MARCINKIEWICZ INTERPOLATION

GUO CHEN

Abstract. In this paper I present some interpolation theorems used in real analysis. Such results allow one to bound the norms of linear or non-linear operators acting on L^p spaces. One such estimate, the Marcinkiewicz interpolation theorem, is effective in establishing bounds for non-linear operators such as the important Hardy-Littlewood maximal operator. Marcinkiewicz's theorem can also be applied to the Hilbert transform, a widely used linear operator in Fourier analysis. Such operators are important, for instance, in proving Carleson's theorem on the almost everywhere convergence of Fourier series of L^p functions.

Contents

1. Introduction

The use of interpolation theorems is the first step in proving Carleson's theorem. Carleson's theorem states that if f is an L^p periodic function for some $p \in (1, \infty)$ with Fourier coefficients $\hat{f}(n)$, then

$$\lim_{N\to\infty} \sum_{|n|\leq N} \hat{f}(n)e^{inx} = f(x)$$

for almost every x. Understanding the estimates involved in various interpolation theorems is crucial in establishing the properties of several important operators used in the proof the Carleson's theorem.

2. Basic Setting

We take f to be a real-valued function defined on the closed interval $[-A, A]$ and suppose that $f \in L^1([-A, A])$. The Lebesgue measure on $\mathbb{R}$ is denoted by m.

Date: August 2nd, 2010.

Definition 2.1. Suppose $y \in \mathbb{R}_+$. The function $\lambda_f : \mathbb{R}_+ \to [0, 2A]$ defined by

$$\lambda_f(y) = m(\{x \in [-A, A] \,|\, |f(x)| > y\})$$

is called the *distribution function* of f.

It is immediate that $0 \le \lambda_f(y) \le 2A$ for any $y \in \mathbb{R}_+$ and $\lambda_f(y) \to 0$ as $y \to 0$. Since $\{x \in [-A, A] \,|\, |f(x)| > a\} \subset \{x \in [-A, A] \,|\, |f(x)| > b\}$ if $a > b > 0$, λ_f is a decreasing function. Moreover λ_f is continuous from the right because

$$\cup_{n=1}^{\infty}\{x \in [-A, A] \,|\, |f(x)| > y + \frac{1}{n}\} = \{x \in [-A, A] \,|\, |f(x)| > y\}$$

for $y \in \mathbb{R}_+$. Therefore λ_f is a measurable function.

We consider an operator T from $L^1([-A, A])$ into the set of all measurable functions on $[-A, A]$. The operator T will not necessarily be defined on all of $L^1([-A, A])$. At the very least, we assume that T is defined on all simple functions and all continuous functions. This ensures that the domain of T is dense in $L^1([-A, A])$.

In the following, T will either be a linear operator or a sublinear operator.

Definition 2.2. An operator T is *sublinear* if it satisfies

$$|T(\alpha f)| = |\alpha||Tf| \text{ for any } \alpha \in \mathbb{R}$$
$$|T(f + g)| \le |Tf| + |Tg|.$$

Here f and g are any functions in the domain of T.

We now classify operators by the type of bounds they satisfy.

Definition 2.3. The operator T is of *(strong) type* p, where $p \in [1, \infty]$ if there exists a constant $A_p \in \mathbb{R}_+$ such that

$$||Tf||_p \le A_p||f||_p$$

for all f in the domain of T.

Note that if the operator T is of type p with $p \in [1, \infty]$, then T can be extended to all of $L^p([-A, A])$ by continuity because the domain of T is dense in $L^p([-A, A])$. Thus T is a bounded operator defined on all of $L^p([-A, A])$.

Definition 2.4. The operator T is of *weak type* p , where $p \in [1, \infty)$ if there exists a constant $A_p \in \mathbb{R}_+$ such that

$$\lambda_{Tf}(y) \le \left(\frac{A_p}{y}\right)^p ||f||_p^p$$

for all f in the domain of T and $y \in \mathbb{R}_+$.

If the operator T is of type p then T is of weak type p. To see this, suppose T is of type p for $p \in [1, \infty)$, and note that for $y \in \mathbb{R}_+$ we have

$$||Tf||_p^p = \int_{-\infty}^{\infty} |Tf(x)|^p dx \ge y^p \lambda_{Tf}(y)$$

by the definition of the distribution function of Tf. Upon rearranging the last inequality and using the type p bound, we have

$$\lambda_{Tf}(y) \le \frac{1}{y^p}||Tf||_p^p \le \left(\frac{A_p}{y}\right)^p ||f||_p^p$$

so T is of weak type p, as required. We also introduce the following related concepts:

Definition 2.5. The operator T is of *restricted type p* , where $p \in [1, \infty)$, if there exists a constant $A_p \in \mathbb{R}_+$ such that

$$||T\chi_E||_p \leq A_p||\chi_E||_p = A_p[m(E)]^{\frac{1}{p}}$$

for all measurable sets $E \subset [-A, A]$.

Definition 2.6. The operator T is of *restricted weak type p* , where $p \in [1, \infty)$, if there exists a constant $A_p \in \mathbb{R}_+$ such that

$$\lambda_{T\chi_E}(y) \leq \frac{A_p}{y}^p ||\chi_E||_p^p = \left(\frac{A_p}{y}\right)^p m(E)$$

for all measurable sets $E \subset [-A, A]$.

Of course an operator of type p is also of restricted type p, and the same is true for an operator of weak type p. In the same way that we showed why an operator being of weak type p is weaker than an operator being of strong type p, we can show that an operator of restricted type p is also of restricted weak type p.

3. Marcinkiewicz Interpolation

Our aim is to prove a result, the Marcinkiewicz Interpolation Theorem, that allows one to bound the norms of operators acting on L^p spaces. We first need to relate the distrbution function λ_f to the p-norm $||f||_p$ of f via the following lemma.

Lemma 3.1. *If $f \in L^1([-A, A])$ then for $p \in [1, \infty)$*

$$||f||_p^p = \int_{-\infty}^{\infty} |f(x)|^p \, dx = \int_0^{\infty} py^{p-1}\lambda_f(y) \, dy,$$

and in particular

$$||f||_1 = \int_0^{\infty} \lambda_f(y) \, dy$$

Proof. We may rewrite $||f||_p^p$ as

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx = \int_{-\infty}^{\infty} \left(\int_0^{|f(x)|} py^{p-1} \, dy\right) dx.$$

Applying Fubini's theorem to the expression on the right, we obtain

$$\int_{-\infty}^{\infty} \left(\int_0^{|f(x)|} py^{p-1} \, dy\right) dx = \int_0^{\infty} py^{p-1}\lambda_f(y) \, dy,$$

as required. □

We now prove the first interpolation result.

Theorem 3.2. *Suppose that T is of restricted weak type p and q, where $1 \leq p \leq q < \infty$. Then T is of restricted type r for all $r \in (p, q)$.*

Proof. Fix a measurable $E \subset [-A, A]$; let $\lambda(y)$ denote the distribution function of $T\chi_E$. By hypothesis there exist constants C_p and C_q such that

$$\lambda(y) \leq \left(\frac{C_p}{y}\right)^p m(E) \text{ and } \lambda(y) \leq \left(\frac{C_q}{y}\right)^q m(E)$$

for all measurable sets E and all $y \in \mathbb{R}_+$. By the previous lemma, we have

$$\begin{aligned} ||T\chi_E||_r^r = r\int_0^\infty y^{r-1}\lambda(y)\,dy &= r\int_0^1 y^{r-1}\lambda(y)\,dy + r\int_1^\infty y^{r-1}\lambda(y)\,dy \\ &\le r\cdot m(E)\left\{\int_0^1 y^{r-p-1}C_p^p\,dy + \int_1^\infty y^{r-q-1}C_q^q\,dy\right\} \\ &= r\cdot m(E)\left\{C_p^p\cdot\frac{1}{r-p} + C_q^q\cdot\frac{1}{q-r}\right\}, \end{aligned}$$

where the inequality follows from the hypotheses and the last line follows by evaluating the integrals in the second line (note that $r-q-1<-1$). Put

$$C_r = r^{\frac{1}{r}}\cdot\left\{C_p^p\cdot\frac{1}{r-p} + C_q^q\cdot\frac{1}{q-r}\right\}$$

for $r \in (p,q)$. Then

$$||T\chi_E||_r \le C_r[m(E)]^{\frac{1}{r}},$$

so T is of restricted type r, as desired. □

Note that C_r is bounded as long as $r \in (p,q)$. Informally, this result tells us that having weak boundedness at the extremes p and q is enough to obtain strong boundedness inside the interval. The Marcinkiewicz result, which we prove now, is of similar flavor. Note that Theorem 2.2 is not a consequence of Marcinkiewicz's theorem. An operator being of restricted weak type p and q does not imply it is of weak type p and q, so if an operator T is of restricted weak type p and q but not of weak type p and q, Theorem 2.2 applies but Marcinkiewicz does not.

Theorem 3.3. *(Marcinkiewicz) Suppose that T is a sublinear operator of weak type p and q, where $1 \le p \le q < \infty$. Then T is of type r for all $r \in (p,q)$.*

Proof. By hypothesis there exist constants C_p and C_q such that

$$\lambda_{Tf}(y) \le \left(\frac{C_p}{y}\right)^p ||f||_p^p \text{ and } \lambda_{Tf}(y) \le \left(\frac{C_q}{y}\right)^q ||p||_q^q.$$

Put $C = C_p^{\frac{p}{q-p}} C_q^{\frac{-q}{q-p}}$. We introduce the functions f^y and f_y for a fixed $y \in \mathbb{R}_+$ by

$$f^y(x) = \begin{cases} f(x) & \text{if } |f(x)| \le Cy \\ 0 & \text{if } |f(x)| > Cy \end{cases}, \quad f_y(x) = \begin{cases} 0 & \text{if } |f(x)| \le Cy \\ f(x) & \text{if } |f(x)| > Cy \end{cases}$$

Clearly $f(x) = f_y(x) + f^y(x)$ and the sublinearity of T gives

$$\lambda_{Tf}(2y) \le \lambda_{Tf_y}(y) + \lambda_{Tf^y}(y).$$

The right-hand side of the above is, by assumption, smaller than

$$C_p^p y^{-p}\int_{-\infty}^\infty |f_y(x)|^p dx + C_q^q y^{-q}\int_{-\infty}^\infty |f^y(x)|^q dx.$$

Then by Lemma 2.1,

$$||Tf||_p^p = \int_0^\infty p(2y)^{p-1}\lambda_{Tf}(2y)\,d(2y) = p\cdot 2^p\int_0^\infty y^{p-1}\lambda_{Tf}(2y)\,dy.$$

By the inequality just established, the last quantity is smaller than

$$p{\cdot}2^p\left\{\int_0^\infty C_p^p y^{-p}y^{p-1}\left(\int_{|f(x)|>Cy}|f(x)|^p\,dx\right)dy+\int_0^\infty C_q^q y^{-q}y^{p-1}\left(\int_{|f(x)|\le Cy}|f(x)|^q\,dx\right)dy\right\}$$

Using Fubini's theorem to interchange the order of integration, this expression is equal to

$$p{\cdot}2^p\left\{C_p^p\int_{-\infty}^\infty|f(x)|^p\left(\int_{y=0}^{\frac{|f(x)|}{C}}y^{r-p-1}\,dy\right)dx+C_q^q\int_{-\infty}^\infty|f(x)|^q\left(\int_{\frac{|f(x)|}{C}}^\infty y^{r-q-1}\,dy\right)dx\right\}$$

$$=p\cdot 2^p\left\{C_p^pC^{r-p}\cdot\frac{1}{r-p}\int_{-\infty}^\infty|f(x)|^r\,dx+C_q^qC^{q-r}\cdot\frac{1}{q-r}\int_{-\infty}^\infty|f(x)|^r\,dx\right\}.$$

Since $C_p^pC^{p-r}=C_q^qC^{q-r}=C_p^{p\frac{q-r}{q-p}}\cdot C_q^{q\frac{r-p}{q-p}}$, the above expression is equal to

$$p\cdot 2^p\cdot C_p^{p\frac{q-r}{q-p}}\cdot C_q^{q\frac{r-p}{q-p}}\cdot\left\{\frac{1}{r-p}+\frac{1}{q-r}\right\}||f||_r^r.$$

Thus we have shown $||Tf||_r^r\le K_r||f||_r^r$, where

$$K_r=p\cdot 2^p\cdot C_p^{p\frac{q-r}{q-p}}\cdot C_q^{q\frac{r-p}{q-p}}\cdot\left\{\frac{1}{r-p}+\frac{1}{q-r}\right\},$$

so T is of strong type r for all $r\in(p,q)$, as desired. □

In a similar way we obtain a result that will be useful in further applications.

Theorem 3.4. *Suppose that T is a sublinear operator of weak type p and of strong type ∞, where $p\in[1,\infty]$ Then T is of type r for all $r\in(p,\infty)$.*

Proof. We use the same notation as in the proof of the last theorem. By hypothesis we have

$$\lambda_{Tf}(y)\le\left(\frac{C_p}{y}\right)^p||f||_p^p$$

and $||Tf||_\infty\le C_\infty||f||_\infty$. Choose the constant $C=\frac{1}{C_\infty}$. Then $||f^y||_\infty\le\frac{1}{C_\infty}y$ and $||Tf^y||_\infty\le y$, and consequently $\lambda_{Tf^y}(y)=0$. Then

$$\lambda_{Tf}(2y)\le\lambda_{Tf_y}(y)\le C_p^py^{-p}\int_{-\infty}^\infty|f_y(x)|^p\,dx,$$

and exactly as in the proof of the last theorem,

$$||Tf||_r^r\le p\cdot 2^p\cdot C_p^p\cdot\left(\frac{1}{C_\infty}\right)^{p-r}\cdot\frac{1}{r-p}||f||_r^r.$$

Thus T is of type r for all $r\in(p,\infty)$, as desired. □

4. Hardy-Littlewood maximal function

We now consider the Hardy-Littlewood maximal function and derive estimates for this function via Theorem 2.4.

Let $f\in L^1(\mathbb{R})$. We define the *maximal function* θ by

$$\theta f(x)=\sup_{t\in\mathbb{R}_+}\frac{1}{2t}\int_{x-t}^{x+t}|f(y)|\,dy,\ \ x\in\mathbb{R}.$$

The function θf is measurable for each $f \in L^1(\mathbb{R})$. To see this, first note that θf is lower semicontinuous because it is the supremum of continuous functions. Moreover the operator θ is a sublinear operator defined on $L^1(\mathbb{R})$.

Theorem 4.1. *The operator θ is of type ∞ and of weak type 1.*

Proof. It is clear that θ is of type ∞ since the average of a function is no larger than its essential supremum. For the weak type 1 bound, we may assume that f is of compact support. We can do this because functions of compact support are dense in L^p and so we may approximate any function in the domain of T with a function of compact support. We may also assume f is nonnegative. Let $y \in \mathbb{R}_+$. Take $x \in \{t \,|\, \theta f(t) > y\}$. By the definition of the least upper bound, there exists $r \in \mathbb{R}_+$ such that

$$\frac{1}{2r}\int_{x-r}^{x+r} |f(t)|\, dt > y.$$

Thus to each $x \in \{t \,|\, \theta f(t) > y\}$ we can find an interval I_x centered at x such that

$$\int_{I_x} f(t)\, dt > y \cdot m(I_x).$$

As f has compact support, the set $\{\theta f > y\}$ is bounded, and we may suppose that all the intervals I_x are contained in the interval $[-B, B]$. By the Vitali covering lemma, there exists a sequence $\{I_n\}$ of pairwise disjoint intervals such that

$$m\left(\cup_{n=1}^{\infty} I_n\right) \geq \frac{1}{4} m\left(\cup_x I_x\right)$$

Then

$$\lambda_{\theta f}(y) \leq m(\{x \,|\, \theta f(x) > y\}) \leq m\left(\cup_x I_x\right) \leq 4m\left(\cup_{n=1}^{\infty} I_n\right) \leq 4\sum_{n=1}^{\infty} m(I_n)$$

$$\leq \frac{4}{y}\sum_{n=1}^{\infty}\int_{I_n} f(y)\, dy = \frac{4}{y}\int_{\cup_n I_n} f(y)\, dy \leq \frac{4}{y}||f||_1,$$

so f is of weak type 1. □

Note that θ is not of type 1. If we let $f(x) = \chi_{(0,1)}(x)$, then θ is not even integrable. Thus an operator of weak type p is not necessarily of type p as well.

Corollary 4.2. *The operator θ is of type p for all $p \in (1, \infty)$.*

Proof. Since θ is of type ∞ and of weak type 1, a direct application of Theorem 2.4 gives the desired result. □

5. Conclusion

As demonstrated by the application in the previous section, Marcinkiewicz's theorem allows one to derive rather strong estimates from a basic hypothesis. Often times it takes some work to establish the basic hypothesis, but once this is done, Marcinkiewicz is used to reveal an entire range of estimates for the operator under consideration. The advantage of using Marcinkiewicz is that it can be applied to non-linear operators such as the maximal operator in the previous section as well as linear operators. Consequently, Marcinkiewicz's theorem can tell us something about any of the important operators used throughout Fourier analyis. This makes it an invaluable tool.

Acknowledgments. I thank my mentors, Timur Akhunov and Olga Turanova for their guidance and their extensive suggestions on improving this paper.

References

[1] O. G. Jorsboe and L. Mejlbro The Carleson-Hunt Theorem on Fourier Series. Springer-Verlag Berlin Heidelberg New York, 1982.