THE RESTRICTION PROBLEM AND THE TOMAS-STEIN THEOREM

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Abstract. E. M. Stein's restriction problem for Fourier transforms is a deep and only partially solved conjecture in harmonic analysis. Below we state the problem and the Tomas-Stein theorem, which solves a particularly useful case of the conjecture. We then introduce the Fourier transform of complex-valued measures and the stationary phase method as tools used in the proof of the Tomas-Stein theorem. We give a proof of the theorem, and then turn to applications in deriving the Strichartz estimate for the Schroedinger equation.

Contents

1. Introduction

We assume the reader is familiar with basic facts about the L^p spaces, L^1 and L^2 Fourier transforms, and convolutions. These can be found in most analysis textbooks, such as [2] or [1]. Given a function $f : \mathbb{R}^n \to \mathbb{C}$ we will use the following notation and normalization to denote the Fourier transform (when defined):

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\mathrm{d}x \tag{1.1}$$

and the inverse Fourier transform:

$$\check{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi\cdot x} f(\xi)\mathrm{d}\xi. \tag{1.2}$$

The restriction problem asks when an inequality of the form

$$\|\hat{f}|_{S^{n-1}}\|_{L^q(S^{n-1})} \leq C\|f\|_{L^p} \tag{1.3}$$

holds, where S^{n-1} is the unit sphere and the constant depends only on p, q, and n. When $q = 2$, the best possible result is given by the following theorem:

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Theorem 1.4 (Tomas-Stein). *Let $f \in L^{p'}$ with $p' \leq \frac{2n+2}{n+3}$. Then $\|\hat{f}|_{S^{n-1}}\|_{L^2(S^{n-1})} \leq C\|f\|_{L^{p'}}$, where C depends only on n and p'.*

We will give a proof for the region $p' < \frac{2n+2}{n+3}$. The argument for the endpoint is quite different, using a complex interpolation method; it can be found in [3]. The next section introduces the Fourier transform of a measure and the convolution of a measure with a function. Section 3 derives estimates for the decay of the Fourier transform of the surface measure of the sphere. Section 4 fills in the remaining details of the proof. Section 5 gives a useful application to differential equations. Sections 2-4 follow the exposition in [4] while 5 follows [3].

2. Fourier transforms of measures

We will want to rephrase the statement of the Tomas-Stein theorem in terms of the surface measure of a sphere. To do this, we will first prove some general statements about complex-valued measures.

Definitions 2.1. Let μ be a complex-valued measure on $\mathbb{R}^n$ with finite total variation. Then $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$, the Fourier transform of the measure, is given by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \mathrm{d}\mu(x).$$

Let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a Schwartz function (i.e. smooth, rapid decay, and rapid decay for all derivatives). Then the convolution of φ and μ is

$$\varphi * \mu(x) = \int_{\mathbb{R}^n} \varphi(x-y)\mathrm{d}\mu(y).$$

It is clear that both the Fourier transform of μ and its convolution with φ are bounded, since

$$|\hat{\mu}| = |\int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \mathrm{d}\mu| \leq \int_{\mathbb{R}^n} |e^{-2\pi i x \cdot \xi}| \mathrm{d}|\mu| = |\mu|(\mathbb{R}^n) < \infty$$

by assumption. For the convolution,

$$|\varphi * \mu| = |\int_{\mathbb{R}^n} \varphi(x-y)\mathrm{d}\mu(y)| \leq \int_{\mathbb{R}^n} |\varphi(x-y)|\mathrm{d}|\mu|(y) \leq (\sup|\varphi|)\, |\mu|(\mathbb{R}^n) < \infty,$$

using that Schwartz functions are bounded.

The following lemma states that the ordinary interaction between convolutions and the Fourier transform carry over to measures.

Lemma 2.2. *Let μ and φ be as above, and ν is another finite measure. Then we have:*

(1) $\widehat{\varphi\check{\mu}} = \hat{\varphi} * \mu$
(2) $\widehat{\varphi\mu} = \hat{\varphi} * \hat{\mu}$
(3) $\int \hat{\mu}\mathrm{d}\nu = \int \hat{\nu}\mathrm{d}\mu$.

Proof. These can be derived from Fubini's theorem. We start with the duality relation (3):

$$\int \hat{\mu}\mathrm{d}\nu = \iint e^{-2\pi i x \cdot \xi}\mathrm{d}\mu(x)\mathrm{d}\nu(\xi) = \iint e^{-2\pi i x \cdot \xi}\mathrm{d}\nu(\xi)\mathrm{d}\mu(x) = \int \hat{\nu}\mathrm{d}\mu,$$

with Fubini's theorem clearly justified since both measures are finite.

To see (1), fix $\xi \in \mathbb{R}^n$. Consider $e^{-2\pi i x\cdot(\xi-y)}\varphi(x)$ as a function from $\mathbb{R}^n \times \mathbb{R}^n$ under the product measure $\mathrm{d}x \times \mathrm{d}\mu(y)$. It is integrable by Tonelli's theorem:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |e^{-2\pi i x\cdot(\xi-y)}\varphi(x)|\mathrm{d}x\times\mathrm{d}|\mu|(y) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\varphi(x)|\mathrm{d}x\mathrm{d}|\mu|(y) \leq \|\varphi\|_{L^1}\cdot|\mu|(\mathbb{R}^n) < \infty.$$

We can therefore apply Fubini's theorem to the following:

$$\begin{aligned}\widehat{\varphi\check{\mu}}(\xi) &= \int e^{-2\pi i x\cdot\xi}\varphi(x)\check{\mu}(x)\mathrm{d}x = \int e^{-2\pi i x\cdot\xi}\varphi(x)\left(\int e^{2\pi i y\cdot x}\mathrm{d}\mu(y)\right)\mathrm{d}x \\ &= \iint e^{-2\pi i x\cdot(\xi-y)}\varphi(x)\mathrm{d}x\mathrm{d}\mu(y) = \int \hat{\varphi}(\xi-y)\mathrm{d}\mu(y) = \hat{\phi}*\mu(\xi).\end{aligned}$$

(2) also falls out from Fubini's theorem, this time using the integrability of $\hat{\varphi}$ to justify switching the order of integration:

$$\begin{aligned}\hat{\varphi}*\hat{\mu}(x) &= \int \hat{\mu}(x-y)\hat{\varphi}(y)\mathrm{d}y = \int\left(\int e^{-2\pi i\xi\cdot(x-y)}\mathrm{d}\mu(\xi)\right)\hat{\varphi}(y)\mathrm{d}y \\ &= \int\left(\int e^{2\pi i\xi\cdot y}\hat{\varphi}(y)\mathrm{d}y\right)e^{-2\pi i\xi\cdot x}\mathrm{d}\mu(\xi) = \int \varphi(\xi)e^{-2\pi i\xi\cdot x}\mathrm{d}\mu(\xi) = \widehat{\varphi\mu}.\end{aligned}$$

□

We have already proven most of the next lemma:

Lemma 2.3. *Let μ be a finite measure and f and g Schwartz functions. Then*

$$\int \hat{f}\bar{\hat{g}}\mathrm{d}\mu = \int (\hat{\mu}*\bar{g})\cdot f\mathrm{d}x.$$

Proof. The only new fact needed is that $\bar{\hat{g}} = \check{\bar{g}}$. This follows from the fact that the integral of the conjugate is the conjugate of the integral:

$$\bar{\hat{g}} = \overline{\int e^{-2\pi i x\cdot\xi}g(x)\mathrm{d}x} = \int \overline{e^{-2\pi i x\cdot\xi}g(x)}\mathrm{d}x = \int e^{2\pi i x\cdot\xi}\overline{g(x)}\mathrm{d}x = \check{\bar{g}}.$$

Now we can simply compute, using first duality (Lemma 2.2(3)) and then Lemma 2.2(2):

$$\int \hat{f}\bar{\hat{g}}\mathrm{d}\mu = \int f\cdot\widehat{\bar{\hat{g}}\mu}\mathrm{d}x = \int f\cdot(\hat{\bar{\hat{g}}}*\hat{\mu})\mathrm{d}x = \int f\cdot(\hat{\check{\bar{g}}}*\hat{\mu})\mathrm{d}x = \int f\cdot(\bar{g}*\hat{\mu})\mathrm{d}x.$$

□

We are now in a position to make the major simplifying restatement of the Tomas-Stein theorem. This is known as the TT^* approach for this reason: the Tomas-Stein theorem as stated above asserts the boundedness of the operator T^* from L^p to $L^2(S^{n-1})$ which is given by taking the Fourier transform and restricting it to the sphere. We claim this is equivalent to its adjoint T being bounded and the composition TT^* being bounded.

Theorem 2.4. *Let p and p' satisfy*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let μ be a finite measure on $\mathbb{R}^n$. Then the following are equivalent:

(1) $\|\widehat{f\mu}\|_{L^p} \leq C\|f\|_{L^2(\mu)}$ *for all* $f \in L^2(\mu)$
(2) $\|\hat{g}\|_{L^2(\mu)} \leq C\|g\|_{L^{p'}}$ *for all Schwartz functions* g

(3) $\|\hat{\mu} * h\|_{L^p} \leq C^2 \|h\|_{L^{p'}}$ *for all Schwartz functions* h.

Proof. The proof uses the following basic fact about dual Banach spaces (self-dual, in this case):

$$\|\hat{g}\|_{L^2(\mu)} = \sup_{f \in L^2(\mu), \|f\|_{L^2(\mu)}=1} \left| \int \hat{g} f \mathrm{d}\mu \right|.$$

Then if (1) holds, we can use the duality relation (Lemma 2.2(c)):

$$\begin{aligned}
&\text{(duality)} & \sup_{\|f\|_{L^2(\mu)}=1} \left|\int \hat{g} f \mathrm{d}\mu\right| &= \sup_{\|f\|_{L^2(\mu)}=1} \left|\int \widehat{f\mu} \cdot g \mathrm{d}x\right| \\
&\text{(Holder)} & &\leq \sup_{\|f\|_{L^2(\mu)}=1} \|\widehat{f\mu}\|_{L^p} \|g\|_{L^{p'}} \\
&\text{(1)} & &\leq \sup_{\|f\|_{L^2(\mu)}=1} C\|f\|_{L^2(\mu)} \|g\|_{L^{p'}} \\
& & &= C\|g\|_{L^{p'}},
\end{aligned}$$

which is (2). Conversely, if we assume (2) by the same reasoning we have

$$\int \widehat{f\mu} \cdot g \mathrm{d}x = \int \hat{g} f \mathrm{d}\mu \leq \|\hat{g}\|_{L^2(\mu)} \|f\|_{L^2(\mu)} \leq C\|g\|_{L^{p'}} \|f\|_{L^2(\mu)}.$$

Since Schwartz functions are dense in $L^{p'}$, we then have

$$\|\widehat{f\mu}\|_{L^p} = \sup_{g \in \mathcal{S}, \|g\|_{L^{p'}}=1} \left|\int \widehat{f\mu} \cdot g \mathrm{d}x\right| \leq \sup_{g \in \mathcal{S}, \|g\|_{L^{p'}}=1} C\|g\|_{L^{p'}} \|f\|_{L^2(\mu)} = C\|f\|_{L^2(\mu)},$$

which is (1) ($\mathcal{S}$ is the Schwartz space).

Next we show (2) and (3) are equivalent. If (3) holds, then

$$\|\hat{h}\|^2_{L^2(\mu)} = \int \hat{h}\bar{\hat{h}} \mathrm{d}\mu = \int (\hat{\mu} * \bar{h}) \cdot h \mathrm{d}x \leq \|\hat{\mu} * \bar{h}\|_{L^p} \|h\|_{L^{p'}} \leq C^2 \|h\|^2_{L^{p'}},$$

which is the square of (2). The second equality is Lemma 2.3 with both functions set to h, the next step is Holder's inequality, and the last inequality is (3) by assumption. Now assume (2) and let h be a Schwartz function. Then, using the same tools,

$$\begin{aligned}
&\text{(dual spaces)} & \|\hat{\mu} * \bar{h}\|_{L^p} &= \sup_{g \in \mathcal{S}, \|g\|_{L^{p'}}=1} \left|\int (\hat{\mu} * \bar{h}) \cdot g \mathrm{d}x\right| \\
&\text{(Lemma 2.3)} & &= \sup_g \left|\int \hat{g}\bar{\hat{h}} \mathrm{d}\mu\right| \\
&\text{(Cauchy-Schwartz)} & &\leq \sup_g \|\hat{h}\|_{L^2(\mu)} \|\hat{g}\|_{L^2(\mu)} \\
&\text{(2)} & &\leq C^2 \sup_g \|h\|_{L^{p'}} \|g\|_{L^{p'}} = C^2 \|h\|_{L^{p'}},
\end{aligned}$$

which is (3). □

3. Stationary phase approximations

The previous section rephrased the restriction problem in terms of the Fourier transform of the surface of the sphere. In order to exploit this, we will need to estimate the decay of this function. First, we consider a more general question. Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be smooth and $\alpha : \mathbb{R}^n \to \mathbb{C}$ be smooth and of compact support. Let

$$I(\lambda) = \int e^{-\pi i \lambda \phi(x)} \alpha(x) \mathrm{d}x.$$

We wish to estimate the decay of $|I(\lambda)|$ as $\lambda \to \infty$. The following computation shows that if such an estimate is independent of α, it is diffeomorphism invariant. More precisely, if ϕ_1 and ϕ_2 are smooth functions with $\phi_1 = \phi_2 \circ G$ for some diffeomorphism G,

$$\int e^{-\pi i \lambda \phi_2(x)} \alpha(x) \mathrm{d}x = \int e^{-\pi i \lambda \phi_1(G^{-1}x)} \alpha(x) \mathrm{d}x = \int e^{-\pi i \lambda \phi_1(y)} \alpha(Gy) |J(G)| \mathrm{d}y,$$

where J takes the determinant of the Jacobian matrix.

The decay on I is related to the degeneracy of ϕ. The simplest case of this is when ϕ is nonstationary. The following is a general fact about smooth functions; the proof is omitted.

Lemma 3.1 (Straightening)**.** *Let f be a smooth complex valued function on a neighborhood of p with $\nabla f(p) \neq 0$. Then there is a diffeomorphism G of neighborhoods of 0 and p with $G(p) = 0$ and $f \circ G(x) = f(p) + x_n$.*

This immediately gives the following:

Theorem 3.2 (Nonstationary phase)**.** *Let ϕ be smooth on a neighborhood of p with $\nabla f(p) \neq 0$. Then for α smooth and supported on a sufficiently small neighborhood of p, for all N there is a C_N (that depends on α and ϕ) such that $|I(\lambda)| \leq C_N \lambda^{-N}$.*

Proof. Let $G : U \to V$ be the diffeomorphism given by applying Lemma 3.1. Assume α is supported inside V. Then by the computation above,

$$|I(\lambda)| = |\int e^{-\pi i \lambda(\phi(p) + y_n)} \alpha(Gy)|J(G)| \mathrm{d}y| = C|\widehat{\alpha \circ G|J(G)}|(\frac{\lambda e_n}{2})$$

where in the last step the phase $e^{-\pi i \lambda \phi(p)}$ was pulled out of the integral. But the function $\alpha(Gy)|J(G)|$ is C_0^∞, so its Fourier transform is Schwartz and can be bounded as required. $\square$

The decay grows weaker for a nondegenerate critical point, but the proof of this is somewhat harder. The replacement for Lemma 3.1 is the Morse lemma. The calculation is more sophisticated and requires the Fourier transform of tempered distributions, but the general format (reducing to a normal form by diffeomorphism invariance) is the same. The result is stated below; the proof can be found in [4]

Theorem 3.3 (Stationary phase)**.** *Let ϕ be smooth on a neighborhood of p with $\nabla f(p) \neq 0$. Assume the Hessian matrix $H_\phi(p)$ is invertible. Then for α smooth and supported on a sufficiently small neighborhood of p, there is a C (that depends on α and ϕ) such that $|I(\lambda)| \leq C\lambda^{-n/2}$.*

We wish to apply this to the surface measure of the sphere, which from now on will be denoted by σ. The first observation is that since σ has rotational symmetry, so does $\hat{\sigma}$. Therefore it is sufficient to estimate the decay of $\hat{\sigma}(\lambda e_n)$. Next, we cover S^{n-1} by coordinate charts as follows: the first is the inverse of $q_1 : D^{n-1}(0, 1/2) \to S^{n-1}$ given by $q_1(x) = (x, \sqrt{1 - |x|^2})$ paired with its image, which is a neighborhood of the north pole. The second is the inverse of $q_2 : D^{n-1}(0, 1/2) \to S^{n-1}$ given by

$q_2(x) = (x, -\sqrt{1-|x|^2})$ paired with its image, which is a neighborhood of the south pole. $D^{n-1}(p,r) = D(p,r)$ here means $\{x \in \mathbb{R}^{n-1} | \ |x-p| < r\}$. The remaining charts (with similar maps $\{q_j\}_{j=1}^k$) avoid the north and south poles.

Let φ_j be a partition of unity subordinate to the covering by charts. Now we have

$$\begin{aligned}\hat{\sigma}(\lambda e_n) &= \int e^{-2\pi i \lambda x_n} \mathrm{d}\sigma(x)\\ &= \sum_{j=1}^k \int e^{-2\pi i \lambda x_n} \varphi_j(x) \mathrm{d}\sigma(x)\\ &= \int_{D(0,1/2)} e^{-2\pi i \lambda \sqrt{1-|y|^2}} \frac{\varphi_1(y, \sqrt{1-|y|^2})}{\sqrt{1-|y|^2}} \mathrm{d}y + \int_{D(0,1/2)} e^{2\pi i \lambda \sqrt{1-|y|^2}} \frac{\varphi_2(y, -\sqrt{1-|y|^2})}{\sqrt{1-|y|^2}} \mathrm{d}y\\ &\quad + \sum_{j=3}^k \int_{U_j} e^{-2\pi i \lambda q_j(y)\cdot e_n} \varphi_j \circ q_j(y) \alpha_j(y) \mathrm{d}y\end{aligned}$$

where U_j are the domains of the q_j and the α_j are smooth functions that depend on q_j. Consider the phase functions of the integrals in the sum. It is easy to see that if $\nabla(e_n \cdot \lambda q_j)(y) = 0$, then the q_j is stationary in the e_n direction; since q_j is a coordinate map, this happens only at two places on the sphere–the north and south poles. But these are not in the support of φ_j for $j > 2$, so the phase there is nonstationary everywhere. This means that Theorem 3.2 applies in a neighborhood of every point, and so, by compactness, on the whole support of φ_j.

The other two integrals are each stationary at exactly one point ($y = 0$), and there the Hessian can be computed to be $-2I$, which is invertible. Using Theorem 3.3 when $y = 0$ and Theorem 3.2 elsewhere and combining with the above, we get

$$|\hat{\sigma}(\lambda e_n)| \leq C\lambda^{-\frac{n-1}{2}}.$$

Recalling from before that the Fourier transform of a finite measure is bounded gives

Theorem 3.4. *For all $\xi \in \mathbb{R}^n$, we have*

$$|\hat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}}.$$

4. Proof of the Tomas-Stein theorem

The following is a well-known interpolation theorem we will use in the proof below. Its proof can be found in [1].

Theorem 4.1 (Reisz-Thorin)**.** *Let T be a linear operator from $L^{p_0} + L^{p_1}$ with $1 \leq p_0 \leq p_1 \leq \infty$. Assume*

$$\|Tf\|_{q_0} \leq A_0 \|f\|_{p_0}$$

and

$$\|Tf\|_{q_1} \leq A_1 \|f\|_{p_1}$$

for some $q_0, q_1 \in [1, \infty]$. Then if for some $\theta \in (0,1)$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then

$$\|Tf\|_q \leq A_1^\theta A_0^{1-\theta}\|f\|_p.$$

A consequence of this is Young's inequality for convolutions:

Theorem 4.2 (Young)**.** *Let $f \in L^p$ and $g \in L^q$ with $\frac{1}{p}+\frac{1}{q} \geq 1$. Then if we take r such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1$,*

$$\|f * g\|_r \leq \|f\|_p\|g\|_q.$$

Proof. Let T be the convolution operator $Th = f * h$. We will prove the inequalities

$$\|Th\|_p \leq \|f\|_p\|h\|_1 \tag{A}$$

and

$$\|Th\|_\infty \leq \|f\|_p\|h\|_{p'} \tag{B}$$

where L^p and $L^{p'}$ are dual. Thinking of T as an operator from $L^{p'}+L^1$ to L^p+L^∞, then, we can apply Reisz-Thorin and set $\theta = 1-\frac{p}{r}$. Then

$$\frac{1-\theta}{1}+\frac{\theta}{p'} = \frac{p}{r}+\frac{1}{p'}\left(1-\frac{p}{r}\right) = \frac{p}{r}\left(1-\frac{1}{p'}\right)+\frac{1}{p'} = \frac{p}{r}\frac{1}{p}+1-\frac{1}{p} = \frac{1}{q}$$

and

$$\frac{1-\theta}{p}+\frac{\theta}{\infty} = \frac{1}{r},$$

so

$$\|f * g\|_r = \|Tg\|_r \leq \|f\|_p^{1-\theta}\|f\|_p^\theta\|g\|_q = \|f\|_p\|g\|_q$$

as desired.

To prove inequality (A), we use Reisz-Thorin again, this time fixing $h \in L^1$ and letting $Rf = f * h$. Then if f is bounded, clearly we have

$$|Rf| \leq \int |f(x-y)||g(y)|\mathrm{d}y \leq \|f\|_\infty\|g\|_1.$$

If f is in L^1, then

$$\|Rf\|_1 \leq \iint |f(x-y)g(y)|\mathrm{d}y\mathrm{d}x = \iint |f(x-y)|\mathrm{d}x|g(y)|\mathrm{d}y = \int \|f\|_1|g(y)|\mathrm{d}y = \|f\|_1\|g\|_1.$$

We tus have R bounded $L^1 \to L^1$ and $L^\infty \to L^\infty$, so by Reisz-Thorin we have $\|f * h\|_p = \|Rf\| \leq \|f\|_p\|h\|_1$ as desired.

To finish the proof, we observe that inequality (B) is a trivial consequence of Holder's inequality. □

Proof of Tomas-Stein. Recall that from Theorem 2.4, it suffices to show that for $p' < \frac{2n+2}{n+3}$ (or, equivalently, $p > \frac{2n+2}{n-1}$) $\|\hat{\sigma} * f\|_{L^p} \leq C\|f\|_{L^{p'}}$ for all Schwartz functions f. Since $\mathcal{S}$ is dense in $L^{p'}$, this will imply Tomas-Stein as stated in Theorem 1.4.

Next, let ψ be a smooth function $\mathbb{R}^n \to [0,1]$ that is 0 when $|x| < 1/2$ and 1 when $|x| > 1$. Then let $\phi(x) = \psi(2x) - \psi(x)$. Then when $|x| > 1$, $\phi(x) = 1-1 = 0$, and when $|x| < 1/4$, $\phi(x) = 0-0 = 0$. Moreover, for $|x| > 1$,

$$\sum_{k \geq 0} \phi(2^{-k}x) = 1.$$

This means that we can write

$$\hat{\sigma} = \left(1 - \sum_{k\geq 0} \phi(2^{-k}x)\right)\hat{\sigma} + \sum_{k\geq 0}(\phi(2^{-k}x)\hat{\sigma}) = K_{-\infty} + \sum_{j=0}^{\infty} K_j,$$

where we use $K_{-\infty} = \left(1 - \sum_{j\geq 0} \phi(2^{-j}x)\right)\hat{\sigma}$ and $K_j = \phi(2^{-j}x)\hat{\sigma}$.

$K_{-\infty}$ has compact support by construction. This means that by Young's inequality,

$$\|K_{-\infty} * f\|_{L^p} \leq \|K_{-\infty}\|_{L^q}\|f\|_{L^{p'}} \tag{4.3}$$

where $\frac{1}{q} = 2 - \frac{2}{p'}$, which works because $p' > \frac{2n+2}{n+3} \geq 2$. As was noted earlier, $\hat{\sigma}$ is bounded, so $\|K_{-\infty}\|_{L^q}$ is finite and depends only on dimension.

In the previous section, we obtained the estimate $|\hat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}}$, where C depends only on n (Theorem 3.4). But because of the conditions on the support of K_j, this gives $\|K_j\|_{L^\infty} \leq C2^{-j\frac{n-1}{2}}$. Using a simple case of Young's inequality,

$$\|K_j * f\|_{L^\infty} \leq \|K_j\|_{L^\infty}\|f\|_{L^1} \leq C2^{-j\frac{n-1}{2}}\|f\|_{L^1}. \tag{4.4}$$

At this point we need to make this observation:

$$\sigma(D(x,r)) \leq Ar^{n-1} \tag{4.5}$$

where $D(x,r) = \{x \in \mathbb{R}^n |\ |x| < r\}$ and A depends only on n. This is clear since S^{n-1} is an $(n-1)$ dimensional submanifold of $\mathbb{R}^n$.

Next, because of the radial symmetry of the sphere σ, and so $\hat{\sigma}$, is invariant under reflection $x \to -x$, and so $\check{\sigma}(\xi) = \hat{\sigma}(-\xi) = \hat{\sigma}(\xi)$. Then we can apply Lemma 2.2(a) to K_j:

$$\hat{K}_j = \widehat{\phi_{2^{-j}}\hat{\sigma}} = \widehat{\phi_{2^{-j}}\check{\sigma}} = 2^{nj}\hat{\phi}_{2^j} * \sigma,$$

where $g_a(x) = g(ax)$. Now, $\hat{\phi}$ is Schwartz, so for each N there is a constant C_N such that $|\hat{\phi}(\xi)| \leq C_N(1+|\xi|)^{-N}$. Letting $N = n$, we can perform the following computation, decomposing the integral into a sum over annular regions:

$$\begin{aligned}
|\hat{K}_j(\xi)| &= 2^{jn}\left|\int \hat{\phi}(2^j(\xi - y))\mathrm{d}\sigma(y)\right| \\
&\leq C_n 2^{jn}\int (1 + 2^j|\xi - y|)^{-n}\mathrm{d}\sigma(y) \\
&= C_n 2^{jn}\left(\int_{D(\xi,2^{-j})}(1 + 2^j|\xi - y|)^{-n}\mathrm{d}\sigma(y) + \sum_{k\geq 0}\int_{D(\xi,2^{k+1-j})-D(\xi,2^{k-j})}(1 + 2^j|\xi - y|)^{-n}\mathrm{d}\sigma(y)\right) \\
\text{(A)}\quad &\leq C_n 2^{jn}\left(\sigma(D(\xi,2^{-j})) + \sum_{k\geq 0} 2^{-nk}\sigma(D(\xi,2^{k+1-j}) - D(\xi,2^{k-j}))\right) \\
\text{(B)}\quad &\leq C_n 2^{jn}\left(A2^{-j(n-1)} + \sum_{k\geq 0} A2^{-nk}2^{(k+1-j)(n-1)}\right) \\
\text{(C)}\quad &= C_n 2^{jn}\cdot 3A2^{n-1}2^{-j(n-1)} = 3AC_n 2^{n-1}2^j.
\end{aligned}$$

Step (A) bounds each integral by the product of the integrand's supremum and the measure of its support. Step (B) uses the observation (4.5) to approximate the measures of the disks. Step (C) evaluates the geometric series on the right and simplifies the expression.

Now we have the following bound:

$$\|K_j * f\|_{L^2} = \|\widehat{K_j * f}\|_{L^2} = \|\hat{K}_j \hat{f}\|_{L^2} \leq \|\hat{K}_j\|_{L^\infty} \|\hat{f}\|_{L^2} \leq Q 2^j \|f\|_{L^2}, \tag{4.6}$$

where Q is the constant $3AC_n 2^{n-1}$ from the computation. We used Plancharel's theorem twice to pass to the Fourier transform and back again.

But this means the convolution operator $T_j f = K_j * f$ is bounded $L^1 \to L^\infty$ by (4.4) and $L^2 \to L^2$ by (4.6). By Reisz-Thorin,

$$\|K_j * f\|_p \leq Q^\theta 2^{j\theta} 2^{-j\frac{n-1}{2}(1-\theta)} \|f\|_{p'}$$

where $\theta = \frac{2}{p}$. This is because

$$\frac{\theta}{2} + \frac{1-\theta}{\infty} = \frac{1}{p}$$

and

$$\frac{\theta}{2} + \frac{1-\theta}{1} = 1 - \frac{1}{p} = \frac{1}{p'}.$$

Then

$$\|K_j * f\|_p \leq Q^{2/p} 2^{2j/p} 2^{-j\frac{n-1}{2}(1-2/p)} \|f\|_{p'} = Q^{2/p} 2^{j(\frac{n+1}{p} - \frac{n-1}{2})} \|f\|_{p'}. \tag{4.7}$$

The exponent is negative when $p > \frac{2n+2}{n-1}$, as desired.

The remaining step is the simple observation that

$$\hat{\sigma} * f = K_{-\infty} * f + \sum_{j \geq 0} K_j * f.$$

Then combining (4.3) and (4.7), both the left and right sides are bounded for $p < \frac{2n+2}{n-1}$ and so $\|\hat{\sigma} * f\|_p \leq C\|f\|_{p'}$, which completes the proof. □

Remark 4.8. No special properties of the sphere were used in the proof. The relevant facts were Theorem 3.4 and equation (4.5). The latter only requires the surface to be a compact submanifold of codimension 1, while the former needs nondegeneracy at the critical points of the phase functions. This turns out to be equivalent to having nonzero Gaussian curvature. We will use this in the next section in deriving Strichartz estimates for the Schroedinger equation.

5. Strichartz Estimates

An application of the Tomas-Stein theorem is the derivation of Strichartz estimates for partial differential equations. These bound the L^p norm of the solution in terms of Banach space norms of the initial data. They are a powerful tool in dealing with the limiting processes often necessary to answer existence and uniqueness questions. A simple example is the homogeneous Schroedinger equation:

$$\begin{cases} -i\partial_t u + \frac{1}{2\pi}\Delta u &= 0 \\ u(x,0) &= f(x) \end{cases} \tag{5.1}$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ depends on position (x) and time (t), ∂_t is the time derivative, and Δ is the Laplacian in the position variables only. We can fix time and take the

x Fourier transform (which we denote with the usual hat symbol) of the equation, which gives the ordinary differential equation

$$\left\{ \begin{aligned} -i\partial_t \hat{u}(\xi,t) - 2\pi|\xi|^2\hat{u}(\xi,t) &= 0 \\ \hat{u}(\xi,0) &= \hat{f}(\xi) \end{aligned} \right. \tag{5.2}$$

where we used the property of the Fourier transform that $\widehat{\partial_{x_j} u}(\xi) = (2\pi i\xi_j)\hat{u}(\xi)$ on the Laplacian. But (5.2) is easy to solve and yields

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{2\pi it|\xi|^2}.$$

Now we just need to take the inverse Fourier transform to get

$$u(x,t) = \int \hat{f}(\xi)e^{2\pi i(\xi\cdot x + t|\xi|^2)}\mathrm{d}\xi. \tag{5.3}$$

This can be reexpressed as the inverse Fourier transform of $\hat{f}\mu$, where μ is the measure in $\mathbb{R}^n \times \mathbb{R}$ given by

$$\int_{\mathbb{R}^n\times\mathbb{R}} \phi(\xi,t)\mathrm{d}\mu = \int_{\mathbb{R}^n} \phi(\xi,|\xi|^2)\mathrm{d}\xi$$

for $\phi \in C^0$. This is a measure supported on the surface of a paraboloid. By Remark 4.8, the Tomas-Stein theorem applies to $\psi\mu$ where $\psi \in C_0^\infty$ and is 1 on the unit ball in $\mathbb{R}^n \times \mathbb{R}$. The paraboloid is n dimensional, so for $p \geq \frac{2n+4}{n} = 2 + \frac{4}{n}$,

$$\|(\hat{f}\mu)\check{}\|_{L^p} \leq C\|\hat{f}\|_{L^2(\mu)} = C\|\hat{f}\|_{L^2(\mathbb{R}^n)} \tag{5.4}$$

for f with $\hat{f}$ supported in the unit ball. The last equality follows from the fact that integrating a time-independent function with respect to μ is the same as integrating it with respect to $\mathbb{R}^n$.

The left hand side of (5.4) is $\|u\|_{L^p}$ by construction, while the right hand side is $C\|f\|_{L^2(\mathbb{R}^n)}$ by Plancharel, so

$$\|u\|_{L^p(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{5.5}$$

Now let $\hat{f}$ have support in $D(0,\lambda)$. If we let $f_\lambda(x) = f(\frac{x}{\lambda})$ and $u_\lambda(x,t) = u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$, (5.1) implies

$$\left\{ \begin{aligned} -i\partial_t u_\lambda + \tfrac{1}{2\pi}\Delta u_\lambda &= 0 \\ u_\lambda(x,0) &= f_\lambda(x) \end{aligned} \right. \tag{5.6}$$

by the chain rule. But $\hat{f}_\lambda(\xi) = \lambda^{-n}\hat{f}(\lambda\xi)$, which has support in $D(0,1)$. By change of variables, $\|f_\lambda\|^2_{L^2(\mathbb{R}^n)} = \lambda^n\|f\|^2_{L^2(\mathbb{R}^n)}$ and $\|u_\lambda\|^p_{L^p(\mathbb{R}^{n+1})} = \lambda^{n+2}\|u\|^p_{L^p(\mathbb{R}^{n+1})}$. Combining this with (5.5) gives

$$\|u\|_{L^p(\mathbb{R}^{n+1})} = \lambda^{\frac{p}{n+2}}\|u_\lambda\|_{L^p(\mathbb{R}^{n+1})} \leq C\lambda^{\frac{p}{n+2}}\|f_\lambda\|_{L^2(\mathbb{R}^n)} = C\lambda^{\frac{p}{n+2}}\lambda^{\frac{n}{2}}\|f\|_{L^2(\mathbb{R}^n)}, \tag{5.7}$$

and the left hand side is $C\|f\|_{L^2(\mathbb{R}^n)}$ when $p = 2 + \frac{4}{n}$. This means that for that value of p, (5.5) holds for all $f \in L^2$ with compact support (uniformly in C). By the density of compactly supported functions, this means it holds for all f in L^2, which is the Strichartz estimate we wanted.

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