphic to $H_m(S^{p-1})$ where p is the degree of x, for all $m \in \mathbb{N}$.

Since homology hasn't been defined for posets in this discussion, the reader should be considering the singular homology of the associated finite T_0 space.

Definition 4.10 (Skeleton)**.** Let X be a cellular poset. The *p-skeleton* X^p is the set of all points with degree less than or equal to p.

The p-skeleta of a poset define a filtration on the associated finite space.

Theorem 4.11. *Let X be a cellular poset. The homology groups of the p-skeleta, $H_n(X^p)$ are isomorphic to the homology groups of the space $H_n(X)$ for all $p > n$.*

A proof is done by Minian [7]. This has a remarkable implication for persistent homology, in that, in context of cellular posets, it appears that persistent homology is the same as normal homology.

Proposition 4.12. *Let X be a cellular poset. For all $n > p$, $H_n(X^p) = 0$.*

Proof. Consider $\mathcal{K}(X)$, $n > p$ implies that there are no n-simplicies, and thus $\Delta_n = 0$. Thus, necessarily $\mathrm{Ker}(\partial_n) = 0$. □

Corollary 4.13. *Let X be a finite topological space and consider the filtration induced by the p-skeleta. The persistent homology groups $H_n^{l,p}(X)$ are zero for all $n > l$.*

Theorem 4.14. *Let X be a cellular poset and consider the filtration defined by the p-skeleta. The persistent homology groups $H_n^{l,p}(X)$ are isomorphic to the homology groups $H_n(X)$ for all $l > n$ and all $p \in \mathbb{N}$.*

Proof. The isomorphism from $H_n(X^m)$ to $H_n(X)$ for all $m > n$ implies that the homology groups of any two p-skeleta, with $p_1 \geq p_2 > n$ will be isomorphic. Consider the elements of $H_n(X^{n+1})$; these must be isomorphic to the elements of the homology group of the next filtration. Therefore, in order for the persistent homology group not to be isomorphic to the homology group of the space, new cycles must be born. However, any addition to the n-skeleton after x^n will be unable to add non-trivial cycles, because it will necessarily include a boundary trivializing those cycles. Therefore, none of the features of $H_n(X^{n+1})$ will be made trivial in further steps of the filtration, and thus have infinite persistence. □

Remark 4.15. Unfortunately, a stronger statement cannot be made. A good example to illustrate this is the non-Hausdorff cone of the four point circle (which is examined in detail by Salvatore [9]). The non-Hausdorff cone of the four point circle is the space defined by the minimal basis $\{\{0\}, \{1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 2, 3, 4\}$. In this space, both the zeroth and the first homology groups of X^0 and X^1 respectively, are not isomorphic to the zeroth and first homology groups of X. This is because X^0 is S^0, which is not connected. Furthermore, the first step in the filtration is $\mathcal{S}S^0$, which has homology groups isomorphic to those of S^1. In constrast, the entire space has homology groups equal to zero.

5. Distance and Finite Topological Spaces

In general, persistent homology requires a choice of a filtration, and it is impossible to discuss the persistent homology of a finite space, as it will differ depending on the filtration. However, the filtration via the p-skeleta of a space gives a means of defining persistence merely from a finite space. This can be used to compare spaces in terms of their persistent homology.

5.1. Persistence Extended Pseudometric.

Unfortunately, p-skeleta are only defined for cellular posets. However, it is possible to construct a cellular poset with isomorphic homology groups from any poset.

Proposition 5.1. *Let K be a finite simplicial complex. The poset $\mathcal{X}(K)$ is cellular.*

Proof. Let σ be an n-simplex in K. The poset $U^X_{\mathcal{X}(\sigma)}$ is the set of all of the subsets of σ, that is, the faces of σ. As such, $\hat{U}^X_{\mathcal{X}(\sigma)}$ is the set of proper faces of σ. Considering the geometric realization of σ, this will have the same homology groups as an $(n-1)$ sphere, as the cycles and boundaries are equivalent. □

Corollary 5.2. *For any finite poset X, the poset $\mathcal{X}(\mathcal{K}(X))$ is cellular. For the purposes of this discussion, $\mathcal{X}(\mathcal{K}(X))$ will be denoted as the barycentric subdivision of X.*

Therefore, a persistence diagram is defined for any finite topological space by considering the filtration associated with the barycentric subdivision of the associated poset.

Theorem 5.3 (Persistence Extended Pseudometric)**.** *The sum over all n of the bottleneck distance between the n-th persistence diagrams of the barycentric subdivisions of posets associated with finite topological spaces defines an extended pseudometric on finite topological spaces.*

Proof. As the bottleneck distance is an extended metric on persistence diagrams, and a persistence diagram is associated with every finite topological space, the result follows. □

This pseudometric will be denoted d_P and referred to as the persistent distance. It has several useful characteristics.

Proposition 5.4. *The persistent distance between two finite topological spaces is infinite if and only if their homology groups are not isomorphic.*

Proof. There are two cases in which the persistent distance is infinite, either it must be impossible to have a bijection mapping all infinite points to infinite points, or there must be infinite off-diagonal points. We are defining a point to be 'infinite' if it is not an element of $\mathbb{R}^2$. The second case is impossible, as the spaces are finite, so there it must be impossible to have a bijection mapping all infinite points to infinite points. The lack of such a bijection is precisely the definition of having differing cardinalities. From the definition of the persistence diagram, we can see that if the sets of infinite points have different cardinalities, the ranks of the homology groups of the finite spaces are not equal. □

Proposition 5.5. *For any two finite spaces X and Y with isomorphic homology groups and whose barycentric subdivisions have the same degree, the persistent distance is at most $\deg(X')\frac{\sqrt{2}}{2}$, where X' is the the barycentric subdivision of X.*

Proof. We know that the n-th homology groups of the two barycentric subdivisions will be isomorphic for all of the p-skeleta other than the n-skeleton, and thus any feature which cannot be exactly matched in a bijection will have distance $\frac{\sqrt{2}}{2}$ from the diagonal. Furthermore, the p-skeleta of any cellular poset X will be equal to X for any p greater than the degree of X, thus no features will be born or die after the $\deg(X)$-skeleton. □

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References

[1] Allen Hatcher. Algebraic Topology. Cambridge University Press. 2002.
[2] J. P. May. Finite Topological Spaces. Preprint (2010). http://math.uchicago.edu/∼may/FINITE/FiniteSpaces.pdf.

[3] James R. Munkres. Elements of Algebraic Topology. Perseus Publishing. 1984.
[4] H. Edelsbrunner, D. Letscher and A. Zomorodian. Topological persistence and simplification. Discrete Comput. Geom. 28 (2002), 511-533.
[5] Cohen-Steiner, D., Edelsbrunner, H., and Harer, J. Stability of Persistence Diagrams. Discrete Comput. Geom. 37, 1 (Jan. 2007), 103-120.
[6] D. Morozov. Homological Illusions of Persistence and Stability. Ph.d. dissertation, Department of Computer Science, Duke University, August 2008.
[7] E. G. Minian. Some Remarks on Morse Theory for Posets, Homological Morse Theory and Finite Manifolds. Preprint (2010). arXiv:1007.1930v1.
[8] M. C. McCord. Singular Homology Groups and Homotopy Groups of Finite Topological Spaces. Duke Math. J. Volume 33, Number 3 (1966), 465-474.
[9] J. Salvatore. The Weak Homotopy Equivalence of S^n and a Space with $n+2$ Points. Preprint (2009). http://math.uchicago.edu/∼may/FINITE/Salvatore.pdf.