SOME POSSIBLE DIRECTIONS FOR REU FINAL PROJECTS

In case it is useful – here is a list of topics which might be interesting to investigate. Difficulty and background knowledge varies. If there are people who already know a nontrivial amount of model theory, they should feel free to ask me for other suggestions.

(1) Investigate the topology of the type spaces $S_n(T)$ and $S_n(A)$ (where $n < \omega$, T is a first-order theory, A is a subset of some model of T). There are many interesting questions here: for instance, what does it mean if the principal types are dense?

(2) Consider the first-order theory of the random graph and of $(\mathbb{Q}, <)$. Prove that each of these theories eliminates quantifiers and is $\aleph_0$-categorical (i.e. has, upto isomorphism, a unique countably infinite model) using a back-and-forth argument.

(3) There's a nice theorem (approx. 2 pages in Chang and Keisler) which gives half-a-dozen conditions on a theory which are equivalent to $\aleph_0$-categoricity. It would be a good exercise in compactness to understand this theorem and look at how it works in some particular theories.

(4) Develop cardinals and basic cardinal arithmetic, enough to begin counting things in $\mathbb{R}$. For instance, prove that transcendental numbers exist, and that most subsets of $\mathbb{R}$ are not open.

(5) Develop the ordinals and transfinite induction, and give some examples of the power of this method. For instance, prove that there exists a set $A \subset \mathbb{R}^2$ which intersects every line in exactly two points. Or suppose that we say a subset of the plane $S \subset \mathbb{R}^2$ is a "circle" if there exists a point s, called the *center*, such that every half-line beginning from s intersects S in a single point. Then $\mathbb{R}^2$ can be covered with countably many "circles."

(6) The Ax-Grothendieck theorem.

(7) ZFC.

(8) Shelah's theorem that an unstable theory either has the independence property or the strict order property (and what this means).

(9) Indiscernible sequences (in models of first-order theories): definition, existence proof using Ramsey's theorem, usefulness.

(10) There is no complete first-order theory which has (up to isomorphism) exactly two countable models.

(11) What types are realized in regular ultrapowers? (This is a deep question but there are accessible pieces, for someone interested in learning about ultraproducts.)