SIMPLE CLOSED CURVES ON SURFACES WITH INTERSECTION NUMBER AT MOST ONE

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ABSTRACT. Given a surface Σ_g, how many loops (distinct up to homotopy) can be placed on it such that no two loops intersect more than once? Benson Farb proposed this question in 2006 and while it is not obvious that this number should even be finite, general upper and lower bounds are known. Though these bounds are very disparate, the lower bound being quadractic and the upper bound being exponential, it is very difficult to obtain anything better: the exact answer is not known for genus more than 1. This paper presents some of the previous results on this problem, introduces some new proofs and corrections to the proofs of these results, and discusses some of the difficulties in generalizing these results to higher genus surfaces.

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1. INTRODUCTION

In this paper, we study the question of how many essential free homotopy classes of simple closed curves can one place on the genus-g surface Σ_g such that any two curves have intersection number at most one. To clarify the statement of the problem, we recall some definitions:

Definitions 1.1. For a topological space X, a *curve* or *path* on X is a continuous map $f : I \to X$ where $I = [0, 1] \in \mathbb{R}$. If $f(0) = f(1)$, then we say that f is a *closed curve* or *loop*; we often write loops as maps $f : S^1 \to X$. If f is injective, then we say that f is *simple*.

For most spaces X, there are many of these loops, though often some of these will be topologically similar, in the sense that we can continuously deform one loop into another. We then define an equivalence relation on loops.

Definition 1.2. For two loops a and b on a space X, we say that a and b are *freely homotopic* if there is a continuous map $H : I \times S^1 \to X$ such that $H(0,t) = a(t)$ and $H(1,t) = b(t)$. We then write $a \simeq b$, or $H : a \simeq b$ and say that H is a *homotopy* between a and b.

Homotopy then forms an equivalence relation and captures the idea of continuously deforming one loop into another via a continuous path of loops. If we consider homotopies which preserve base points (i.e. $H(s,0) = H(0,0)$ for all $s \in I$), then the set of homotopy classes of loops starting at a given base point $x_0 \in X$ forms a group under concatenation of loops. This group is called the *fundamental group* and is denoted $\pi_1(X, x_0)$.

Theorem 1.3 ([5]). *The fundamental group of the genus-g surface with base point x_0 is $\pi_1(\Sigma_g, x_0) = \langle \alpha_1, \ \beta_1, \cdots, \ \alpha_g, \ \beta_g \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1 \rangle$ where α_i is the canonical loop based at x_0 going through the ith hole and β_i is the canonical loop based at x_0 going around the ith hole.*

If a space X is path connected, $\pi_1(X, x_0) \cong \pi_1(X, y_0)$ for any $x_0, y_0 \in X$. Since we are dealing only with path connected spaces, we will often write the fundamental group as $\pi_1(X)$ for simplicity.

While the fundamental group is often useful and seems to relate to our question, there are loops which are distinct in π_1 which are identical under free homotopy.

Example 1.4. Consider Σ_2 and $\alpha_1\beta_2\alpha_1^{-1} \in \pi_1(\Sigma_2)$. In the fundamental group, this is not equal to β_2, yet once we can move the base point via free homotopy, then these two are equivalent, as can be seen in Figure 1.
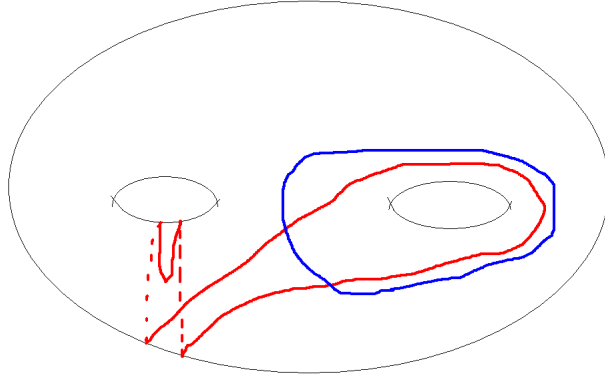

FIGURE 1. Loops that are distinct in the fundamental group but identical under free homotopy

Proposition 1.5. *The set of free homotopy classes of loops on the genus-g surface Σ_g is equal to the set of conjugacy classes of elements of the fundamental group $\pi_1(\Sigma_g)$.*

Proof. First we show that an element of the fundamental group is freely homotopic to any conjugate of itself. If $\gamma \in \pi_1(\Sigma_g)$ is a homotopy class of loops then we can express γ as a product of the canonical loops on Σ_g, so $\gamma = x_1 \cdots x_n$, where each x_k is a canonical loop. Then for any $1 < k < n$,

$$x_1 \cdots x_n \simeq x_k \cdots x_n x_1 \cdots x_{k-1}$$

up to free homotopy via shifting the base point of the loop. Thus for any α and $\beta \in \pi_1(\Sigma_g)$, $\alpha \simeq \beta\alpha\beta^{-1}$ up to free homotopy.

Now we show that any two loops which are freely homotopic are conjugate in the fundamental group, or at least have representatives of their free homotopy classes which are so. Suppose that a and b are freely homotopic loops on Σ_g, that is $a, b : S^1 \to \Sigma_g$ and $a \simeq b$ via some homotopy $H : I \times S^1 \to \Sigma_g$. Consider then a basepoint $t_0 \in S^1$ and $a(t_0), b(t_0) \in \Sigma_g$ and let $a(t_0) = x_0$. Then, since Σ_g is path connected, there is a path $p : I \to \Sigma_g$ such that $p(0) = a(t_0)$ and $p(1) = b(t_0)$. So a and $p^{-1}bp$ are loops with a shared base point. But $p^{-1}bp \simeq b$, so we can assume that a and b are representatives of basepoint preserving homotopy classes $\alpha, \beta \in \pi_1(\Sigma_g, x_0)$ with $a \in \alpha$ and $b \in \beta$. Consider then the loop f based at x_0 defined by $f(t) = H(t, *)$ for $t \in I$ and where $*$ is the basepoint of S^1 [3]. Then a and $f^{-1}bf$ are homotopic via a basepoint preserving homotopy, and so are representatives of the same class in $\pi(\Sigma_g)$. Thus a and b are conjugate in the fundamental group. $\square$

Definition 1.6. A loop is *essential* if it is not *homotopicaly trivial*, that is it is not homotopic to a constant map.

Remark 1.7. For the purpose of our question, we will not consider a loop and its inverse (that is the same loop traveled backwards) to be distinct. While a loop and its inverse are distinct under homotopy (both base point preserving and free), we will only count one such loop.

Now we need only define the intersection number of two classes of loops.

Definition 1.8. The *geometric intersection number* of two homotopy classes of loops α and β is $i(\alpha, \beta) = \min|a \cap b|$ across all representatives $a \in \alpha$ and $b \in \beta$.

Definition 1.9. We say that two loops a and b are in *minimal position* if $|a \cap b| = i(\alpha, \beta)$ where $\alpha, \beta \in \pi_1(\Sigma_g)$ with $a \in \alpha$ and $b \in \beta$. This is to say that a and b actually realize the intersection number of their homotopy classes.

For the torus, we know that $\pi_1(\Sigma_1) \cong \mathbb{Z}^2$, so any homotopy class of loops on Σ_1 is $a^n b^m$ for $n, m \in \mathbb{Z}$ and where a is the loop through the hole and b is the loop around the hole. Then we can represent any homotopy class of loops as some point $(n, m) \in \mathbb{Z}^2$, making it easy to compute the intersection number of two homotopy classes of loops.

Theorem 1.10 ([4])**.** *For the torus $T^2 = \Sigma_1$, the intersection number of two homotopy classes of loops (p, q) and (p', q') is $i((p, q), (p', q')) = |pq' - qp'|$.*

Notation 1.11. For simplicity, for a surface Σ_g, let $C(g)$ be the maximum size of a set of homotopy classes of essential simple loops on Σ_g such that any two loops in that set have intersection number at most one. We say that a set A of loops on Σ_g is a *best* or *maximal* collection of loops if $|A| = C(g)$ and A satisfies the appropriate properties of loops and intersections. Thus what we are interested in is finding more information on $C(g)$.

2. Upper and Lower Bounds

Finding a closed solution to this problem seems to be very difficult, so much of what has been accomplished towards a solution has been in the form of bounds on the answer. As for these bounds, proving a lower bound is as simple as constructing a large number of loops which satisfy the appropriate properties; proving an upper bound is generally more difficult.

The initial lower bound on $C(g)$ was presented by Farb.

Proposition 2.1. *There is a quadratic lower bound on $C(g)$, namely*

$$C(g) \geq \frac{(g-1)(g-2)}{2}.$$

Proof. For g even, divide Σ_g evenly into two submanifolds with boundary of genus $g/2$ connected by a small annulus. Then any curve on Σ_g is a pair of arcs on this partition, one arc on each submanifold with boundary. On each submanifold, we can construct $(g/2 - 1)$ arcs from the boundary to itself by traveling along the top of the surface, going around a hole, then returning to the boundary; another $(g/2 - 1)$ using the same process but on the bottom of the surface; and one that goes to the end of the submanifold, goes around the end hole, then returns. These can intersect only on the boundary, and so if we pair any arc from one submanifold with any arc from the other which is "interior" to it, we mantain the intersection property. This gives $\frac{(g-1)(g-2)}{2}$ loops. For g odd, the argument is similar, but we divide our surface into two submanifolds where one has genus $\frac{g-1}{2}$ and the other has genus $\frac{g+1}{2}$. $\qquad\square$

A better lower bound was latter shown by Constantin in her 2006 paper using the $(4g + 2)$-gon representation of Σ_g

Proposition 2.2. [1] *For $g > 1$, $C(g) \geq 2g^2 + 2g$.*

Proof. Consider the $(4g+2)$-gon representation of Σ_g, where one identifies all opposite sides in an orientation preserving manner. This presentation, after the identifications, leaves two distinct vertices, which we can view as the even vertex and the odd one based on which vertices were identified in our process. Then connecting the odd vertices into a $(2g + 1)$-gon gives $2g + 1$ curves; we have another $2g - 2$ diagonals emanating from a fixed odd vertex; one curve which encircles the entire polygon; finally there are $2g(g - 1)$ loops which Constantin calls "bounces." These she describes as starting at an odd vertex, following the perimeter of the polygon for at least three sides, reaching an even vertex, then returning to the initial odd vertex. All together, this gives $2g^2 + 2g$ loops. $\qquad\square$

Note that this formula does not apply to the torus. The torus has a hexagon as its $(4g + 2)$-gon representation, so when we create the initial three loops by drawing lines between odd vertices, we get get a triangle, which does not allow for the diagonals or bounces.

In general, there is an exponential upper bound on $C(g)$.

Proposition 2.3 ([1])**.** $C(g) \leq O(2^{3g})$

3. Dehn Twists and the Torus

Before discussing the value of $C(1)$, we would like to introduce a particular type of homeomorphism of surfaces known as a Dehn twist, which we will make heavy use of.

Definition 3.1. For the annulus $A = S^1 \times I$, the function $T : A \to A$ given by $T(\theta, t) = (\theta + 2\pi t, t)$ is an orientation preserving homeomorphism that fixes ∂A, the boundary of A, pointwise. For some simple loop α on Σ_g, we can find some small regular neighborhood N of α together with an orientation preserving embedding $\phi : A \to \Sigma_g$ such that $\phi(A) = N$. We then use this neighborhood to define the *Dehn twist about* α to be the homeomorphism $T_\alpha : \Sigma_g \to \Sigma_g$ given by

$$T_\alpha(x) = \begin{cases} x & \text{if } x \in S \setminus N \\ \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N. \end{cases}$$

.

The Dehn twist about α then can be viewed as cutting the surface along α, then twisting one side of this cut once around, then gluing the two side back together. These homeomorphisms are very useful, particularly in how they interact with loops on a surface. To see this, we review some basic properties about Dehn twists and intersection numbers of loops.

Proposition 3.2 ([2])**.** *If a and b are homotopy classes of essential simple loops and $k \in \mathbb{Z}$, then*

$$i(T_a^k(b), b) = |k| i(a, b)^2.$$

For the interaction of twists around different loops, we have the following formula.

Proposition 3.3 ([2])**.** *Let $a_1, \cdots, a_n$ be pairwise disjoint classes of simple loops on Σ_g, and let $e_1, \cdots, e_n \in \mathbb{Z}$ be either all positive or all negative. Define $M = \prod_{j=1}^{n} T_{a_j}^{e_j}$, and let b and c be arbitrary classes of simple loops on Σ_g. Then*

$$\left| i(M(c), b) - \sum_{j=1}^{n} |e_j| i(a_j, b) i(a_j, c) \right| \leq i(c, b).$$

Remark 3.4. One might be tempted to think the intersection formula should be one such as

$$i(T_\alpha^{n'} T_\beta^{m'}(a), a) \geq i(T_\alpha^n T_\beta^m(a), a)$$

for $n' \geq n$ and $m' \geq m$, but this is false. Consider two disjoint loops α and β in Σ_g; for example, let α be the loop though one hole in Σ_g and β be the loop through the other hole, and let a be the loop around both holes. Then $i(T_\alpha^2(a), a) = 2$ but $i(T_\alpha^2 T_\beta^1(a), a) = 1$.

Lastly, we cover a few basic facts about Dehn twists and intersections.

Proposition 3.5 ([2])**.** *For any two loops a and b on a surface, $T_a = T_b$ if and only if $a = b$.*

Proposition 3.6 ([2]). *For any two loops a and b on a surface, the following are equivalent:*

$$1)\ i(a,b) = 0$$
$$2)\ T_a(b) = b$$
$$3)\ T_a T_b = T_b T_a.$$

Proposition 3.7. *For any simple loops a, b, c, we have the interesection property that $i(a,b) = i(T_c(a), T_c(b))$.*

Proof. Consider the points of intersection of a and b. If any of these occur on c, we can homotope these away from c via a point pushing map. Thus at any point at which a and b cross c, we can assume that a and b are parallel in some small neighborhood of c. So, when we twist about c, we do not affect the intersection on a and b. $\square$

For now we will recall the following theorem.

Theorem 3.8 (Dehn [2]). *For $g \geq 0$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists about nonseparating simple loops.*

This then allows allows us to send any nonseparating loop on Σ_g to any other nonseparating loop via Dehn twists. We then have the following result.

Proposition 3.9. *For the torus $T^2 = \Sigma_1$, we have $C(g) = 3$.*

Proof. Let γ be any loop on T^2, so in the fundamental group $\gamma = \alpha^n \beta^m$ where α is the canonical curve going through the center of the torus and β is the canonical curve going around the torus. The using Dehn twists about β, we can send γ to α^n via the homeomorphism T_β^{-m}. Likewise, via T_α^{1-n}, we can send this to α, so we can assume that our first loop is just α. Then cut along this curve, so that we now have a cylinder with boundary (the ends along which we cut). Now take any simple curve δ from one boundary component to the other. This corresponds to any second simple loop on T^2 which intersects α at most once. Again using Dehn twists, we can assume that this second curve is just β, the straight line from the base point of α on one end of the cylinder to the base point on the other end. Cut along this curve, so that we now have a square where one pair of opposite sides is defined by α and the other pair is defined by β. To make a third simple loop which intersects α and β at most once, we again cannot cross the boundary of the square except at vertices. Then we must draw a line across a diagonal of this square, since going between vertices of the same edge would be homotopic to α or β. Thus we can obtain three curves but cannot obtain any more. $\square$

4. Subtleties in the genus-2 case

In general, the exact values of $C(g)$ for higher genus is not known, though Constantin's paper presented a proof for an exact value for $C(2)$. Unfortunately, this paper contained many difficulties which we will discuss. For now, we will walk through the idea of the proof to get a better sense of the difficulties involved in it.

Proposition 4.1 ([1]). $C(2) = 12$.

This statement was shown using several lemmas, many of which are very useful in finding solutions to this problem, though there are reasons to doubt the validity of

some of these. For now we will summarize the lemmas, leaving detailed discussions of them for later in the paper.

Proposition 4.2 ([1]). *In any collection of simple loops on Σ_g such that any two loops have intersection number at most one, we can replace any separating curve with two nonseparating curves while preserving this intersection property. Therefore any best collection A of loops on Σ_g has no separating curves.*

Conjecture 4.3 ([1]). *If A is a collection of loops on Σ_g such that all loops in A have pairwise intersection exactly one, then $|A| \leq 2g$.*

When this is combined with the quadratic lower bound, we get the following Corollary.

Corollary 4.4 ([1]). *For $g > 1$, any best collection of loops has at least two disjoint loops.*

Conjecture 4.5 ([1]). *For $g = 2$, if there are two disjoint loops in a best collection, then there is a third disjoint loop in that collection.*

These then allow for a pants decomposition of Σ_2 via curves in our collection, which allows us a proof of Proposition 4.1. To clarify the matter, we present a reduction of Proposition 4.1 to the above conjectures.

Theorem 4.6. *If there are three disjoint loops in some best collection of loops on Σ_2, then that collection has at most 12 loops in it.*

Proof. Let A be a best collection of loops on Σ_2 that has three disjoint loops. Let α, β, and γ be three disjoint loops on Σ_2. Since these are disjoint, we can assume that α is the basic loop through the left hole, γ is the basic loop through the right hole, and β is the basic loop through both holes.

We first show that any other loop on Σ_2 intersects exactly two of α, β, and γ. Suppose δ is some other loop, and consider the pants decomposition of Σ_2 obtained by cutting along α, β, and γ. Then on this decomposition, δ is composed of arcs between boundary components on a pair of pants and loops within a pair of pants. But any simple loop on the interior of a pair of pants is homotopic to one of the boundary components, so there is at least one arc between boundary components in this decomposition of δ. This arc then forces δ to intersect two of α, β, and γ. Then on this pants decomposition, δ is a collection of arcs on each pair of pants, where δ starts on a boundary component of one pair of pants, move to another boundary component on that pair of pants, then goes from the corresponding boundary component on the second pair of pants to another boundary component on that pair of pants, and so on. So to intersect all three of α, β, and γ, δ must have at least two arcs on some pair of pants. But this forces δ to intersect one of α, β, and γ twice, contradicting our assumption about intersection number. Thus δ intersects at most two of α, β, and γ..

So consider the subcollection $A_{\alpha,\beta} \subseteq A$ of loops which intersect α and β. Then any loop in $A_{\alpha,\beta}$ can be written as $T_\alpha^n T_\beta^m(a)$ for some $n, m \in \mathbb{Z}$, where a is the basic loop going around the left hole. Since there are only finitely many of these, we can apply Dehn twists until each loop in $A_{\alpha,\beta}$ is made by positive Dehn twists around α and β applied to a. Pick some $T_\alpha^n T_\beta^m(a)$ and consider any other loop

$T_\alpha^{n'} T_\beta^{m'}(a) \in A_{\alpha,\beta}$. Then by Proposition 3.3,

$$
\begin{aligned}
i(T_\alpha^n T_\beta^m(a), T_\alpha^{n'} T_\beta^{m'}(a)) \geq \Big| & i(T_\alpha^{n'-n} T_\beta^{m'-m} T_\alpha^n T_\beta^m(a), T_\alpha^{n'} T_\beta^{m'}(a)) \\
& - |n'-n| i(\alpha, T_\alpha^n T_\beta^m(a)) i(\alpha, T_\alpha^{n'} T_\beta^{m'}(a)) \\
& - |m'-m| i(\beta, T_\alpha^n T_\beta^m(a)) i(\beta, T_\alpha^{n'} T_\beta^{m'}(a)) \Big| \\
= \Big| & i(T_\alpha^{n'} T_\beta^{m'}(a), T_\alpha^{n'} T_\beta^{m'}(a)) \\
& - |n'-n| i(\alpha, T_\alpha^n T_\beta^m(a)) i(\alpha, T_\alpha^{n'} T_\beta^{m'}(a)) \\
& - |m'-m| i(\beta, T_\alpha^n T_\beta^m(a)) i(\beta, T_\alpha^{n'} T_\beta^{m'}(a)) \Big| \\
= & |n'-n| + |m'-m|
\end{aligned}
$$

since we know that

$$
i(\alpha, T_\alpha^n T_\beta^m(a)) = 1 = i(\beta, T_\alpha^{n'} T_\beta^{m'}(a))
$$

by our hypotheses on A and

$$
i(T_\alpha^{n'} T_\beta^{m'}(a), T_\alpha^{n'} T_\beta^{m'}(a)) = 0
$$

by definition. Then any other loop in $A_{\alpha,\beta}$ can be at most one Dehn twist different from $T_\alpha^n T_\beta^m(a)$. But $i(T_\alpha^{n+1} T_\beta^m(a), T_\alpha^{n-1} T_\beta^m(a)) = 2$, so we cannot have both $T_\alpha^{n+1} T_\beta^m(a), T_\alpha^{n-1} T_\beta^m(a) \in A_{\alpha,\beta}$. Likewise, only one of $T_\alpha^n T_\beta^{m+1}(a)$ and $T_\alpha^n T_\beta^{m-1}(a)$ can be in $A_{\alpha,\beta}$. So there can only be three loops in $A_{\alpha,\beta}$, and similarly for $A_{\alpha,\gamma}$ and $A_{\beta,\gamma}$. Thus there are at most 12 loops in A. $\square$

Notice that this proof relies on being able to find three disjoint loops in some best collection of loops on Σ_2, and as we will see, this is not assured.

For the intersection of two loops, Constantin's paper says that we can decompose Σ_g into g punctured tori and consider the intersection of two loops on each of these punctured tori to find the intersection of those two loops on Σ_g. The claim is that one can then use the simple formula $i((p,q),(p',q')) = |pq' - qp'|$ for intersection on a torus, and sum over all punctured tori in our decomposition to find the intersection of the two loops. From her discussion, it seems that this should be formulated as such:

Conjecture 4.7. *For two loops* $(a_1, b_1, \cdots, a_g, b_g)$, $(a_1', b_1', \cdots, a_g', b_g') \in H_1(\Sigma_g)$, *their intersection number is*

$$
i((a_1, b_1, \cdots, a_g, b_g), (a_1', b_1', \cdots, a_g', b_g')) = \sum_{j=1}^{g} i((a_j, b_j), (a_j', b_j')).
$$

While this formula is very powerful and would be very useful in this problem, the validity of it is uncertain. In addition to it being based on the homological representation, which we will see does not accurately represent free homotopy classes, it is unclear where this formula is coming from. This author has been unable to find any other reference to such a formula, including in the sources which Constantin cites as references. It should be noted that a similar formula does hold for algebraic intersection, which might be used to prove results like those in discussion. Despite

this, algebraic intersection only gives us a lower bound on geometric intersection, as there might be geometric intersections which cancell out due to their sign as algebraic intersections.

Beyond the difficulties inherent in using homology to solve this problem, the argument in Constantin's paper has more immediate problems. To show that $C(2) = 12$, one needs to show that one can never get more than twelve loops on Σ_2; Constantin does this by showing first that every best collection of loops on any surface has two disjoint loops. She then claims that if a best collection of loops on Σ_2 has two disjoint loops, then there is a third disjoint loop in that collection. This then allows us to show that there are at most twelve loops in any best collection on Σ_2, as we saw earlier. Unfortunately, her proof that any best collection of loops on Σ_g has at least two disjoint loops relies on her lemma stating that there are at most $2g$ loops on Σ_g which each have intersection exactly one, but this statement is false.

Example 4.8. Consider the classes of loops $a_1, b_1b_2, a_1b_1, b_1b_2a_2^{-1}, b_1a_2 \in \pi_1(\Sigma_2)$ in the fundamental group. These classes represent distinct homology classes, being distinct under abelianization, and so represent distinct free homotopy classes. These loops have pairwise intersection of one, as can be seen from Figure 2.
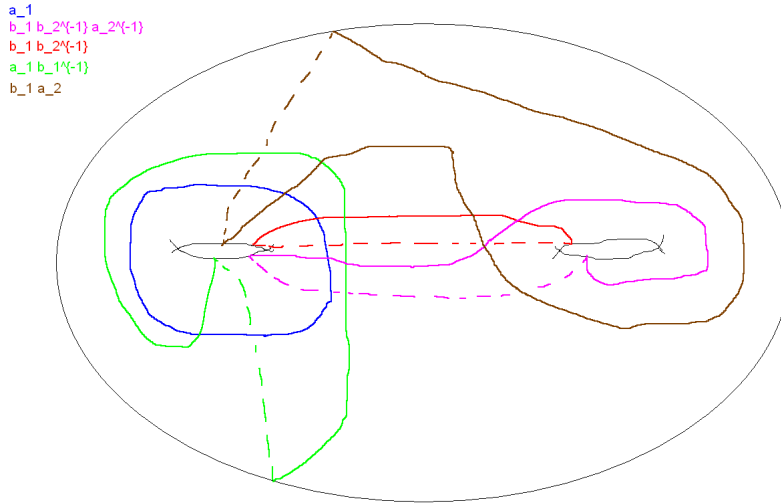

FIGURE 2. Five loops on Σ_2 intersecting exactly once.

It seems likely that one can find two disjoint loops in any best collection of loops; intuitively, given a large number of loops in a collection, there should be two disjoint loops in that collection.

Conjecture 4.9. *There is some $n \in \mathbb{N}$ such that for any $g \geq n$ and best collection A of loops on Σ_g, there are two disjoint loops in A.*

It has been suggested that the above conjecture is correct, and that in fact, a modified version of Conjecture 4.3 holds.

Conjecture 4.10. *If A is a collection of loops on Σ_g such that all loops in A have pairwise intersection exactly one, then $|A| \leq 2g + 1$.*

This conjecture combined with the quadratic lower bound on $C(g)$ would give Corollary 4.4, and Example 4.8 shows that this upper bound can be realized. Unfortunately, this author has yet to see a clear and correct proof of Conjecture 4.10. Furthermore, even if such a result applied to the genus two case, it is not clear that it would imply that a best collection of loops on Σ_2 contains three disjoint loops, as Constantin's proof of Conjecture 4.5 relies on the formula from Conjecture 4.7, which we are skeptical of.

Question 4.11. *Is there some best collection of loops on Σ_2 that has three disjoint loops?*

If this were true, then we would have the result that $C(2) = 12$, but it is not intuitive that the answer to this question should be yes.

We now examine some of the techniques these proofs used and how they might generalize to higher genus surfaces.

5. Methods

There are several existing technologies and tools in mathematics which can, and have been, applied to this problem. We first consider a more geometric approach to this problem, that of covering spaces.

Definition 5.1. For a space X, a *covering space* of X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the property that there is an open cover $\{U_\alpha\}$ of X such that for any α, $p^{-1}(U_\alpha) = \cup V_j$ where the V_j are disjoint open sets in $\tilde{X}$ and the restriction $p|_{V_j} : V_j \to U_\alpha$ is a homeomorphism.

These covering spaces are useful in studying loops, since any loop (or homotopy between loops) in the base space lifts to a path (or homotopy between paths) in the covering space, and the covering map p induces an injective homomorphism from the fundamental group of the covering space into that of the base space. In general, there are a lot of covering spaces of any given space, though here we are mainly concerned with the universal cover.

Definition 5.2. A covering space $p : \tilde{X} \to X$ is the *universal cover* of X if $\pi_1(\tilde{X}) = 1$, that is, the fundamental group of the covering space is trivial. The universal cover of a space is unique up to homeomorphism.

Consider then the genus 1 case. The universal cover of T^2 is $\mathbb{R}^2$ and any loop on T^2 lifts to a path in $\mathbb{R}^2$ starting and ending on the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Furthermore we can homotope any such path to a straight line between vertices on the integer lattice. Different lifts of any given map result from a different selection of base point in the cover, so we can consider all homotopy classes of lifts of a loop on T^2 by considering all translations of a given straight line lift. Then two loops on T^2 intersect if and only if two such straight line lifts intersect. This allows us to consider only lines between vertices on the integer lattice, checking for intersection between all integer translations of any given lifts of two loops. From this it can be shown that the intersection of two loops on T^2 is given by the earlier formula $i((p, q), (p', q')) = |pq' - qp'|$. Furthermore, a loop (p, q) is simple if and only if $\gcd(p, q) = 1$.

Here we constructed the universal cover by tessellating a square, the fundamental polygon of the torus, which clearly defines how to map $\mathbb{R}^2$ onto T^2 as well as indicating how to lift loops to the plane. We can use a similar process in general, taking the fundamental polygon of Σ_g, which is a $4g$-gon, and tessellating it on a hyperbolic plane. Then we have the universal cover $\mathbb{H}^2 \to \Sigma_g$. Again one must have some criterion to determine the intersection number of two loops, and so we have the following.

Theorem 5.3 (Bi-gon Criterion [2])**.** *Two loops a and b on Σ_g are in minimal position if and only if a and b create no bigons, that is if one cuts along each loop, the resulting space (which is a disjoint union of manifolds) has no component homeomorphic to D^1.*

Corollary 5.4 ([2])**.** *When considered as lifts of loops under the covering $\mathbb{H}^2 \to \Sigma_g$, distinct geodesics on the hyperbolic plane represent loops in minimal position on Σ_g.*

Of course, lacking the coordinate system of $\mathbb{R}^2$, it is more difficult to determine whether such a lift represents a simple loop, or even to construct collections of loops which satisfy our desired properties. Due to these difficulties we turn to a more algebraic representation of the problem: homology.

The definition of the homology groups is very long and complicated, though any reader wishing to see such a definition should refer to [5]. Despite the complexity of the definition, it can be said that the kth homology group measures the number of k dimensional holes in X. Fortunately, for the sake of this problem, we are mainly interested in the first homology group, which is easier for us to find.

Proposition 5.5 ([5])**.** *If X is a path connected space then the first homology group of X is the abelianization of the fundamental group of X, that is*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

The first homology group is then much easier to work with than the fundamental group and is easy to find for common, well behaved spaces. Unfortunately, the first homology group does not exactly represent the objects we are interested in.

Example 5.6. Consider Σ_2 and the elements of the fundamental group $b_1 a_1 b_1^{-1} a_2$ and $a_1 a_2$ in $\pi_1(\Sigma_g)$. These elements are not conjugate, and so are not freely homotopic, but they both are sent to $(1, 0, 1, 0) \in H_1(\Sigma_2)$ under abelianization.

While for any homology class, there is a homotopy class which is sent to that homology class under abelianization, this map from homotopy classes to homology classes is not injective. Any homology class has representatives which are distinct under homotopy. It is also unclear whether every homology class has a simple loop representative.

Question 5.7. *If there is some homology class in $H_1(\Sigma_g)$ which does not have a simple loop representative, which homology classes do have such a representative?*

Despite this, we can translate much of what we are interested in into the language of abelian groups via homology.

Proposition 5.8. *A homotopy class of loops α on Σ_g is separating if and only if $\alpha = 0 \in H_1(\Sigma_g)$.*

Thus we need only consider non-trivial elements of H_1.

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