Limit Computable Sets and Degrees

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Abstract. This paper will study sets and degrees containing sets that are determined as limits of computable approximations. By the Shoenfield Limit Lemma, the limit computable sets are precisely the degrees below $\emptyset'$. In particular, the paper will study limit computable sets by classifying them according to bounds to the number of changes to elements in various approximations of the sets. This leads to the n-c.e., ω-c.e., and Δ_2^0 classifications. The paper will show these characterizations of sets and degrees are proper at various levels. Properties of n-c.e. sets particularly concerning the non-existence of n-c.e. minimal degrees will be developed in this paper. These classifications will also provide some insights into the structures of the Truth-Table Turing Degrees.

Contents

1. Basics

All sets are subset of $\omega = \{0, 1, ..., \}$, the natural numbers. A set A is identified with its characteristic function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Also, one writes $A(x) =_{\text{dfn}} \chi_A(x)$.

P_e denotes the e^{th} Turing program. Φ_e denotes e^{th} Turing functionals which is associated to P_e. Φ_e^A denotes the e^{th} A-partial computable function which has oracle A. $\Phi_e^{\emptyset}$ is the e^{th} partial computable functions. The total A computable functions are the total partial A computable functions. $\Phi_e^A(x)\downarrow$ denotes the function converges; $\Phi_e^A(x)\uparrow$ denotes the function diverges. If $\Phi_e^A(x)\downarrow$, then $\varphi_e^A(x)$ is defined to be the largest number used in the oracle for the computation of $\Phi_e^A(x)$. If $\Phi_e^A(x)\uparrow$, then $\varphi_e^A(x)\uparrow$. If $e,x,y\le s$ and P_e computes $\Phi_e^A(x)=y$ in less than or equal s steps, then $\Phi_{e,s}^A(x)=_{\text{dfn}}\Phi_e^A(x)$; otherwise $\Phi_{e,s}^A(x)\uparrow$. More generally, if the set A is also given in stages (such as some enumeration) $\{A_s\}_{s\in\omega}$, one writes concisely, $\Phi_e^A(x)[s]=_{\text{dfn}}\Phi_{e,s}^{A_s}(x)$. Similarly for $\varphi_e^A(x)$. The domain of Φ_e^A is denote W_e^A. $W_{e,s}^A$ is the domain of $\Phi_{e,s}^A$.

A relation relation R is Σ_0^0, Π_0^0, and Δ_0^0 if and only if R is computable. A relation $R(x)$ is Σ_n^0 if $R(x)=(\exists y_1)(\forall y_2)...(Qy_n)C(x,y_1,...,y_n)$ for some computable relation $C(x,y_1,...,y_n,x)$ where $Q=\forall$ if n is even and $Q=\exists$ if n is odd. $R(x)$ is Π_n^0 if $R(x)=\neg S(x)$ for $S\in\Sigma_n^0$. A relation $R(x)$ is Δ_n^0 if $R\in\Sigma_n^0\cap\Pi_n^0$. A relation $R(x)$ if arithmetical if $R\in\Sigma_n^0$ for some n. A set A is Σ_n^0, Π_n^0, Δ_n^0, or arithmetical if $A=\{x:R(x)\}$, where R has that property, respectively.

Proposition 1.1 The function $\langle\cdot,\cdot\rangle:\omega\times\omega\to\omega$ defined by $\langle x,y\rangle=\frac{(x+y)^2+3x+y}{2}$ is a bijection.

Proof : Note the numerator is always even so the function is well-defined. Say that (x,y) is in level n if $x+y=n$. Within a level n, define the ordering $(x_1,y_1)<_n(x_2,y_2)$ if and only if $x_1<_n x_2$. Define an ordering on $\omega\times\omega$ having $(x_1,y_1)<(x_2,y_2)$ if and only if $x_1+y_1<x_2+y_2$ (i.e. (x_1,y_1) is at a lower level than (x_2,y_2)) or $(x_1,y_1)<_n(x_2,y_2)$ where $n=x_1+y_1=x_2+y_2$.

To show $\langle\cdot,\cdot\rangle$ is bijective, it suffice to show that $\langle 0,0\rangle=0$ (value on the least element), and the function monotonically increases by 1. First, $\langle 0,0\rangle=\frac{(0+0)^2+3(0)+0}{2}=0$. Next, within every level, the function monotonically increases by 1. This holds for level 0. Inductively assume that it holds for $n-1$ for $n>0$. In the defined ordering, $(i,n-I)<(i+1,n-i-1)$, and there are no ordered pairs in between, where $0\le i\le n$. Thus, $\langle i,n-i\rangle=\frac{n^2+2i+n}{2}<\frac{n^2+2i+n}{2}+1=\frac{n^2+2i+n+2}{2}=\langle i+1,n-i-1\rangle$. Thus this holds for all levels.

Next, the value of function on any element of higher level is greater than the value of the function on any element of a lower level and increase of the value of the function between levels is by 1. It is clear this holds for level 0 and level 1. Suppose this holds for all level less than n. By the previous paragraph, the value of the function occurs on $(n,0)$. The smallest value of the function on level $n+1$ occurs at $(0,n+1)$. $\langle n,0\rangle=\frac{n^2+3n}{2}<\frac{n^2+2n+1}{2}+1=\frac{n^2+3n+2}{2}=\frac{(n+1)^2+n+1}{2}=\langle 0,n+1\rangle$. Note, that between levels, the function increases by 1. Thus the function increases monotonically by 1. The function is bijective. □

A bijective function from $\omega\times\omega\to\omega$ is called a pairing function. Proposition 1.1 proves there exists a pairing function. Furthermore, there exists a bijection from $\omega^n\to\omega$ for all n. It is given by $\langle x_1,x_2,...,x_n\rangle=\langle\langle...\langle\langle x_1,x_2\rangle,x_3\rangle....\rangle,x_n\rangle$.

Define the function $\pi_i^n(x)=x_i$ for $i\le n$ where x_i is the number such that $\langle x_1,...,x_i,...,x_n\rangle=x$. Call this function the projection onto the i^{th} coordinate of the n-tuple representation of x.

Definition 1.2 Let $F=\{x_0,...,x_k\}$ be a finite set ($x_i\ne x_j$ for all $0\le i<j\le k$). The strong index for F is the number $n=_{\text{dfn}}2^{x_0}+...+2^{x_k}$, and one says that $F=D_{f(n)}$. A computable approximation for A is a computable function f such that for all s, $D_{f(s)}\subseteq[0,s)$ and for all x, $A(x)=\lim_s D_{f(s)}(x)$. A is limit computable if there is a computable approximation for A. Also, one says $\{A_s\}_{s\in\omega}$ is a computable approximation for A if and only if there is a computable approximation f such that $D_{f(s)}=A_s$ for all s.

In practice, one may have $D_{f(0)}\ne\emptyset$ (in particular be the set $\{0\}$) and allow $s\in D_{f(s)}$.

Definition 1.3 Define the Turing functional J as follows:

$$J^X(e) = \begin{cases} 0 & \Phi_e^X(e) \uparrow \\ 1 & \Phi_e^X(e) \downarrow \end{cases}$$

$X' = \mathrm{dom}(J^X)$

Definition 1.4 A set A is ω-c.e. if and only if there is a computable approximation $\{A_s\}_{s\in\omega}$ of A and a computable function f such that $|\{s : A_s(x) \neq A_{s+1}\}| \leq g(x)$. For some $n \in \omega$, a set A is n-c.e. if and only if A is ω-c.e. via a computable function g such that $g(x) \leq n$ for all x.

Definition 1.5 A set A is c.e. in a set X if and only if there exists a e such that $A = W_e^X$ for some $e \in \omega$. A is c.e. if $X = \emptyset$.

Theorem 1.6 The following are equivalent :
(1) A is c.e.
(2) $A = \emptyset$ or $A = f(\omega)$, i.e. is the range of a computable function f.
(3) A is Σ_1^0

Proof : Suppose $A = W_e^\emptyset$. $\emptyset$ is a c.e. set. Suppose $A \neq \emptyset$, then let $b \in \emptyset$. Define

$$f(\langle x, s\rangle) = \begin{cases} b & \Phi_{e,s}^\emptyset(x) \uparrow \\ x & \Phi_{e,s}^\emptyset(x) \downarrow \end{cases}$$

then $A = f(\omega)$.

If $A = \emptyset$, then A is Σ_1^0. Suppose $A = f(\omega)$ for some computable function f, then $A = \{x : (\exists y) f(y) = x\}$.

Suppose $A = \{x : (\exists y) R(x, y)\}$ where R is computable. Define the computable function Θ as follows:

$$\Theta(x) = \begin{cases} 0 & (\exists y) R(x, y) \\ \uparrow & \text{otherwise} \end{cases}$$

then $A = \mathrm{dom}(\Theta)$. □

Proposition 1.7 A set A is computable if and only if A and $\bar{A}$ are c.e.

Proof : Suppose $A = \Phi_e^\emptyset$. Define Θ as follows:

$$\Theta(x) = \begin{cases} 0 & \text{if } \Phi_e^\emptyset(x) = 1 \\ 1 & \text{if } \Phi_e^\emptyset(x) = 0 \end{cases}$$

Θ is computable and $\bar{A} = \Theta$. Thus $\bar{A}$ is computable. All computable sets are c.e.

Suppose A and $\bar{A}$ are c.e. That is, $A = W_e^\emptyset$ and $\bar{A} = W_f^\emptyset$. Define Ψ as follows:

$$\Psi(x) = \begin{cases} 1 & \text{if } \Phi_e^\emptyset(x) \downarrow \\ 0 & \text{if } \Phi_f^\emptyset(x) \downarrow \end{cases}$$

Ψ is well defined since $A \cap \bar{A} = \emptyset$, and it is total computable because $A \cup \bar{A} = \omega$. Thus A is computable since $A = \Psi$. $\square$

Proposition 1.8 A set A is 1-c.e. if and only if A is c.e.

Proof : Suppose A is 1-c.e. via the computable approximation $\{A_s\}_{s\in\omega}$. Then $A = \{x : (\exists s)(A_s(x) = 1)\}$. The defining property is a Σ^0_1 relation.

Suppose A is c.e., then $A = W^{\emptyset}_e$. Then $\{W^{\emptyset}_{e,s}\}_{s\in\omega}$ is the required computable approximation. $\square$

If a set A is c.e. and $A = W^{\emptyset}_e$, then define $A_s =_{\text{dfn}} W^{\emptyset}_{e,s}$. $\{A_s\}_{s\in\omega}$ is the standard computable enumeration of A.

Definition 1.9 A set D is d-c.e. if and only if $D = A - B$ for some c.e. sets A and B.

Proposition 1.10 A set D is 2-c.e. if and only if D is d-c.e.

Proof : Suppose D is 2-c.e. by the computable approximation $\{D_s\}_{s\in\omega}$. Define $A = \{x : (\exists s)(x \in D_s)\}$. Define $B = \{x : (\exists s)(\exists t)((t > s) \wedge (x \in D_s) \wedge (x \notin D_t))\}$. A and B are c.e. because they are Σ^0_1. $D = A - B$.

Suppose $D = A - B$ for c.e. set A and B. Then $D_s = A_s - B_s$ is the required computable approximation of D. $\square$

Theorem 1.11 For all $k \in \omega$. C is $2k$-c.e. if and only C is the union of k d-c.e. sets. C is $2k+1$-c.e. if and only if C is the union of k d-c.e. sets and a c.e. set. Moreover, if C is $2k$-c.e. if and only if C is of the form $C = (A_1 - B_1) \cup ... \cup (A_k - B_k)$ where A_i, B_i are c.e. and $A_1 \supset B_1 \supset A_2 \supset ... \supset A_k \supset B_k$. C is $2k+1$-c.e. if and only if $C = (A_1 - B_1) \cup ... \cup (A_k - B_k) \cup A_{k+1}$ where A_i, B_i are c.e. and $A_1 \supset B_2 \supset A_2 \supset ... \supset A_{k+1}$.

Proof : Suppose C is $2k$-c.e. with computable approximation $\{C_s\}_{s\in\omega}$. Define $A_i = \{x : (\exists s_1)...(\exists s_{2i-1})((s_1 < ... < s_{2i-1}) \wedge (C_{s_1}(x) = 1) \wedge (C_{s_2}(x) = 0) \wedge ... \wedge (C_{s_{2i-1}}(x) = 1))\}$. Define $B_i = \{x : (\exists s_1)...(\exists s_{2i})((s_1 < ... < s_{2i}) \wedge (C_{s_1}(x) = 1) \wedge (C_{s_2}(x) = 0) \wedge ... \wedge (C_{s_{2i}}(x) = 0))\}$. For each i, A_i and B_i are c.e. because they are Σ^0_1. $C = (A_1 - B_1) \cup ... \cup (A_k - B_k)$. Similarly for the odd case.

Suppose C is union of k d-c.e. sets, say $C = (A_1 - B_1) \cup ... \cup (A_k - B_k)$. Let $C_s =_{\text{dfn}} (A_{1,s} - B_{1,s}) \cup ... \cup (A_{k,s} - B_{k,s})$. $\{C_s\}_{s\in\omega}$ is the required computable approximation. Similarly for the odd cases. $\square$

The following is a well-known result by Joseph R. Shoenfield. Δ^0_2 refers to the arithmetic hierarchy.

Theorem 1.12 The following are equivalent:
(1) A is limit computable.
(2) $A \in \Delta^0_2$.
(3) $A \leq_T \emptyset'$.

Proof : Suppose A is limit computable via a computable approximation $\{A_s\}_{s\in\omega}$. Then $A = \{x : (\exists m)(\forall n)(m \leq n \Rightarrow A_n(s) = 1)\}$ and $\bar{A} = \{x : (\exists m)(\forall n)(m \leq n \Rightarrow A_n(s) = 0)\}$. $A \in \Delta^0_2$ since $A \in \Sigma^0_2$ and $\bar{A} \in \Pi^0_2$.

Suppose $A \in \Delta^0_2$. That is, $A = \{x : (\exists m)(\forall n)R(m, n, x)\}$ and $\bar{A} = \{x : (\exists m)(\forall n)S(m, n, x)\}$. To compute whether $x \in A$, use $\emptyset'$ to find the first m such that $(\forall n)R(m, n, x)$ or $(\forall n)(S(m, n, x)$ holds. If the first holds, then $x \in A$; if the latter, then $x \notin A$. One of the case must hold since $x \in A$ or $x \in \bar{A}$.

Suppose $A \leq_T \emptyset'$. Since $\emptyset'$ is c.e., there exists a computable enumeration $\{\emptyset'_s\}_{s\in\omega}$ of A . Let $A_s = \{x : \Phi^{\emptyset'_s}_{e,s}(x) = 1\}$. □

A set that satisfies any of the three equivalent condition in Theorem 1.4 is called Δ^0_2.

Definition 1.13 $2^{<\omega}$ is the set of finite binary string. Let $\sigma, \tau \in 2^{<\omega}$, $\sigma \preceq \tau$ denotes that σ is a prefix or initial segment of τ. $\sigma\tau$ denotes the concatenation of the two strings. There is a bijection $\omega \to 2^{<\omega}$ given by $l(\sigma) = n$ such that 1σ is the binary digit representation of $n + 1$. $|\sigma|$ is the length of σ.

2^ω is the set of infinite binary strings or functions $\omega \to \{0, 1\}$. If $\sigma \in 2^{<\omega}$ and $f \in 2^\omega$, then $\sigma \prec f$ denotes that σ is a prefix or initial segment of σ. For all n, $f \upharpoonright n$ is the finite string σ of length n such that $\sigma(x) = f(x)$ for all $0 \leq x < n$. $f \upharpoonright\upharpoonright n$ is the finite string σ of length $n + 1$ such tha $\sigma(x) = f(x)$ for all $0 \leq x \leq n$. Let $A \subseteq 2^{<\omega}$. $[\![A]\!] = \{f : (f \in 2^\omega) \wedge (\exists n)(f \upharpoonright n \in A)\}$. If $\sigma \in 2^{<\omega}$, then $[\![\sigma]\!] = [\![\{\sigma\}]\!]$. If $T \subseteq 2^{<\omega}$ such that if $\sigma \in T$ and $\tau \prec \sigma$ then $\tau \in T$, then T is called a tree. $[T] = \{f : (\forall n)(f \upharpoonright n \in T\}$. The elements of $[T]$ are called paths through T. Let $\mathcal{T} = \{[\![A]\!] : A \subset 2^{<\omega}\}$. Then $(2^\omega, \mathcal{T})$ is a topological space called Cantor Space. Equivalently, $\mathcal{T}$ is the topology generated by $[\![\sigma]\!]$ for all $\sigma \in 2^{<\omega}$. The closed sets are the $[T]$ for T a tree.

Theorem 1.14 (1) (Weak König's Lemma) If T is an infinite tree, then $[T] \neq \emptyset$.

(2) (Compactness and Effective Compactness) 2^ω is compact. If A is computable and $[\![A]\!] = 2^\omega$, then there exists $F \subset 2^\omega$ and F finite such that $[\![F]\!] = 2^\omega$ and F can be found uniformly from A.

Proof : Since T is infinite, there exists infinitely many $\sigma \in T$ such that $0 \preceq \sigma$ or there exists infinitely many $\sigma \in T$ such that $1 \preceq \sigma$. Define $f(0) = 0$ if 0 have infinitely many extension on T or $f(x) = 1$ if 1 has infinitely many extensions on T. By induction, suppose that $f \upharpoonright n$ has been defined and for all $k \leq n - 1$, $f \upharpoonright k$ has infinitely many extension on T. Thus $(f \upharpoonright n)0$ or $(f \upharpoonright n)1$ has infinitely many extensions on T. Define $f(n) = 0$ if the first holds and $f(n) = 1$ if the latter holds. Thus $f \in [T]$.

Assume that there are no finite subset $F \subset A$ such that $[\![F]\!] = 2^\omega$. Define for all n, $F_n = \{\sigma : (\sigma \in A) \wedge (|\sigma| \leq n)\}$. This set is computable since A is computable. Since $[\![F]\!] \neq 2^\omega$, there exists a string σ of length n such that no initial segment of σ is in A. Let σ_n be the first string (in lexicographic order) of length n with the above property. Define $T_n = \{\tau : \tau \preceq \sigma_n\}$, and note that $T_n \cap A = \emptyset$. Let $T = \cup_{n\in\omega} T_n$. $T \cap A = \emptyset$. T is an infinite tree; thus, by part 1, there exists $f \in [T]$. $f \notin [\![A]\!]$. Thus $[\![A]\!] \neq 2^\omega$. Therefore, such a finite set F exists. To find T, find the least n such that for all σ such that $|\sigma| = n$, there exists a $\tau \in A$ such that $\tau \preceq \sigma$. Then $[\![F_n]\!] = 2^\omega$. This process is uniform since everything could be computably determined. □

Definition 1.15 $A \leq_m B$ ("many to one" reduction) if and only if there is a computable function f such that $x \in A$ if and only if $f(x) \in B$.

Definition 1.16 $A \leq_{wtt} B$ ("weak truth table" or "bounded Turing" reduction) if and only if there exists e and a computable function f such that $A \leq_T B$ via $A = \Phi^B_e$ and $\varphi^B_e(x) \leq f(x)$ for all x.

Definition 1.17 $A \leq_{tt} B$ ("truth table" reduction) if and only if there exists a e such that $A \leq_T B$ via

$A = \Phi_e^B$ where Φ_e is a total Turing functional, .i.e. for all $Z \in 2^\omega$, Φ_e^Z is a total function.

Proposition 1.18 If $A \leq_m B$, then $A \leq_T B$, $A \leq_{wtt} B$, and $A \leq_{tt} B$.

Proposition 1.19 A is c.e. in X if and only if $A \leq_m X'$.

Theorem 1.20 (1) $A \leq_{tt} B$ if and only if there exists a computable function f such that,

$$(\forall x)\left(x \in A \Leftrightarrow \bigvee_{\sigma \in D_{f(x)}} (\sigma \prec B)\right)$$

D_n is the finite set introduced in Definition 1.2.
(2) $A \leq_{tt} B$ if and only $A \leq_{wtt} B$.

Proof : Suppose $A \leq_{tt} B$ by the total Turing function Φ_e. Then let $B_x = \{\sigma : \Phi_{e,|\sigma|}^\sigma(x)\downarrow\}$. Since, Φ_e is a total functional, $[\![B_x]\!] = 2^\omega$. By Theorem 1.14 (2), there is a finite $F_x \subset B_x$ such that $[\![F_x]\!] = 2^\omega$. Further, since B_x is computable, this F_x is known uniformly. Thus, the finite set $F'_x = \{\sigma : (\sigma \in F_x) \wedge (\Phi_{e,|\sigma|}^\sigma(x)\downarrow= 1)\}$ is also known uniformly. That is, there is a computable function f such that $D_{f(x)} = F'_x$.

Conversely, suppose given f, then define the total Turing functional Θ as follows

$$\Theta^X(x) = \begin{cases} 1 & \text{if } (\exists\sigma)((\sigma \in D_{f(x)}) \wedge (\sigma \prec X)) \\ 0 & \text{if otherwise} \end{cases}$$

$A \leq_{tt} B$ via Θ^B.

By (1), let f be function as above. Define $g(x) = \max\{|\sigma| : \sigma \in D_{f(x)}\}$. Then $A \leq_{wtt} B$ via Θ^B defined above and the function $g(x)$ bounds the use. □

Theorem 1.21 The following are equivalent
(1) A is ω-c.e.
(2) $A \leq_{tt} \emptyset'$
(3) $A \leq_{wtt} \emptyset'$
The reduction and sets are effectively obtained.

Proof : Suppose A is ω-c.e. via $\{A_s\}_{s\in\omega}$ and the computable function g. Define the c.e. set C in stages. $C_0 = \emptyset$. Suppose C_{s-1} has been defined, for each $x < s$, if $A_{s-1}(x) \neq A_s(x)$, then enumerate $\langle x, i\rangle$ where i is the least number such that $\langle x, i\rangle \notin C_{s-1}$. Let $C = \cup_{s\in\omega} C_s$. (Note C is c.e. since $\{C_s\}_{s\in\omega}$ makes C 1-c.e. and using Proposition 1.9). Define the total Turing functional Θ as follows

$$\Theta^X(x) = \begin{cases} 0 & (\mu i)((i \leq g(x)) \wedge (\langle x, i\rangle \in X) \text{ is odd or no } i \leq g(x) \text{ exists.} \\ 1 & (\mu i)((i \leq g(x)) \wedge (\langle x, i\rangle \in X) \text{ is even.} \end{cases}$$

where $(\mu i)R(i)$ means the least i satisfying R. $A \leq_{tt} C$ and $C \leq_m \emptyset'$ since C is c.e. Thus $C \leq_{tt} \emptyset'$ by Proposition 1.18. $A \leq_{tt} \emptyset'$ by transitivity of $\leq_{tt}$.

$A \leq_{tt} \emptyset'$, then $A \leq_{wtt} \emptyset'$ by Theorem 1.20.

Suppose $A \leq_{wtt} \emptyset'$ via $\Phi_e^{\emptyset'}$ and a computable function $h(x)$ that bounds the use. $\emptyset'$ is c.e. since $\emptyset' \leq_m \emptyset'$ and Proposition 1.19. Let $\{\emptyset'_s\}_{s\in\omega}$ be 1-c.e. approximation of $\emptyset'$. Let $A_s(x) =_{\text{dfn}} \{x : (x \leq s) \wedge (\Phi_{e,s}^{\emptyset'_s \restriction h(x)+1}(x)\downarrow= 1\}$. The computation only changes if a number less or equal to $h(x)$ enters $\emptyset'$. $A_s(x)$

can change at most $h(x)$ times. A is ω-c.e. via $\{A_s\}_{s\in\omega}$ and bound $h(x)$. □

2. Properly n-c.e., Properly ω-c.e., and Δ^0_2-not-ω-c.e. Sets

Suppose $n = 2k$ for $k \in \omega$. For each z, suppose $z = \langle x^z_1, y^z_1, ..., x^z_k, y^z_k\rangle$. Let $E^n_z = W_{x^z_1} - W_{y^z_1} \cup ... \cup W_{x^z_k} - W_{y^z_k}$. By the bijection in Proposition 1.1 and Theorem 1.12, all $2k$-c.e. sets occur as E_z for some z. Similarly for the odd $n = 2k+1$. In this way, all n-c.e. sets occur as E^n_z for some z. Define $E^n_{z,s} = W_{x^z_1,s} - W_{y^z_1,s} \cup ... \cup W_{x^z_k,s} - W_{y^z_k,s}$.

Definition 2.1 A n-c.e. set which is not $n-1$ c.e. is called a properly n c.e. set. A ω-c.e. set which is not n-c.e. for all $n \in \omega$ is called a properly ω-c.e. set.

Theorem 2.2 For all n, there exists there exists a properly $n+1$-c.e. set.

Proof : This theorem shall be proved by creating a set $C = A_1 - B_1 \cup ... \cup A_k - B_k \cup A_{k+1}$, where A_i $(1 \leq i \leq k+1)$ and B_j $(1 \leq j \leq k)$ are c.e. sets, and C satisfies the following requirements for all z:

$$R_z : C \neq E^n_z$$

$$P : C \text{ is } n+1\text{-c.e.}$$

The construction is as follows: At stage 0, define $A_1 = \{0\}$ and $A_i = \emptyset$ for all $1 < i \leq k+1$ and $B_j = \emptyset$ for all $1 \leq j \leq k$. At stage s, do the following for each $z \leq s$:
(Case 1) If $s - z = 0$, then enumerate z into $A_{1,s}$.
(Case 2) If $s - z > 0$ and $z \in E^n_{z,s-z} - E^n_{z,s-z-1}$, then find the largest j such that $z \in A_{j,s-1}$ and enumerate z into $B_{j,s}$.
(Case 3) If $s - z > 0$ and $z \in E^n_{z,s-z-1} - E^n_{z,s-z}$, then find the largest j such that $z \in B_{j,s-1}$ and put z into $A_{j+1,s}$.

After this has been done for all $z \leq s$, define $A_{i,s}$ and $B_{j,s}$ $(1 \leq i \leq k+1, 1 \leq j \leq k)$ to be the set formed by putting in all the elements specified by the construction.

Let $C =_{\text{dfn}} A_1 - B_1 \cup ... \cup A_k - B_k \cup A_{k+1}$. Note all A_i and B_j are c.e. by Proposition 1.9. C satisfies P by Theorem 1.12. R_z is satisfied for all z since by construction, $C_s(z) \neq E^n_{z,s-z}(z)$, $C(x) = \lim_s C_s(x)$, and this limit exists since $C_s(z) \neq C_{s+1}(z)$ if and only if $E^n_{z,s-z}(z) \neq E^n_{z,s-z+1}(z)$. Since $(E^n_{z,s}(z))_{s\in\omega}$ reaches a limit, so does $(C_s(z))_{s\in\omega}$. The odd case is handled similarly. □

In the above proof, the constructed set was given by an explicit construction of the union of difference of c.e. sets (and one c.e. set). Later constructions will merely give the computably approximation of the set to be constructed. The terms "put into" or "remove" will frequently be used to indicate the action to define $A_{s+1}(x) = 1$ (with implicit assumption $A_s(x) = 0$) and the action to define $A_{s+1}(x) = 0$ (with implicit assumption $A_s(x) = 1$), respectively.

Theorem 2.3 There exists a properly ω-c.e. set.

Proof : This theorem is proved by constructing a set C which satisfies the following requirements for all z,

$$R_z : C \neq E^{\pi^2_1(z)}_{\pi^2_2(z)}$$

$$P : C \text{ is } \omega\text{-c.e.}$$

The construction is as follows: At stage 0, let $C_0 = \{0\}$.

At stage $s > 0$, do the following for all $z \leq s$:

(Case 1) If $s - z = 0$, then put z into C.

(Case 2) If $s - z > 0$ and $z \in E^{\pi_1^2(z)}_{\pi_2^2(z),z-s} - E^{\pi_1^2(z)}_{\pi_2^2(z),z-s-1}$, then remove z from C_s.

(Case 3) If $s - z > 0$ and $z \in E^{\pi_1^2(z)}_{\pi_2^2(z),z-s-1} - E^{\pi_1^2(z)}_{\pi_2^2(z),z-s}$, then put z into C_s.

After doing this for all $z \leq s$, define C_s to be the set formed by enumerating or removing the specified elements of stage s of the construction.

Let $C(x) = \lim_{s\in\omega} C_s(x)$. The limit exists, since each E^n_y is a n-c.e. set. For each z, R_z is satisfied since $C_s(z) \neq E^{\pi_1^2(z)}_{\pi_2^2(z),z-s}(z)$ for all s. C is ω-c.e. by the computable approximation $\{C_s\}$ and the computable function π_1^2. □

Theorem 2.4 There exists a Δ^0_2 set that is not a ω-c.e. set.

Proof : The construction will produce a set C. By the Shoenfield Theorem 1.4, one must have $C \leq_T \emptyset'$. By Theorem 1.21, a set E is ω-c.e. if and only if it is $E \leq_{tt} \emptyset'$. By Theorem 1.20 (1), all truth table reduction are associated with the "truth table" given the computable function f. Hence, the theorem shall be proved by constructing a set $C \leq_T \emptyset'$ which satisfies the following requirements for all z.

$$R_z : (\Phi^{\emptyset}_z(z)\downarrow) \Rightarrow \left(\left((z \in C) \wedge \bigwedge_{\sigma\in D_{\Phi^{\emptyset}_z(z)}} (\sigma \not\prec \emptyset')\right) \vee \left((z \notin C) \wedge \bigvee_{\sigma\in D_{\Phi^{\emptyset}_z(z)}} (\sigma \prec \emptyset')\right)\right)$$

Satisfying all the requirement R_z will suffice. If $\Phi_z(z)\uparrow$, then $\Phi^{\emptyset}_z$ is not total and does not correspond to any truth table reduction. If $\Phi_z(z)\downarrow$, then if $\Phi_z(z)$ is a total function (which it may not be), then z will witness that $A \not\leq_{tt} \emptyset'$ via the truth table associated with $\Phi^{\emptyset}_e(z)$.

Define the set C as follows:

$$C(z) = \begin{cases} 0 & \text{if } z \notin \emptyset' \text{ or if } (z \in \emptyset') \wedge \bigvee_{\sigma\in D_{\Phi^{\emptyset'}_e(z)}} (\sigma \prec \emptyset') \\ \\ 1 & \text{if } (z \in \emptyset') \wedge \bigwedge_{\sigma\in D_{\Phi^{\emptyset'}_z(z)}} (\sigma \not\prec \emptyset') \end{cases}$$

Note that $z \in \emptyset'$ if and only if $\Phi^{\emptyset}_z(z)\downarrow$. It is clear that $C \leq_T \emptyset'$ and satisfies R_z for all z. □

Corollary 2.5 There exists a set $A \leq_T \emptyset'$, $A \not\leq_{wtt} \emptyset'$, and $A \not\leq_{tt} \emptyset'$.

3. Properly n-c.e. Degrees

Definition 3.1 A n-c.e. degree is a degree which contains a n-c.e. set. A properly $n+1$-c.e. degree is a $n+1$-c.e. degree which is not a n-c.e. degree.

Let Θ and Ψ be Turing Functionals, X and Y be sets, and $z \in \omega$. $\Theta^X \upharpoonright\upharpoonright z = \Psi^Y \upharpoonright\upharpoonright z$ if and only if for all $b \leq z$, $\Theta^X(b)\downarrow = \Psi^Y(b)\downarrow$.

Let A be a set. $A^{[<z]} = \{\langle x, y\rangle : y = z\}$.

The following theorem is known as the n-c.e. Hierarchy Theorem proved by S. Barry Cooper in his Ph.D Thesis, *Degrees of Unsolvability* (1971).

Theorem 3.2 For all n, there exists a properly $n+1$-c.e. degree.

Proof : The construction will produce a set C that satisfies the following requirements:

$$R_{\langle d,e,z\rangle} : C \neq \Phi_d^{E_z^n} \vee E_z^n \neq \Phi_e^C$$

$$P : C \text{ is } n+1\text{-c.e.}$$

This construction shall use the following functions which shall be defined in the construction. $r(\xi, s)$ is the restraint on C. $l(\xi, s)$ is the desired agreement on E_z^n. $w(\xi, s)$ is the witness function. $c(x, s)$ is the change counting function.

At stage 0, for all ξ, define $r(\xi, 0) = l(\xi, 0) = w(\xi, 0) = -1$ and $c(\xi, 0) = 0$.

At stage s + 1, if it exists, find the least $\langle d, e, z\rangle < s+1$ such that

$$C_s \upharpoonright\upharpoonright w(\langle d,e,z\rangle, s) = \Phi_{d,s+1}^{E_{z,s+1}^n} \upharpoonright\upharpoonright w(\langle d,e,z\rangle, s)$$

$$E_{z,s+1}^n \upharpoonright\upharpoonright l(\langle d,e,z\rangle, s) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright l(\langle d,e,z\rangle, s)$$

Define $C_{s+1}(w(\langle d,e,z\rangle, s)) = 1 - C_s(w(\langle d,e,z\rangle, s))$, and for all $x \neq w(\langle d,e,z\rangle, s)$, $C_{s+1}(x) = C_s(x)$. Define $c(w(\langle d,e,z\rangle, s)) = c(w(\langle d,e,z\rangle, s)) + 1$, and for all $y \neq w(\langle d,e,z\rangle, s)$, $c(y, s+1) = c(y, s)$. For all $\xi \leq \langle d,e,z\rangle$, define $r(\xi, s+1) = r(\xi, s)$, $l(\xi, s+1) = l(\xi, s)$, and $w(\xi, s+1) = w(\xi, s)$. For all $\xi > \langle d,e,z\rangle$, define $r(\xi, s+1) = l(\xi, s+1) = w(\xi, s+1) = -1$. Go to stage $s+2$.

If no such $\langle d,e,z\rangle < s+1$ exists, then find the least $\langle d,e,z\rangle$ such that there exists $x \leq s+1$ such that $x \in \omega^{[\langle d,e,z\rangle]}$, $x > \max\{r(\xi, s) : \xi < \langle d,e,z\rangle\}$, $c(x, s) = 0$ and

$$C_s \upharpoonright\upharpoonright x = \Phi_{d,s+1}^{E_{z,s+1}^n} \upharpoonright\upharpoonright x$$

$$E_{z,s+1}^n \upharpoonright\upharpoonright \varphi_{d,s+1}^{E_{z,s+1}^n}(x) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright \varphi_{d,s+1}^{E_{z,s+1}^n}(x)$$

Define $C_{s+1}(x) = 1$ and for all $y \neq x$, define $C_{s+1}(y) = C_s(y)$. Define $c(x, s+1) = 1$, $w(\langle d,e,z\rangle, s+1) = x$, $l(\langle d,e,z\rangle, s+1) = \varphi_{d,s+1}^{E_{z,s+1}^n}(x)$, $r(\langle d,e,z\rangle, s+1) = \max\{x, \max\{\varphi_{e,s+1}^{C_s}(y) : y \leq l(\langle d,e,z\rangle, s+1)\}\}$. For all $y \neq x$, $c(y, s+1) = c(y, s)$. For all $\xi < \langle d,e,z\rangle$, $w(\xi, s+1) = w(\xi, s)$, $l(\xi, s+1) = l(\xi, s)$, and $r(\xi, s+1) = r(\xi, x)$. For all $\xi > \langle d,e,z\rangle$, $w(\xi, s+1) = l(\xi, s+1) = r(\xi, s+1) = -1$.

If no such $\langle d,e,z\rangle$ exists, then go to stage $s+2$. This ends the construction.

Each $R_{\langle d,e,z\rangle}$ is satisfied. Suppose not. Let $\langle d,e,z\rangle$ be the least that is not satisfied. Then $C = \Phi_d^{E_z^n}$ and $E_z^n = \Phi_e^C$. Choose a stage s' such that for all $\xi < \langle d,e,z\rangle$, R_ξ has been satisfied. At stage s', at most a finite number (at most s many) of elements from $\omega^{[\langle d,e,z\rangle]}$ have ever entered C. Furthermore, elements from $\omega^{[\langle d,e,z\rangle]}$ can enter C only in an attempt to satisfy $R_{\langle d,e,z\rangle}$, i.e. there is a stage t such that $w(\langle d,e,z\rangle, t) = x$. Thus there must exists an x and a stage $t > s'$ such that $x \in \omega^{[\langle d,e,z\rangle]}$, $c(s', t) = 0$, and $\Phi_{e,t}^{E_{z,t}^n}(x) = 0$. It is clear that there is a x such that $c(s', t) = 0$ since only finitely many element from $\omega^{[\langle d,e,z\rangle]}$ has ever enter C. Furthermore, if there is never a stage $\Phi_{e,t}^{E_{z,t}^n}(x) = 0$, then one must have that $\Phi_e^{E_z^n}(x) = 1$ for all $x \in \omega^{[\langle d,e,z\rangle]}$ such that (at least for) $x > s'$. Hence from the construction, any $x \in \omega^{[\langle d,e,z\rangle]}$ will never satisfy the condition to become a witness for $\langle d,e,z\rangle$. Thus $C(x) = 0$ for all $x \in \omega^{[\langle d,e,z\rangle]}$ such that $x > s'$, yet $\Phi_e^{E_z^n}(x) = 1$. Thus $R_{\langle d,e,z\rangle}$ is satisfied. Contradiction.

Thus there must exists an x and a stage $t > s$ such that $x \in \omega^{[\langle d,e,z \rangle]}$, $c(s,t) = 0$, and $\Phi_{e,t}^{E^n_{z,t}}(x) = 0$. Furthermore, since the assumption is $C = \Phi_d^{E^n_z}$ and $E^n_z = \Phi_e^C$, there must exists a stage $s > s'$ such that

$$C_s \upharpoonright\upharpoonright x = \Phi_{d,s+1}^{E^n_{z,s+1}} \upharpoonright\upharpoonright x$$

$$E^n_{z,s+1} \upharpoonright\upharpoonright \varphi_{d,s+1}^{E^n_{z,s+1}}(x) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright \varphi_{d,s+1}^{E^n_{z,s+1}}(x)$$

holds. This x becomes the witness for $\langle d,e,z \rangle$ at this stage. x is placed into C_{s+1} and the various functions are defined as specified by the construction.

By enumerating x into C_{s+1}, C_{s+1} differs from $\Phi_{d,s+1}^{E^n_{z,s+1}}$. Now there are two cases.

(1) $\Phi_{e,t}^{C_t} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, t) = \Phi_{e,t}^{C_s} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, s+1)$. Let $t \geq s+1$ be the least stage such that $\Phi_{e,t}^{C_t}$ is defined on all $y \leq l(\langle d,e,z \rangle, t) = l(\langle d,e,z \rangle, s+1)$ (since this value can only change if a requirement R_ξ acts for $\xi < \langle d,e,z \rangle$). Furthermore, by the construction, $C_s = C_t$ since C can not change unless all the sets and Turing functions become equal once again on the specified portion. Since $l(\langle d,e,z \rangle, s+1)$ is defined to be $\varphi_{d,s+1}^{E^n_{z,s+1}}$ and $r(\langle d,e,z \rangle, t)$ is at least as great as the maximum of the use on $\Phi_e^{C_s}$, one has that C_t will not change up to $r(\langle d,e,z \rangle, s+1)$, hence $\Phi_{e,t}^{C_t}$ will up to $l(\langle d,e,z \rangle, s+1)$. Thus if $\Phi_{e,t}^{C_t} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, t) = \Phi_{e,t}^{C_s} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, s+1)$, then for any stage such that $E^n_{z,t} \upharpoonright\upharpoonright l(\langle d,e,t) = \Phi_{e,t}^{C_t} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, t)$, one must have since $\Phi_{d,t}^{E^n_{z,t}}(x) = \Phi_{d,s}^{E^n_{z,t}}(x) = 0$ since $E^n_{z,t}$ is the same up to the $l(\langle d,e,z \rangle, t)$ as it was as stage s. However, $C_t(x) = C_s(x) = 1$. Hence, it is impossible for $E_z = \Phi_e^C$ and $C = \Phi_d^{E^n_z}$.

(2) $\Phi_{e,t}^{C_t} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, t) \neq \Phi_{e,t}^{C_s} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, s+1)$. Hence, for all stages $u \geq t$ such that

$$C_u \upharpoonright\upharpoonright w(\langle d,e,z \rangle, u) = \Phi_{d,u+1}^{E^n_{z,u+1}} \upharpoonright\upharpoonright w(\langle d,e,z \rangle, u)$$

$$E^n_{z,u+1} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, u) = \Phi_{e,u+1}^{C_u} \upharpoonright\upharpoonright l(\langle d,e,z \rangle, u)$$

one must have that some $\lambda \leq l(\langle d,e,z \rangle)$ such that $E^n_{z,s}(x) \neq E^n_{z,s}(x)$ so that it matches with $\Phi_{e,u}^{C_u} = \Phi_{e,t}^{C_t}$. This shows first time the sets and functional matches up to the desired part, the some λ changes once. By the construction, the witness $w(\langle d,e,z \rangle, u)$ is then extracted from C_u. Since all $\xi < \langle d,e,z \rangle$ has been satisfied, $r(\langle d,e,z \rangle, u)$ keeps C from changes under the restraint except for the witness. From the definition of the restraint, the $\Phi_{e,u}^{C_u}$ matches up with $E_t \neq E_u$ since they differ at least at λ. Similarly, for all later stages that $E_{z,u'}$ matches up with $\Phi_{e,u'}^{C_{u'}}$ (which has not change since the action of the last time the sets and function matched), at least the element λ must change again. This proves that every time the sets and functions are equal up to the desired portion, some element of $E^n_{z,t}$, say λ, must change.

As specified in the construction, every time the sets and function matches up, C_s changes on the witness element $w(\langle d,e,z,s)$. However, every match causes λ to change. Since E_z is n-c.e., λ can change at most n times. Thus, the sets and functions can match up at most n times (not counting the first match). Thus C_s needs to change the witness at most $n+1$ times to obtain a permanent difference. Thus $R_{\langle d,e,z \rangle}$ is satisfied. Contradiction.

Thus R_ξ is met for all ξ. P is satisfied since by the above, $\lim_s c(x,s) = n+1$ for all x. □

4. Properly ω-c.e. Degrees

Definition 4.1 A ω-c.e. degree is a degree which contains a ω-c.e. set. A properly ω-c.e. degree is a ω-c.e.

degree which is not a n-c.e. degree for any n.

Theorem 4.2 There exists a properly ω-c.e. degree.

Proof : The construction will produce a set C which satisfies the following requirements:

$$R_{\langle d,e,n,z\rangle} : C \neq \Phi_d^{E_z^n} \wedge E_z^n \neq \Phi_e^C$$

$$P : C \text{ is } \omega\text{-c.e.}$$

The construction shall use the functions $r(\xi, s)$, $l(\xi, s)$, $w(\xi, s)$ and $c(x, s)$ as in Theorem 3.2.

At stage 0, for all ξ, define $r(\xi, 0) = l(\xi, 0) = w(\xi, 0) = -1$ and $c(\xi, 0) = 0$.

At stage s + 1, if it exists, find the least $\langle d, e, n, z\rangle < s + 1$ such that

$$C_s \restriction\restriction w(\langle d, e, n, z\rangle, s) = \Phi_{d,s+1}^{E_{z,s+1}^n} \restriction\restriction w(\langle d, e, n, z\rangle, s)$$

$$E_{z,s+1}^n \restriction\restriction l(\langle d, e, n, z\rangle, s) = \Phi_{e,s+1}^{C_s} \restriction\restriction l(\langle d, e, n, z\rangle, s)$$

Define $C_{s+1}(w(\langle d, e, n, z\rangle, s)) = 1 - C_s(w(\langle d, e, n, z\rangle, s))$, and for all $x \neq w(\langle d, e, n, z\rangle, s)$, $C_{s+1}(x) = C_s(x)$. Define $c(w(\langle d, e, n, z\rangle, s)) = c(w(\langle d, e, n, z\rangle, s)) + 1$, and for all $y \neq w(\langle d, e, n, z\rangle, s)$, $c(y, s + 1) = c(y, s)$. For all $\xi \leq \langle d, e, n, z\rangle$, define $r(\xi, s + 1) = r(\xi, s)$, $l(\xi, s + 1) = l(\xi, s)$, and $w(\xi, s + 1) = w(\xi, s)$. For all $\xi > \langle d, e, z\rangle$, define $r(\xi, s + 1) = l(\xi, s + 1) = w(\xi, s + 1) = -1$. Go to stage $s + 2$.

If no such $\langle d, e, n, z\rangle < s + 1$ exists, then find the least $\langle d, e, n, z\rangle$ such that there exists $x \leq s + 1$ such that $x \in \omega^{[\langle d,e,n,z\rangle]}$, $x > \max\{r(\xi, s) : \xi < \langle d, e, n, z\rangle\}$, $c(x, s) = 0$ and

$$C_s \restriction\restriction x = \Phi_{d,s+1}^{E_{z,s+1}^n} \restriction\restriction x$$

$$E_{z,s+1}^n \restriction\restriction \varphi_{d,s+1}^{E_{z,s+1}^n}(x) = \Phi_{e,s+1}^{C_s} \restriction\restriction \varphi_{d,s+1}^{E_{z,s+1}^n}(x)$$

Define $C_{s+1}(x) = 1$ and for all $y \neq x$, define $C_{s+1}(y) = C_s(y)$. Define $c(x, s+1) = 1$, $w(\langle d, e, n, z\rangle, s+1) = x$, $l(\langle d, e, n, z\rangle, s + 1) = \varphi_{d,s+1}^{E_{z,s+1}^n}(x)$, $r(\langle d, e, n, z\rangle, s + 1) = \max\{x, \max\{\varphi_{e,s+1}^{C_s}(y) : y \leq l(\langle d, e, n, z\rangle, s + 1)\}\}$. For all $y \neq x$, $c(y, s + 1) = c(y, s)$. For all $\xi < \langle d, e, n, z\rangle$, $w(\xi, s + 1) = w(\xi, s)$, $l(\xi, s + 1) = l(\xi, s)$, and $r(\xi, s + 1) = r(\xi, x)$. For all $\xi > \langle d, e, n, z\rangle$, $w(\xi, s + 1) = l(\xi, s + 1) = r(\xi, s + 1) = -1$.

If no such $\langle d, e, n, z\rangle$ exists, then go to stage $s + 2$. This ends the construction.

The verification that $R_{\langle d,e,n,z\rangle}$ is satisfied is similar to Theorem 3.2. P is satisfied since $\lim_s c(x, s) \leq \pi_4^5(x) + 1$. This because $x = \langle y, d, e, n, z\rangle$ for some d, e, n, z. Hence $x \in \omega^{[\langle d,e,n,z\rangle]}$. Thus x can only be a witness to $R_{\langle d,e,n,z\rangle}$. Therefore, $c(x, s) = c(\langle y, d, e, n, z\rangle) =\leq n + 1 = \pi_4^5(\langle y, d, e, n, z\rangle) + 1 = \pi_4^5(x) + 1$. □

5. Properties of n-c.e. Sets

Definition 5.1 A set A is immune if and only if A is infinite and does not contain any c.e. subset. That is, A is immune if and only if A is infinite and for all e, $W_e^{\emptyset} \not\subseteq A$.

A set A is simple if and only if $\bar{A}$ is immune.

Proposition 5.2 An immune set A is not c.e. A simple set is not computable.

Proof : Let A be immune. If A is c.e., then $A \subseteq A$. So A would not be immune.

If A is simple, then $\bar{A}$ is immune and by the above $\bar{A}$ is not c.e. However, a set A is computable if and if A and $\bar{A}$ are c.e. (by Proposition 1.8) $\square$

The following theorem is the Standard Permitting Theorem.

Theorem 5.3 Let $f(x)$ be a computable function. Let A and B be c.e. sets with computable approximation $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$, respectively. If $A_{s+1} \upharpoonright\upharpoonright x \neq A_s \upharpoonright\upharpoonright x \Rightarrow (\exists y \leq f(x))(B_{s+1}(y) \neq B_s(y))$, then $A \leq_T B$.

Proof : For each x, using a B oracle, find the least stage s_x such that $B_s \upharpoonright\upharpoonright f(x) = B \upharpoonright\upharpoonright f(x)$. Define $\Theta(x) = A_{s_x}(x)$. Then $A = \Theta$. Thus $A \leq_T B$. $\square$

The next theorem shall be proved using the permitting method and the method of the Sack's Avoiding Cone theorem.

Theorem 5.4 If $\mathbf{b} > \mathbf{0}$ is a c.e. degree, then there exists a c.e. degree $\mathbf{a} > \mathbf{0}$ such $\mathbf{0} < \mathbf{a} < \mathbf{b}$. If $\mathbf{b}$ is a properly 1-c.e. degree, then there exists a properly 1-c.e. degree $\mathbf{a}$ such that $\mathbf{a} < \mathbf{b}$.

Proof : Let $B \in \mathbf{b}$ and $\{B_s\}_{s\in\omega}$ be a computable enumeration of B. This theorem will be proved by constructing a c.e. set A which satisfies the following requirements.

$$P : A \leq_T B$$

$$R_e : B \neq \Phi_e^A$$

$$N_e : |W_e| = \infty \Rightarrow W_e \not\subseteq \bar{A}$$

The construction shall use the following functions:

$$l(e,s) = \max\{x : (\forall y < x)(\Phi_{e,s}^{A_s}(y) \downarrow = B_s(y))\}$$

$$r(e,s) = \max\{\varphi_{e,s}^{A_s}(y) : y \leq l(e,s)\}$$

At stage 0, let $A_0 = \emptyset$.

At stage $s+1$, for all $e \leq s$, define $l(e,s)$ and $r(e,s)$. Then find the least $e \leq s$ such that $W_{e,s} \cap A_s = \emptyset$ and there exists $x \in W_{e,s}$ such that $x > 2e$, $x > \max\{r(i,s) : i \leq e\}$, and there exists $y \leq x$ such that $B_{s-1}(y) \neq B_s(y)$. Enumerate this x into A_{s+1}.

Let $A = \lim_s A_s$.

Say that R_e is injured if $x \leq r(e,s)$ and $x \in A_{s+1}$. Note that R_e, for all e, R_e is injured at most e times, in order to satisfy requirement R_i for $i < e$.

Next, the claim is that R_e is satisfied for all e. Let e be the least such that R_e is not satisfied. That is, $B = \Phi_e^A$. Let s be a stage in which R_e is never injured afterward. Because $B = \Phi_e^A$, $\lim_s l(e,s) = \infty$. To determine whether $x \in B$ or $x \notin B$, find a stage $t \geq s$ such that $l(e,s) > x$. Because R_e is never injured after s, $\Phi_e^A(x) = \Phi_{e,t}^{A_t}(x)$. Thus $B(x) = B_t(x) = \Phi_{e,t}^{A_t}(x)$. Thus B is computable. Contradiction. Furthermore, $\lim_s r(e,s) < \infty$.

Next, the claim is that each N_e is satisfied. Suppose not. Let e, be the smallest such that N_e is not satisfied. By the above, suppose that $r = \lim_s r(e,s)$ and t is a stage such that for all $s > t$, $r(e,s) = r$. Since N_e fails, one has that $|W_e| = \infty$ and $W_e \subseteq \bar{A}$. Let $x \in \omega$, since W_e is infinite, there exists a $u > t$ and $y > x$ such that $y \in W_{e,u}$. Now $B(x) = B_u(x)$. If not, then there is a stage $v > u$ such that $B_v(x) \neq B_{v-1}(x)$. Then y would be enumerated into A_{v+1}, and N_e would have been satisfied. Thus B is computable. Contradiction.

$\bar{A}$ is immune. By construction, elements used to satisfy N_e must be greater than $2e$. Furthermore, elements are enumerated into A only to satisfy N_e. Thus, for any e, $|A \restriction 2e| \leq e$. Thus $\bar{A}$ is infinite. Since each N_e is satisfied, $\bar{A}$ is immune. A is simple.

$A \leq_T B$ by Theorem 5.3 (Standard Permitting Theorem) via the identity function.

Let $\mathbf{a}$ be the degree of A. $\mathbf{a} > \mathbf{0}$ by Proposition 5.2. $\mathbf{a} \leq \mathbf{b}$ by the above paragraph. $\mathbf{b} \not\leq \mathbf{a}$ by requirement R_e. Thus $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

The second assertion follows from the fact that $\mathbf{a}$ a c.e. degree is properly 1-c.e. if and only if $\mathbf{a} > 0$. □

Theorem 5.5 For $n \geq 2$, if $\mathbf{b}$ is a properly n-c.e. degree, then there exists a c.e. degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

<u>Proof</u> : Let $B \in \mathbf{b}$ be a n-c.e. set. By Theorem 1.12, $B = (C_1 - D_1) \cup ... \cup (C_k - D_k)$, if $n = 2k$, or $B = (C_1 - D_1) \cup ... \cup (C_k - D_k) \cup C_{k+1}$, if $n = 2k + 1$, where the C_i and D_i are c.e. In the former case, let $\mathbf{a}$ be the degree of D_k; in the latter, let $\mathbf{a}$ be the degree of C_{k+1}. As in Theorem 1.12, one has that $C_1 \supset D_1 \supset ... \supset C_k \supset D_k$ (similarly the $2k+1$ case). In the $n = 2k$ case, suppose D_k is computable. Then define $C'_k = \{x : x \in C_k \wedge x \notin D_k\}$. C'_k is c.e.; thus, $B = (C_1 - D_1) \cup ... \cup C'_k$. Hence B is $n-1$-c.e. However $\mathbf{b}$ is a properly n-c.e. degree. Contradiction. Similar argument for the $n = 2k + 1$ case. D_k (or C_{k+1} in other case) is computable from B and an computable approximation of B (refer to proof of Theorem 1.12). $D_k <_T B$ since $\mathbf{b}$ is properly n-c.e. for $n > 1$. □

Definition 5.6 A degree $\mathbf{b}$ is minimal if and only if $\mathbf{b} > \mathbf{0}$ and there is no degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

Corollary 5.7 For all n, no n-c.e. degree is minimal.

<u>Proof</u> : Let $\mathbf{a}$ be a properly n-c.e. degree. If $\mathbf{a}$ contains a 0-c.e. degree (i.e. contains $\emptyset$), then $\mathbf{a} = \mathbf{0}$. $\mathbf{a}$ is not minimal. If $\mathbf{a}$ is properly 1-c.e., then by Theorem 5.4, it is not minimal. If $\mathbf{a}$ is neither computable or properly 1-c.e., then it is properly n-c.e. for some n. By Theorem 5.5, $\mathbf{a}$ is not minimal. □

Theorem 5.8 Let $\mathbf{b}$ be a properly 1-c.e. degree, then for all $n \geq 1$, there exists a properly $n+1$-c.e. degree $\mathbf{a}$ such that $\mathbf{a} < \mathbf{b}$.

<u>Proof</u> : Let $B \in \mathbf{b}$ and $\{B_s\}_{s \in \omega}$ a 1-c.e. computable approximation. The construction will produce a set A that satisfies the following requirements:

$$R_{\langle d,e,z \rangle} : A \neq \Phi_d^{E_z^n} \vee E_z^n \neq \Phi_e^A$$

$$P : A \text{ is } n+1\text{-c.e.}$$

$$Q : A \leq_T B$$

The construction will use the following functions which shall be defined in the construction. However, for each ξ, there may be several witness. The witness function takes the form, $w(\xi, s) = \langle n, \langle w_1^\xi, ..., w_n^\xi \rangle \rangle$, where n indicate the current number of witnesses and $w_1^\xi < ... < w_n^\xi$ are the n witnesses. $r(x, s)$ is the restrain on C. $l(x, s)$ is the desired agreement on E_z^n. $c(x, s)$ is the change counting function.

At stage 0, for all ξ, define $r(\xi, 0) = l(\xi, 0) = w(\xi, 0) = -1$ and $c(\xi, 0) = 0$.

At stage $s+1$, find the least $\langle d, e, z \rangle < s+1$ where $w(\langle d, e, z \rangle, s) = \langle n, \langle w_1^{\langle d,e,z \rangle}, w_2^{\langle d,e,z \rangle}, ..., w_n^{\langle d,e,z \rangle} \rangle \rangle$ such that there exists a witness $w_i^{\langle d,e,z \rangle}$ $(1 \leq i \leq n)$ such that there exists $y \leq w_i^{\langle d,e,z \rangle}$ such that $B_s(y) \neq B_{s+1}(y)$

and

$$C_s \upharpoonright\upharpoonright w_i^{\langle d,e,z\rangle} = \Phi_{d,s+1}^{E^n_{z,s+1}} \upharpoonright\upharpoonright w_i^{\langle d,e,z\rangle}$$

$$E^n_{z,s+1} \upharpoonright\upharpoonright l(w_i^{\langle d,e,z\rangle}, s) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright l(w_i^{\langle d,e,z\rangle}, s)$$

Define $C_{s+1}(w_i^{\langle d,e,z\rangle}) = 1 - C_s(w_i^{\langle d,e,z\rangle})$, and for all $x \neq w_i^{\langle d,e,z\rangle}$, $C_{s+1}(x) = C_s(x)$. Define $c(w_i^{\langle d,e,z\rangle}, s+1) = c(w_i, s) + 1$. For all $y \neq w_i$, define $c(y, s+1) = c(y, s)$. Define $w(\langle d, e, z\rangle, s+1) = \langle i, \langle w_1^{\langle d,e,z\rangle}, ..., w_i^{\langle d,e,z\rangle}\rangle\rangle$. For $\xi > \langle d, e, z\rangle$, $w(\xi, s+1) = -1$. For $\xi < \langle d, e, z\rangle$, $w(\xi, s+1) = w(\xi, s)$. For all x, $r(x, s+1) = r(x, s)$ and $l(x, s+1) = l(x, s)$. Go to stage $s + 2$.

If no such $\langle d, e, z\rangle$ exists, then find the least $\langle d, e, z\rangle$ such that there exists $x \leq s+1$ such that $x \in \omega^{[\langle d,e,z\rangle]}$, $x > \max\{r(w_i^{\xi}) : \xi \leq \langle d, e, z\rangle \wedge w(\xi, s) = \langle n\langle w_1^{\xi}, ..., w_i^{\xi}, ..., w_n^{\xi}\rangle\rangle\}$, $c(x, s) = 0$, and there exists $y \leq x$ such that $B_s(y) \neq B_{s+1}(y)$, and

$$C_s \upharpoonright\upharpoonright x = \Phi_{d,s+1}^{E^n_{z,s+1}} \upharpoonright\upharpoonright x$$

$$E^n_{z,s+1} \upharpoonright\upharpoonright \varphi_{d,s+1}^{E^n_{z,s+1}}(x) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright \varphi_{d,s+1}^{E^n_{z,s+1}}(x)$$

Define $C_{s+1}(x) = 1$ and for all $y \neq x$, define $C_{s+1}(y) = C_s(y)$. Define $c(x, s+1) = 1$ and for all $y \neq x$, define $c(y, s+1) = c(y, s+1)$. If $w(\langle d, e, z\rangle, s) = \langle n, \langle w_1^{\langle d,e,z\rangle}, ..., w_n^{\langle d,e,z\rangle}\rangle\rangle$, then define $w(\langle d, e, z\rangle, s+1) = \langle n+1, \langle w_1^{\langle d,e,z\rangle}, ..., w_n^{\langle d,e,z\rangle}, x\rangle\rangle$. Define $l(x, s+1) = \varphi_{d,s+1}^{E^n_{d,s+1}}(x)$. Define $r(x, s+1) = \max\{x, \max\{\varphi_{e,s+1}^{C_s}(y) : y \leq l(x, s+1)\}\}$. For all $\xi < \langle d, e, z\rangle$, let $w_(\xi, s+1) = w(\xi, s)$. For all $\xi > \langle d, e, z\rangle$, let $w(\xi, s+1) = -1$. For all $y \neq x$, let $r(x, s+1) = r(x, s)$ and $l(x, s+1) = l(x, s)$. This ends the construction.

The proof that $R_{\langle d,e,z\rangle}$ is satisfied is the similar to that of Theorem 3.2. The difference is the proof that $\{B_s\}_{s\in\omega}$ permits the removal or addition of a witness as specified in the construction. Suppose $\langle d, e, z\rangle$ is the least such that $R_{\langle d,e,z\rangle}$ is not satisfied. Let s be a stage by which R_ξ has been satisfied for all $\xi < \langle d, e, z\rangle$. By a similar argument to Theorem 3.2, one can show that $n+1$ changes is sufficient to satisfy $R_{\langle d,e,z\rangle}$. Thus, it must be proved that $\{B_s\}_{s\in\omega}$ permits these changes.

If $R_{\langle d,e,z\rangle}$ is not satisfied, one claims that there are infinitely many witnesses such that $\lim_s c(x, s) \geq 1$. Suppose there are only finitely many such that $\lim_s c(x, s) \geq 1$. Let x' be the greatest witness with $\lim_s c(x', s) \geq 1$ To compute whether $y \in B$, let $x = \max\{x' + 1, y\}$ and let s' be a stage such that $c(k, u) = \lim_s c(k, s)$ for all $k \leq x'$ and $u > s'$, and find a stage $t > s$ and $t > s'$ such that

$$C_t \upharpoonright\upharpoonright x = \Phi_{d,t}^{E^n_{z,t}} \upharpoonright\upharpoonright x$$

$$E^n_{z,t} \upharpoonright\upharpoonright \varphi_{d,t}^{E^n_{z,t}}(x) = \Phi_{e,t}^{C_s} \upharpoonright\upharpoonright \varphi_{d,t}^{E^n_{z,t}}(x)$$

This t can be found since it is assumed that $R_{\langle d,e,z\rangle}$ is not satisfied. Then $y \in B$ if and only if $y \in B_t$. Since if $y \in B_u$ for $u > t$, then at this stage some existing witness would make a further change or a new witness greater than x' would appear. Contradicting $t > s'$. However, this proves that B is computable. Contradiction.

Now the claim is for all $k \leq n+1$, if $R_{\langle d,e,z\rangle}$ is not satisfied, then there exists infinitely many witnesses x such that $\lim_s c(x, s) \geq k$. Suppose there were only finitely many. Let x' be largest such that $\lim_s c(x, s) \geq k$. To compute whether $y \in B$, let $x = \max\{x' + 1, y\}$, let s' be a stage such that $c(k, u) = \lim_s c(k, u)$ for all $k \leq x'$ and $u > s'$, and find a stage $t > s$ and $t > s'$ such that

$$C_t \upharpoonright\upharpoonright x = \Phi_{d,t}^{E^n_{z,t}} \upharpoonright\upharpoonright x$$

$$E^n_{z,t} \upharpoonright\upharpoonright \varphi_{d,t}^{E^n_{z,t}}(x) = \Phi_{e,t}^{C_s} \upharpoonright\upharpoonright \varphi_{d,t}^{E^n_{z,t}}(x)$$

Then $y \in B$ and only if $y \in B_t$. This is because if at a later stage $u > t$, y enters B_u, then either a witness will change or a new witness will appear which contradicts $t > s'$. Thus B is computable. Contradiction.

It has been proved that if $R_{\langle d,e,z\rangle}$ is not satisfied, then there are infinitely many witnesses that $\lim_s c(x, s) \geq$

$n+1$. However, like in Theorem 3.2, $n+1$ changes is enough to satisfy $R_{\langle d,e,z\rangle}$. Thus $R_{d,e,z}$ is satisfied. Contradiction.

P is satisfied because $\lim_s c(x,s) = n+1$ for all x. This is proved in the same manner as in Theorem 3.2. Q is satisfied by a permitting argument like Theorem 5.3, with the appropriate changes for $n+1$-c.e. sets. □

Corollary 5.9 For all $n \geq 1$, if $\mathbf{b}$ is a properly n-c.e. degree, then there exists a properly n-c.e. degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{b}$.

Proof : This is true for $n = 1$ by Theorem 5.4. For $n > 1$, by Theorem 5.5, there exists a properly 1-c.e. degree $\mathbf{b}'$ such that $\mathbf{0} < \mathbf{b}' < \mathbf{b}$. By Theorem 5.8, there is a properly n-c.e. degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{b}' < \mathbf{b}$. □

6. Hyperimmune Sets and Degrees

Definition 6.1 Let R be a relations. $(\forall^\infty x)R(x)$ if and only if $(\exists n)(\forall x)(x > n \Rightarrow R(x))$. $(\exists^\infty x)R(x)$ if and only if $(\forall n)(\exists x)(x > n \wedge R(x))$.

Definition 6.2 Let g and f be functions. g dominates f if and only if $(\forall^\infty x)(f(x) \leq g(x))$. f escapes g if and only if $(\exists^\infty x)(f(x) > g(x))$.

A function g majorizes f if and only if $(\forall x)(f(x) \leq g(x)$.

A function g computably dominates f if and only if g is dominates g and g is computable (analogously for computably majorize).

A function f is computably dominated (or computably majorized) if there is a computable function g such that g dominates (or majorizes) f.

Proposition 6.3 A function f is computable majorized if and only if f is computably dominated.

Proof : If is f is computably majorized by a computable function g, then f is computable dominated by g. Suppose f is computable dominated by a computable function g. That is there exists a n such that for all $x > n$, $g(x) \geq f(x)$. Let $m = \max\{f(x) : x \leq n\}$. Define a new computable function h as follows:

$$h(x) = \begin{cases} g(x) & \text{if } x > n \\ m & \text{if } x \leq n \end{cases}$$

h is computable and majorizes f. □

Definition 6.4 Let A be a set. Let $A = \{a_1, a_2, ...\}$ be an enumeration of A such that $a_1 < a_2 < ...$ The principal function of A is defined as $p_A(x) = a_x$.

Definition 6.5 An infinite set A is hyperimmune (h-immune) if and only if A is infinite and p_A is not computably dominated. A set A is hypersimple (h-simple) if and only if $\bar{A}$ is hyperimmune.

Definition 6.6 A strong array is a sequence of finite sets $\{D_{f(n)}\}_{n\in\omega}$ where f is a computable function. A strong array is disjoint if and only if for all m, n such that $n \neq m$, $D_{f(m)} \cap D_{f(n)} = \emptyset$.

Theorem 6.7 *A is hyperimmune if and only if A is infinite and for all disjoint strong arrow $\{D_{f(n)}\}_{n\in\omega}$ there exists n such that $D_{f(n)} \cap A = \emptyset$.*

Proof : Suppose there exists a disjoint strong arrow $\{D_{f(n)}\}_{n\in\omega}$ such that for all n, $D_{f(n)} \cap A \neq \emptyset$. Define $g(n) = \max\{D_{f(n)}\}$. Since the strong arrow is disjoint, $g(n)$ is injective. Define $h(n) = \max\{g(k) : k \leq n\}$. Then $h(x)$ is computable and dominates p_A. A is not hyperimmune; contradiction.

Suppose A is not hyperimmune. If A is not infinite, then A does not satisfy the condition on the right. Suppose p_A is computably dominated. Using Proposition 6.3, suppose p_A is computably majorized by a computable function g. Define a function h as follows. Let $h(0) = g(0)$. For $n > 0$, let $h(n) = g(h(n) + 1)$. Define $l(n)$ to be the strong index for the set $[h(n) + 1, h(n + 1)]$. $\{D_{l(n)}\}_{n\in\omega}$ is the disjoint strong array such that for all n, $D_{l(n)} \cap A \neq \emptyset$. By definition it is a disjoint strong array. For all n, $D_{l(n)} \cap A \neq \emptyset$. Let $A = \{a_0 < a_1 < ....\}$. For 0, $D_{l(n)} = [0, g(0)]$. Since $g(0) \geq p_A(0)$, $a_0 \in [0, g(0)]$. For $n + 1$, $| \cup_{k<n+1} D_{l(k)}| = |[0, h(n)]| = h(n) + 1$. By definition, $D_{l(n+1)} = [h(n) + 1, h(n + 1)]$. Since $h(n + 1) = g(h(n) + 1) > p_A(h(n) + 1)$, $a_0, ..., a_{h(n)+1} \in [0, g(h(n) + 1)]$. There are $h(n) + 2$ many elements of A in $[0, h(n + 1)]$ and $|[0, h(n)]| = h(n) + 1$. Thus $a_{h(n)+1} \in D_{l(n+1)} = [h(n) + 1, h(n + 1)]$. $\{D_{l(n)}\}_{n\in\omega}$ is a disjoint strong array such that for all n, $D_{l(n)} \cap A \neq \emptyset$. Contradiction. □

Definition 6.8 A degree **a** is hyperimmune if and only if it contains a hyperimmune set.

A degree **a** is hyperimmune-free if and only if it contains no hyperimmune sets.

Proposition 6.9 *If* **b** *is a hyperimmune-free degree, then all* **a** $\leq$ **b** *are hyperimmune-free.*

Proof : Let $A \in \mathbf{A}$ and $B \in \mathbf{b}$. Define a function g as follows: $g(0) = 0$. For all n, $g(2n+1) = g(2x)+p_B(x)+1$ and $g(2n + 2) = g(2x + 1) + p_A(x) + 1$. g is a strictly increasing function so it is the principal function of $C = g(\omega)$. $g \leq_T A$ because g is defined with $p_A \leq_T A$ and $p_B \equiv_T B \leq_T A$. $A \leq_T g$ because $p_A(x) = g(2x + 2) - g(2x + 1) - 1$. Thus $g \equiv_T A$. $C \equiv_T p_C = g \equiv_T A$. Therefore $C \equiv_T A$ and $C \in \mathbf{a}$. Since **a** is hyperimmune-free, there exists a computable function h which computably dominates $p_C = g$. Thus $h'(x) = h(2x + 1)$ computably dominates p_B. B is not hyperimmune. Since $B \in \mathbf{b}$ chosen arbitrarily, **b** is hyperimmune-free. □

Corollary 6.10 *If* **a** *is hyperimmune, then for all* **b** $\geq$ **a** *is hyperimmune.*

Proof : Suppose **a** is hyperimmune. Suppose **b** is hyperimmune-free. By Proposition 6.9, **a** is hyperimmune-free. Contradiction. □

Proposition 6.11 *There exists $g \leq_T A$, g is not computably dominated if and only if* deg(A) *is hyperimmune.*

Proof : Suppose $g \leq_T A$ is not computably dominated. Define a new function h as follows: $h(0) = g(0)$. $h(x + 1) = h(x) + g(x + 1) + 1$. $h \leq_T g \leq_T A$ and h is not computably dominated. h is strictly increasing so it is the principal function of $E = h(\omega)$. Since $p_E = h$, E is hyperimmune. By Corollary 6.10, $\deg(A)$ is hyperimmune.

Suppose $E \in \deg(A)$ and E is hyperimmune. $p_E \leq_T E \equiv_T A$ and p_E is not computably dominated. $\square$

Definition 6.12 Let $A \in \Delta^0_2$ with computable approximation $\{A_s\}_{s\in\omega}$. Define c_A as follows: Let $c_A(0) = (\mu s)(A_s \upharpoonright\upharpoonright 0 = A \upharpoonright\upharpoonright 0)$. Define $c_A(x+1) = (\mu s > c_A(x))(A_s \upharpoonright\upharpoonright x = A \upharpoonright\upharpoonright x)$.

Proposition 6.13 $c_A \equiv_T A$

Proof : To compute $c_A(0)$, find the least s such that $(A_s \upharpoonright\upharpoonright 0 = A \upharpoonright\upharpoonright 0$. Given $c_A(x)$, to compute $c_A(x+1)$ find the first $s > c_A(x)$ such that $A_s \upharpoonright\upharpoonright x = A \upharpoonright\upharpoonright x$. $c_A \leq_T A$.

$\chi_A(x) = A_{c_A(x)}(x)$. Thus $A \leq_T c_A$. $\square$

Theorem 6.14 If $A \in \Delta^0_2$, $\{A_s\}_{s\in\omega}$ a computable approximation of A, and c_A is computably dominated, then A is computable.

Proof : First, $c_A(x) > x$ since c_A is strictly increasing. Suppose c_A is computable dominated by a computable function f. Since $x < c_A(x) \leq f(x)$. Thus for all x, there is a stage t between x and $f(x)$, such that $A_t \upharpoonright\upharpoonright x = A \upharpoonright\upharpoonright x$. Thus, for all $y > x$, there exists a stage t such that $y < t \leq f(y)$ such that $A_t \upharpoonright\upharpoonright x = A \upharpoonright\upharpoonright x$.

To compute if $x \in A$, find a stage $z > x$ such that for all t, $z < t \leq u \leq f(z)$, $A_t \upharpoonright\upharpoonright x = A_u \upharpoonright\upharpoonright x$. Such a z exists. Let n_y be a stage such that $(\forall n > n_y)(A_n(y) = A(y))$. n_y exists since $\lim_s A_s(y) = A(y)$. Let $z' = \max\{x, n_0, n_1, ..., n_x\}$. z' is such a stage.

Then $A(x) = A_{f(z)}$. This is because, as stated above, for all $z > x$ there exists a stage t such that $A_t(x) = A(x)$. However since for all t such that $z < t \leq f(z)$, one has that $A_t(x) = A_u(x)$, one must have that $A(x) = A_{f(x)}$. A is computable. $\square$

Theorem 6.15 If $\mathbf{0} < \mathbf{a} \leq \mathbf{0}'$, then $\mathbf{a}$ is hyperimmune.

Proof : Let $A \in \mathbf{a}$. By Theorem 6.14, one has that c_A is not computable bounded. Since c_A is strictly increasing, it is the principal function of the set $B = c_A(\omega)$. $B \equiv_T c_A \equiv_T A$ by Proposition 6.13. Thus B is hyperimmune and $B \in \mathbf{a}$. $\square$

Theorem 6.16 If $\mathbf{0} < \mathbf{a} \leq \mathbf{0}'$ or $\mathbf{a} \geq \mathbf{0}'$, then $\mathbf{a}$ is hyperimmune.

Proof : By Theorem 6.15, every $\mathbf{a} \leq \mathbf{0}'$ is hyperimmune. By Corollary 6.10, all $\mathbf{a} \geq \mathbf{0}'$ are hyperimmune. $\square$

Theorem 6.17 If $\emptyset' \leq_T A$, then there exists B such that $B \equiv_T A$ and $B \not\equiv_{tt} A$.

Proof : At stage 0, let $\sigma_0 = A(0)$. At stage $s = 2e$ for $e > 0$, let $\sigma_s = \sigma_{s-1}A(e)$. At stage $s = 2e+1$ for $e \geq 0$, if $\Phi^\emptyset_e(s)\uparrow$ or $\Phi^\emptyset_e(s)\downarrow$ and

$$\bigvee_{\sigma \in D_{\Phi^\emptyset_e(s)}} \sigma \prec A$$

then let $\sigma_s = \sigma_{s-1}0$. If

$$\bigwedge_{\sigma \in D_{\Phi^\emptyset_e(s)}} \sigma \not\prec A$$

then define $\sigma_s = \sigma_{s-1}1$. Define $B = \cup_s \sigma_s$.

$A \leq_T B$ since $A(e) = B(2e)$. $B \leq A$ since the even steps require A and the odds step require $\emptyset' \leq A$. $B \not\leq_T A$ since $s = 2e+1$ is witness that $\Phi_e^{\emptyset}$ is not the computable function which produce the truth table as in Theorem 1.20. □

The above theorem shows that there are sets whose Turing and Truth-table degrees are not the same. The following two results gives a more general relationship between the two types of degrees.

Theorem 6.18 Let B be a set. $\deg(B)$ is hyperimmune-free if and only if for all set A, $A \leq_T B \Rightarrow A \leq_{tt} B$.

Proof : Suppose $A = \Phi_e^B$. Define $f(x) = (\mu s)(\Phi_{e,s}^B(x)\downarrow)$. $f \leq_T B$ and total since Φ_e^B is total. Thus by Proposition 6.9, f is computably dominated. Let g be the computable function that computably majorizes f. Define a Turing function Θ as follows:

$$\Theta^X(x) = \begin{cases} \Phi^X_{e,g(x)}(x) & \text{if } \Phi^X_{e,g(x)}(x)\downarrow \\ 0 & \text{if otherwise} \end{cases}$$

Θ is a total functional. $A \leq_{tt} B$ via Θ^B.

Conversely, suppose that for all $A \leq_T B \Rightarrow A \leq_{tt} B$. $p_A \equiv_T A \leq_T B$, so $p_A \leq_{tt} B$. Let $p_A = \Theta^B$ where Θ is a total Turing functional. Define a function h as follows. Given x, find n such that forall $\sigma \in 2^{<\omega}$ such that $|\sigma| = n$, $\Theta^\sigma(x)\downarrow$. Such an n exists, because Θ is a total functional. Let $h(x) = \max\{\Theta^\sigma(x) : |\sigma| = n\}$. Then h computably dominates p_A. No $A \leq_T B$ is hyperimmune. □

Corollary 6.19 Let $\deg_T(A)$ and $\deg_{tt}(A)$ denote the Turing and Truth Table degree containing A. $\deg_T(A) = \deg_{tt}(A)$ if and only if $\deg_T(A)$ is hyperimmune-free.

Proof : Immediate from Theorem 6.18. □

7. Δ_2^0-not-ω-c.e. Degrees

Proposition 7.1 There exists a enumeration $\{G_n\}_{n\in\omega}$ of the ω-c.e. sets.

Proof : By Theorem 1.21, a set A is ω-c.e. if and only if $A \leq_{tt} \emptyset'$. By Theorem 1.20, $A \leq_{tt} B$ if and only if there is an associated computable function that induces a truth table, as described in that theorem. let $\{\emptyset'_s\}_{s\in\omega}$ be a 1-c.e. enumeration of $\emptyset'$. Define

$$G_{n,s} = \left\{ x : (\forall y \leq x) \left(\Phi^{\emptyset}_{n,s}(y)\downarrow \wedge \bigvee_{\sigma \in D_{\Phi^{\emptyset}_{n,s}(x)}} \sigma \prec \emptyset'_s \right) \right\}$$

Let $G_n = \lim_n G_{n,s}$. If $\Phi_n^{\emptyset}$ is computable, then G_n is the ω-c.e. set associated with the truth table induced by $\Phi_n^{\emptyset}$. If $\Phi_n^{\emptyset}$ is not total, then G_n will be finite set, which is still ω-c.e. Moreover, $\{G_{n,s}\}_{s\in\omega}$ gives an ω-c.e. computable approximation for the ω-c.e. set. Define $h(x) = \max\{|\sigma| : \sigma \in D_{\Phi_n^{\emptyset}(x)}\}$. $h(x)$ is a computable function if $\Phi_n^{\emptyset}$ is total. If Φ_n is not total, then let k be the least such that $(\forall y \leq k)(\Phi_n^{\emptyset}(y)\downarrow)$. Then

$$h'(x) = \begin{cases} \max\left\{|\sigma| : \sigma \in D_{\Phi_n^{\emptyset}(x)}\right\} & \text{if } x \leq k \\ 0 & \text{if } x > k \end{cases}$$

$G_{n,s}(x)$ can change only if an element less than or equal to $h(x)$ enters $\emptyset'_s$. Since the enumeration $\{\emptyset'_s\}_{s\in\omega}$ is 1-c.e., at most $h'(x)+1$ elements can enter. Thus the computable bound is $h(x)+1$. □

Theorem 7.2 There exists an degree $\mathbf{c} < \mathbf{0}'$ such that $\mathbf{c}$ is not ω-c.e.

Proof : The construction will produce a set C which satisfies the following requirements:

$$R_{\langle d,e,n,z\rangle} : C \neq \Phi_d^{G_z^n} \wedge G_z^n \neq \Phi_e^C$$

$$P : C \leq_T \emptyset'$$

The construction shall use the functions $r(\xi, s)$, $l(\xi, s)$, $w(\xi, s)$ and $c(x, s)$ as in Theorem 3.2.

At stage 0, for all ξ, define $r(\xi, 0) = l(\xi, 0) = w(\xi, 0) = -1$ and $c(\xi, 0) = 0$.
At stage s + 1, if it exists, find the least $\langle d, e, n, z\rangle < s+1$ such that

$$C_s \upharpoonright\upharpoonright w(\langle d,e,n,z\rangle, s) = \Phi_{d,s+1}^{G_{z,s+1}^n} \upharpoonright\upharpoonright w(\langle d,e,n,z\rangle, s)$$

$$G_{z,s+1}^n \upharpoonright\upharpoonright l(\langle d,e,n,z\rangle, s) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright l(\langle d,e,n,z\rangle, s)$$

Define $C_{s+1}(w(\langle d,e,n,z\rangle, s)) = 1 - C_s(w(\langle d,e,n,z\rangle, s))$, and for all $x \neq w(\langle d,e,n,z\rangle, s)$, $C_{s+1}(x) = C_s(x)$. Define $c(w(\langle d,e,n,z\rangle, s)) = c(w(\langle d,e,n,z\rangle, s)) + 1$, and for all $y \neq w(\langle d,e,n,z\rangle, s)$, $c(y, s+1) = c(y, s)$. For all $\xi \leq \langle d,e,n,z\rangle$, define $r(\xi, s+1) = r(\xi, s)$, $l(\xi, s+1) = l(\xi, s)$, and $w(\xi, s+1) = w(\xi, s)$. For all $\xi > \langle d,e,z\rangle$, define $r(\xi, s+1) = l(\xi, s+1) = w(\xi, s+1) = -1$. Go to stage $s+2$.

If no such $\langle d,e,n,z\rangle < s+1$ exists, then find the least $\langle d,e,n,z\rangle$ such that there exists $x \leq s+1$ such that $x \in \omega^{[\langle d,e,n,z\rangle]}$, $x > \max\{r(\xi, s) : \xi < \langle d,e,n,z\rangle\}$, $c(x, s) = 0$ and

$$C_s \upharpoonright\upharpoonright x = \Phi_{d,s+1}^{G_{z,s+1}^n} \upharpoonright\upharpoonright x$$

$$G_{z,s+1}^n \upharpoonright\upharpoonright \varphi_{d,s+1}^{G_{z,s+1}^n}(x) = \Phi_{e,s+1}^{C_s} \upharpoonright\upharpoonright \varphi_{d,s+1}^{G_{z,s+1}^n}(x)$$

Define $C_{s+1}(x) = 1$ and for all $y \neq x$, define $C_{s+1}(y) = C_s(y)$. Define $c(x, s+1) = 1$, $w(\langle d,e,n,z\rangle, s+1) = x$, $l(\langle d,e,n,z\rangle, s+1) = \varphi_{d,s+1}^{E_{z,s+1}^n}(x)$, $r(\langle d,e,n,z\rangle, s+1) = \max\{x, \max\{\varphi_{e,s+1}^{C_s}(y) : y \leq l(\langle d,e,n,z\rangle, s+1)\}\}$. For all $y \neq x$, $c(y, s+1) = c(y, s)$. For all $\xi < \langle d,e,n,z\rangle$, $w(\xi, s+1) = w(\xi, s)$, $l(\xi, s+1) = l(\xi, s)$, and $r(\xi, s+1) = r(\xi, x)$. For all $\xi > \langle d,e,n,z\rangle$, $w(\xi, s+1) = l(\xi, s+1) = r(\xi, s+1) = -1$.

If no such $\langle d,e,n,z\rangle$ exists, then go to stage $s+2$. This ends the construction.

Let $C = \lim_s C_s$. The verification that $R_{d,e,n,z}$ is satisfied is similar to Theorem 3.2 or Theorem 4.2. P is satisfied because C is limit computable via $\{C_s\}_{s\in\omega}$. For all x, $\lim_s C_s$ exists because if $x = \langle y, d, e, n, z\rangle$. Since G_n is ω c.e., there is a computable function f which bounds the number of changes for the computably approximation $\{G_{n,s}\}_{s\in\omega}$. Thus C_s can change at most $f(x)$ times. Hence the $\lim_s C_s(x)$ exists. □

Corollary 7.3 There exists a a degree $\mathbf{c} < \mathbf{0}'$ such that for all $C \in \mathbf{c}$, $C \leq_T \emptyset'$, $C \not\leq_{wtt} \emptyset'$, and $C \not\leq_{tt} \emptyset'$.

Proof : Let $\mathbf{c}$ be a Δ_2^0-not-ω-c.e. degree, which exists by Theorem 7.2. $\mathbf{c} \leq \mathbf{0}'$ since it is Δ_2^0. If there exists a $C \in \mathbf{c}$ such that $C \leq_{wtt} \emptyset'$ or $C \leq_{tt} \emptyset'$, then C is ω-c.e. by Theorem 1.21. □

8. Further Developments

In this section, some possible further directions, developments, and questions concerning the study of the computably approximation of Δ^0_2 sets and degrees are informally discussed. These ideas may or may not have been developed in the literature of Computability Theory.

There are several generalizations of the idea of bounding the number of changes of a computable approximation of Δ^0_2 sets. Let f be an arbitrary function, A set A is f-c.e. if and only if there exists a computably approximation $\{A_s\}_{s\in\omega}$ such that for all x, $|\{s : A_s(x) \neq A_{s+1}(x)\}| \leq f(x)$. Let $\mathcal{T}$ be a set of total computably functions (or a set of their indices). A set A is $\mathcal{T}$-c.e. if and only if there exists there exists a computably approximation $\{A_s\}_{s\in\omega}$ of A such that for all x, $|\{s : A_s(x) \neq A_{s+1}(x)\}| \leq f(x)$ for some $f \in \mathcal{T}$. The basic development of this idea can be found in Epstein, Haas, and Kramer, *Hierarchies of Sets and Degrees below* $\mathbf{0'}$. With a proof similar to those found in this paper, one can prove that if there exists a function g which enumerates the index of all the total computable function of $\mathcal{T}$ and g is a total computable function which dominates all $f \in \mathcal{T}$, then there exists a set A which is g-c.e. but not $\mathcal{T}$-c.e. The analogous question can be asked about degrees.

In this paper, ω-c.e. sets were studied. One can also develop limit approximation via computably and constructive ordinals. Computable and Constructive Ordinals are discussed in *Higher Recursion Theory* by Sacks and *Computably Structures and the Hyperarithmetic Hierarchy* by Ash and Knight. The concept of the Ordinal c.e. can be found in Epstein, Haas, and Kramer, *Hierarchies of Sets and Degrees below* $\mathbf{0'}$ and *Computably Structures and the Hyperarithmetic Hierarchy* by Ash and Knight. In this development, the ordinal ω-c.e. coincides with ω-c.e. developed in this paper.

Finally, for every Δ^0_2 set A and every computable approximation $\{A_s\}_{s\in\omega}$ there exists a $f \in \mathbf{0'}$ which bounds the number of changes. For example, $m(x) = (\mu s)(\forall t, u > s)(A_t(x) = A_u(x))$ is one such function. Say a set A is $\mathbf{a}$ -c.e. (for a degree $\mathbf{a}$) if and only if there exists a $f \in \mathbf{a}$ and a computable approximation of A such that f bounds the number of changes for every x.

This theory of $\mathbf{a}$-c.e. sets is meaningful if there is a set that is $\mathbf{a}$-c.e. for some $\mathbf{0} < \mathbf{a}$ and $\mathbf{a} \neq \mathbf{0'}$. Note $\mathbf{0}$-c.e. is ω-c.e. Before, this one has to prove if there are degrees which have functions which are not computably dominated by computably functions. Else $\mathbf{a}$-c.e. sets will all $\mathbf{0}$-c.e. sets for all $\mathbf{a}$. Luckily, the hyperimmune degree are the degrees which contain a function which is not computably dominated. That is there exists a function g in the degree such that for all computably functions f there exists infinitely many x such that $f(x) < g(x)$. Thus there is a possibility that there is a degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0'}$, $f \in \mathbf{a}$ and a set A such that A is f-c.e. and not ω-c.e. The theory of $\mathbf{a}$-c.e. sets could prove to be very interesting if various existence theorem for sets and degrees could be discover. Especially for this last question, the author is uncertain to what extend this theory has been developed in the literature.

Acknowledgement

The author thanks Eric Astor and Damir Dzhafarov for carefully listening and commenting on the results

and proofs that appear in this paper.

References

S.B. Cooper, *Computability Theory*, Chapman & Hall / CRC Mathematics, New York, 2004.

R.L. Epstein, R. Haas, and R. Kramer, Hierarchies of Sets and Degrees below $\mathbf{0}'$, *Lecture Notes in Mathematics*, 859, Springer-Verlag, Heidelberg, 1981.

A. Nies, *Computability Theory and Randomness*, Oxford University Press, New York, 2009.

R. I. Soare, *Computability Theory and Applications, The Art of Classical Computability*, Springer-Verlag, Heidelberg, to appear.