NIM

KAREN YE

ABSTRACT. In this paper I analyze the game Nim, starting with an introduction of the game and ending with the solution. Within, I explain some of the theory behind Nim and other combinatorial games, give a cursory overview of combinatorial game theory, and describe my interactions with the game.

CONTENTS

1. NIM

Nim is a two-person combinatorial game played with piles of beans. The game is set up with an arbitrary number of beans in an arbitrary number of piles. Players take alternating turns. When it is his turn to move, a player picks a pile of beans and removes beans from it. The player may take as many beans as he wants—the whole pile, even—provided that he moves at least one, and only from that pile. The game ends when there are no beans left, and all of the piles are empty. The player who removes the last bean of the entire game wins.
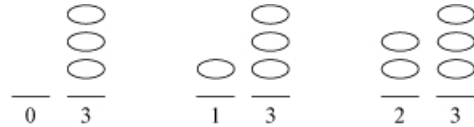
2. 3–3

Consider the following Nim game. There are two piles, and each pile holds three beans. It will be denoted in this manner—
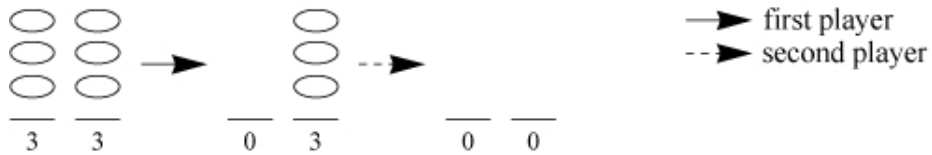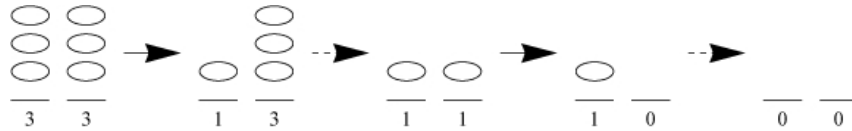

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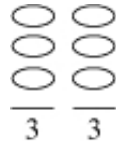
Here are the options the first player may choose.

Suppose the first player chooses 0–3; the second player is able to win the game by removing three beans from the remaining pile.

If the first player tries again, and chooses 1–3,

The second player wins again.

The first player has only one other option he has not made before, 2–3.
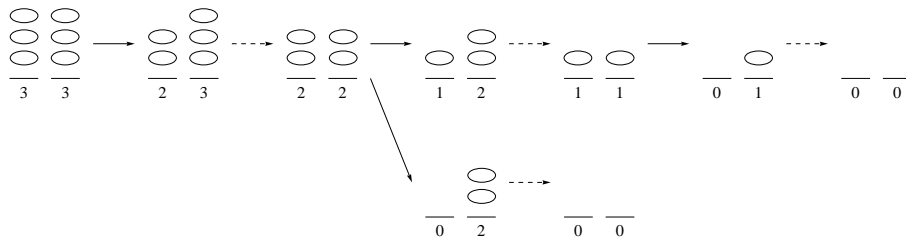
When he reaches 2–2, the first player may play 1–2 and 0–2, but we see that in both cases, the second player has found some way to win.

In fact, here we see that no matter which move the first player makes, there is always a move the second player can make that will force the first player to lose. The first player does not have such a luxury, as with any move he makes he is able to lose, and he does not have a sure-fire way of forcing the second player to lose.

Definition 2.1. A *winning strategy* is a set of moves a player can make that will force his opponent to lose the game. If a player has a winning strategy for a specific game, then that game is a *winning game* for the player with the winning strategy, and a *losing game* for his opponent.

By now, we can tell that 3–3 is a losing game for the first player to move, and a winning game for the second. It would probably be a good idea to force one's opponent into such a position.

We see that there are definite losing games. There are also definite winning games. In fact, in the next section we will see that all games must fall into one of these two categories.

3. Winning and Losing Games

Proposition 3.1. *For all games G, G must be either a winning game or a losing game.*

Definition 3.2. By *game*, I don't mean Nim but a certain starting position of Nim, from which players then take turns moving until no beans are left.

Let us start with all of the games in which the first player can win (go to 0–0) with a single move.
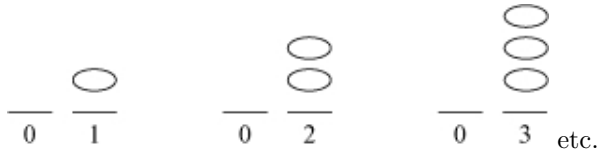
These are all winning games—the first player has a winning strategy, namely, making the move 0–0.

Consider the game 3–3 previously mentioned. We know that if a player makes a move that results in giving this game to his opponent, then he has a winning strategy, because he then has a set of moves that will force his opponent to lose.

So all of the games in which the first player can make the game to 3–3 with a single move are winning games.
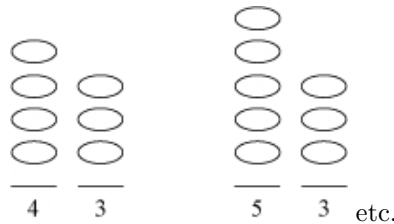
Generalizing this,

Definition 3.3. A *winning game* is a game in which the first player is able to give his opponent a losing game with a single first move.

Definition 3.4. A *losing game* is a game in which any move the first player makes will not give his opponent a losing game. As a game is a winning game if it is not a losing game, any of the first player's options for the first move results in a winning game for his opponent.

Assuming both players play optimally, a winning game will result in a win for the first player and a losing game a win for the second player.

Theorem 3.5. *For all games G, G must be either a winning game or a losing game.*

Proof. Let N be the set of all winning games, and let P be the set of all losing games.

For each game, we introduce its game-graph $R = (V, E)$, a finite acyclical digraph where V is the set of game positions and $(u, v) \in E$ if and only if there exists a move from u to v.

Define the set of *followers* of $u \in V$ as

$$F_G(u) = F(u) = \{v \in V : (u, v) \in E\}$$

The definitions of winning and losing games are that

(1) $u \in P$ iff $F(u) \subseteq N$.
(2) $u \in N$ iff $F(u) \cap P \neq \emptyset$.

Suppose u_1 is in neither P nor N. Therefore, by (1) $F(u_1) \nsubseteq N$, and by (2) $F(u_1) \cap P = \emptyset$. We have now the next move $u_2 \in F(u_1)$ in neither P nor N, and consequently a chain of $u_3, u_4, \ldots u_i$ in neither P nor N where $u_i \in F(u_{i-1})$, $i \geq 2$. But we know that the end game (the game 0) $u_i \in P$, and the penultimate game $u_{i-1} \in N$. There is a contradiction, and therefore u_j, where $1 \leq j \leq i$, must be either in P or N.

$\square$

4. Combinatorial Games

Nim belongs to a class of games called combinatorial games. Combinatorial game theory is a mathematical theory dealing with two-player games of perfect information and no chance moves. Known to both players are the rules, moves available for both players, and defined winning condition, and combinatorial game theorists attempt to find the optimal strategy for each position of a game.

In contrast is classical game theory, in which events may happen simultaneously, and alliances and coalitions are allowed.

Definition 4.1. A *combinatorial game* is a game with the following properties:

(1) There are only two players, and no coalitions. The two players are commonly called **L**eft and **R**ight.
(2) There are a finite number of positions the game can take, and often a starting position. Each position may be defined as a new game.
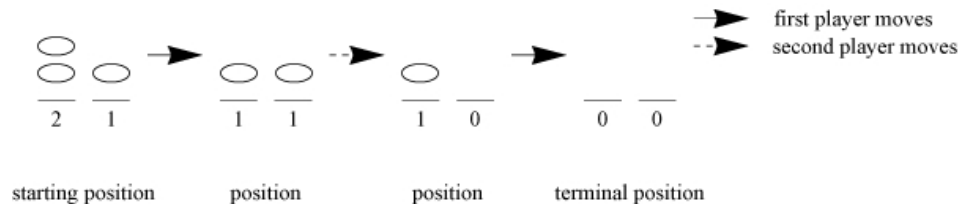
(3) There are clearly defined rules, and at each position P Left player has a set of moves he can make, and Right has a set of moves he can make.
(4) Left and Right alternate turns.
(5) In the normal play convention, which is the one we are analyzing, a player who cannot move loses.
(6) The game will end, as some player will find himself unable to move. Draws by repetition are not allowed.
(7) Both players have complete information and neither is able to bluff.
(8) No part of the game is decided by chance (i.e. rolling dice).

Some other examples of combinatorial games are Go, Dots-and-Boxes, chess, and checkers.

A generalization of the previously mentioned theorem—proved similarly—is

Theorem 4.2 (Fundamental Theorem of Combinatorial Game Theory)**.** *Let Γ be a 2-person game with perfect information and no chance moves, whose game-graph may be infinite. Then for every position of Γ there are precisely two possibilities: (i) There exists a winning move for precisely one of the two players. (ii) There is a winning move for neither player, but both can maintain a draw.*

The implications of this theorem are that there are optimal moves for all positions in combinatorial games. Imagine playing chess with a specific winning strategy—knowing exactly which moves will guarantee your win by forcing a losing game upon your opponent! It is just that many games are too complicated to analyze.

5. Numbers

We will be dealing with John H. Conway's surreal numbers.

Conway starts with—

Dedekind constructed the real numbers by splitting the rational numbers into two sets, left and right; the number x is between the set of all rational numbers less than it and the set of all rational numbers greater than it, written as $\{L \mid R\}$.

Cantor constructed the infinite ordinal numbers by supposing the integers $\{1, 2, 3, ...\}$ given, their order-type w is the infinite number greater than all of the integers, and the order-type $w + 1$ is the infinite number greater than $\{1, 2, 3, ..., w\}$ and so on.

Von Neumann adjusted Cantor's construction, making each ordinal number the set of all ordinal numbers before it. In his construction,

$\{\}$ is 0,

$\{0\}$ is 1,

$\{0, 1\}$ is 2,

$\{0, 1, 2, ...\}$ is w, and so on.

The surreal numbers are a number system encompassing both the real and ordinal numbers; the surreal number x is the simplest number between two sets of numbers, x^L and x^R, where no element of x^L is greater than or equal to any element of x^R.

$$x = \{x^L \mid x^R\}$$

Simplest, here, means earliest created.

Construction—

Before any numbers are created, we have the set of numbers $\emptyset$. So plugging $\emptyset$ into $\{L \mid R\}$ (where L and R are both sets of numbers), we get $\{\mid\}$. We call this number 0.

Now we have two elements that we can make sets of numbers from, 0 and $\emptyset$. We get possible numbers $\{0 \mid 0\}$, $\{0 \mid\}$, $\{\mid 0\}$, and $\{\mid\}$.

One of the conditions of being a number is that no member of x^L can be greater than or equal to a member of x^R. Since $0 = 0$, $\{0 \mid 0\}$ is not a number. Also, $\{\mid\}$ we have seen before.

The two numbers that are born are $\{0 \mid\} = 1$ and $\{\mid 0\} = -1$.

Now, with $\emptyset$, 0, 1, -1, we give birth to:

$\{\mid -1, 0, 1\} = -2$,
$\{-1 \mid 0, 1\} = -\frac{1}{2}$,
$\{-1, 0 \mid 1\} = \frac{1}{2}$,
$\{-1, 0, 1 \mid\} = 2$.

Note that only the least element of x^R and greatest element of x^L are important. $\{\mid 0, 1, 2\}$ produces the same number as $\{\mid 0, 1\}$ and $\{\mid 0\}$.

We later on get numbers such as $\{0 \mid \frac{1}{2}\} = \frac{1}{4}$, $\{\frac{1}{2} \mid \frac{3}{4}\} = \frac{5}{8}$, $\{0, 1, 2 \mid\} = 3$ and so on.

We can draw a tree of births.

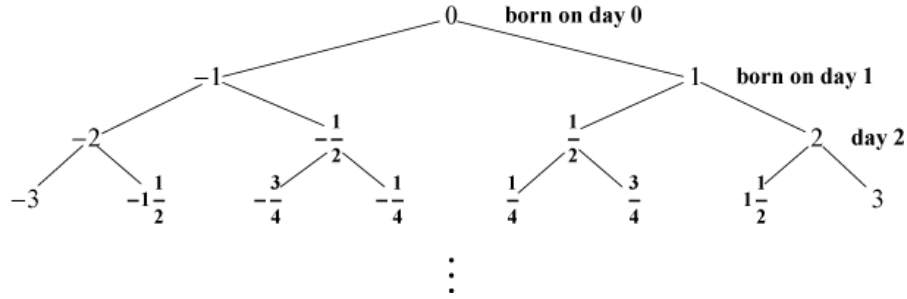FIGURE 1. Tree of Number Births

This is how the surreal numbers were born.
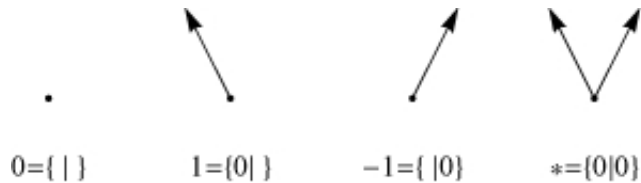
6. GAMES AS NUMBERS

The mathematical definition of a game G is

Definition 6.1.
$$G = \{G^L \mid G^R\}$$

where G^L is the set of all of the moves Left can make, and G^R the set of all of Right's moves.

The games born earliest are:

$0 = \{\mid\}$ $1 = \{0 \mid\}$ $-1 = \{\mid 0\}$ $* = \{0 \mid 0\}$

Here, we see that the 0 game is the end game; neither player has a move, so as the first player who cannot make a move loses, the first player to go in this case loses, and the second player wins.

In the case of the game with assigned value 1, we see that Left has a move, namely, to bring the game to the 0 game, while Right has no move. Right loses.

It is the opposite in the game assigned -1. Right wins, left loses.

In the * game, both players can make the move 0. It depends entirely on which player moves first, so the game is a first-player win.

The four games belong in four outcome classes—

If the game is 0, the *second player* wins.

If the game is *positive*, *Left* wins.

If the game is *negative*, *Right* wins.

If the game is *, *first player* wins.

7. Nim as Numbers

I wanted to see if I could assign the same types of numbers to Nim. What followed was a process of trial-and-error as I tried to work out a number system that would represent the game.

I assigned numbers to Nim games using the $G = \{G^L \mid G^R\}$ construction.

Nim is an impartial game; that is, both players are allowed to make the same types of moves. The winner and loser are determined by who goes first, as a game is either a winning game or a losing game. As such, I decided to make the Left player the first player, and the Right player the second. Otherwise, the game would be $G^L = G^R$, $G = \{G^L \mid G^R\} = *$, an arrangement that I thought would be boring[1].

I started from the beginning (or the end, depending on which way you look at it).

7.1. **One Pile.** I started with hypothetically one pile of beans, with x beans in the pile.

This is the end game

Player one has no moves, and player two has no moves. The value I assigned to this game is $\{\mid\} = 0$.

Then

We already know it is a winning game, so player one should win, and the number should be positive.

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[1]

Many mathematicians have analyzed Nim before, most notably Charles L. Bouton. Ironically, they used $G^L = G^R$, assigning $*$ and an ordinal number to each Nim-pile.

Player one has one move—he can make the move 0, taking away one bean and ending the game. Player two, on the other hand, has no moves. He moves second, and there are no beans left. This game receives the value $\{0 \mid\} = 1$.

Let us look at

There are two moves that the first player can make.
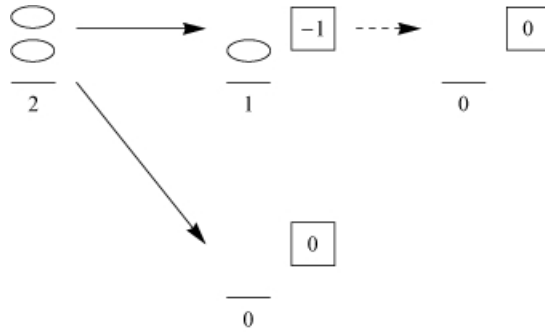
I made the 1 game -1, because although the game has a value 1, it is a value 1 for the second player, leading to a second-player win. So it is a -1 for the first player, in the context of the game 2.

Definition 7.1. The *negative* of a game,

$$-G = \{G^R \mid G^L\}$$

where the roles of the first and second player are switched.

Grouping all of the moves together, $G = \{-1, 0 \mid 0\}$.

Here, I ran into a problem. $0 \in G^L$, $0 \geq 0 \in G^R$. $G = \{-1, 0 \mid 0\}$ is not a number.

I felt very strongly, however, that the Nim game of two beans in one pile should be a positive number, a first-player win. I decided to follow the game along the path of player one's best move, assuming that player one played optimally.

Between
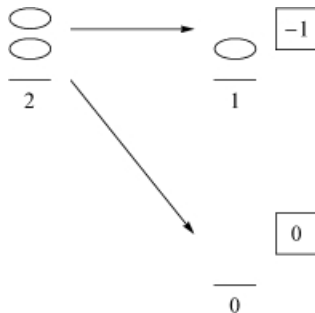
I picked the move 0 and the path that ensued, because this path led to player one winning, whereas he would lose taking the other. The game becomes $\{0 \mid \} = 1$.

The way in which these numbers are assigned, the game of one bean is equivalent to the game of two beans, because the two games share the same best move, 0.
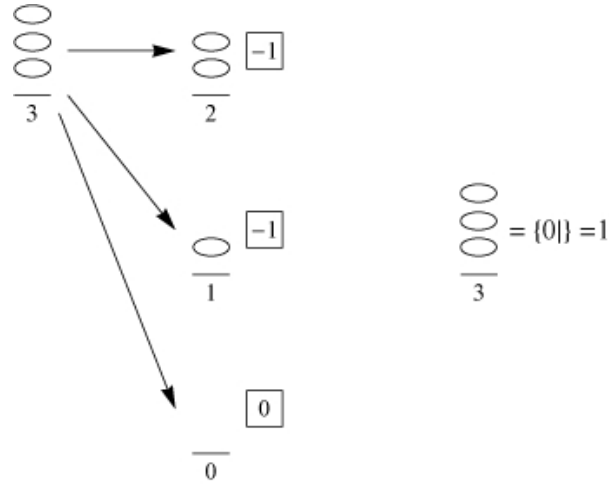
Next is the game 3, which came out with the value 1 as well.

So, I assigned the value $(+)1$ to all Nim-games of one pile, with the exception of the 0 game. The only game that is a losing game in one-pile Nim is the game 0. All others are winning games.

With one pile done, I moved on to two piles.

7.2. **Two Piles.** With two piles, we have x beans in one pile and y in the other.

The end game is 0–0, which is equivalent to 0–0–0, 0–0–0–0, etc. and 0.

Next, 1–1. We know this game is a losing game, so hopefully it is negative. $\{-1 \mid 0\} = -\frac{1}{2}$. It is!
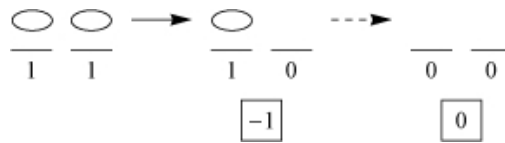
2–1, played right, should be a winning game, as there is a winning strategy for player one.

Note that here, the game 1–1 is valued $+\frac{1}{2}$, because it is a $-\frac{1}{2}$ to player two (player two becomes the first player in the game 1–1). 1–0 is again $+1$ because it is a game to first's advantage and first player is the first player in the game 1–0. The game 2–1 becomes $\{0, \frac{1}{2} \mid 1\} = \frac{3}{4}$.

3–1 has the same best move (only good move) as 2–1, so has the same value, $\frac{3}{4}$ as 2–1.
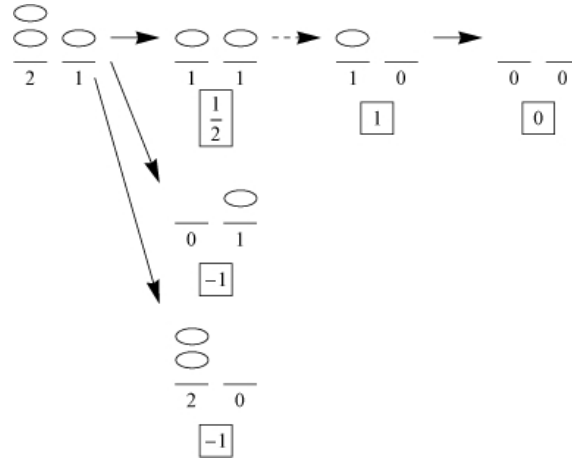
Lemma 7.2. *If games G and H have the same best move, then they have the same value.*

As a consequence, two-pile games of Nim in which $x = 1$ and $y \geq 2$ are all valued $\frac{3}{4}$.

Best move just means which path I chose to follow in order to come up with values for the games. In winning games, the best move would be to move to a losing game. Sometimes there is more than one losing game you can go to, so I decided to follow the path of the losing game with the higher value (which would be the highest positive valued move for the first player). In losing games, any move the first player can make would be negative, so I went with the path that was least negative, still the highest value. Following the highest valued path became an easy rule to follow, as I encountered more games.

Now suppose $x = 2$. We have seen these before: 2–0= $\{0 \mid\} = 1$. 2–1= $\{0, \frac{1}{2} \mid 1\} = \frac{3}{4}$.

The 2–2 game, however, is new.
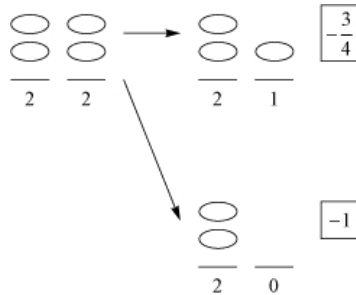
Player one may choose 2–1 and 2–0, but both paths are negative. Between $-\frac{3}{4}$ and -1, I chose to follow the path of $-\frac{3}{4}$, the higher value.

Assuming player one makes the move 2–1,
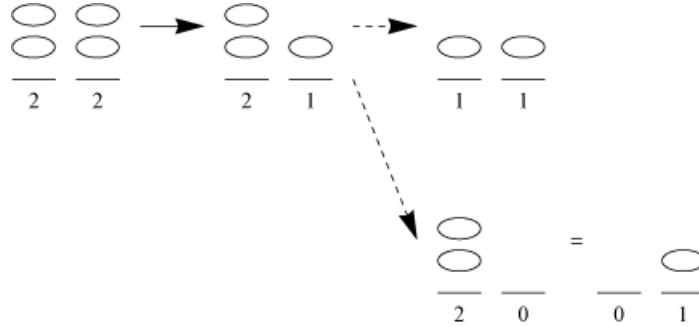
We run into the same problem, unless we assume that player two plays optimally as well.

The path thus far becomes
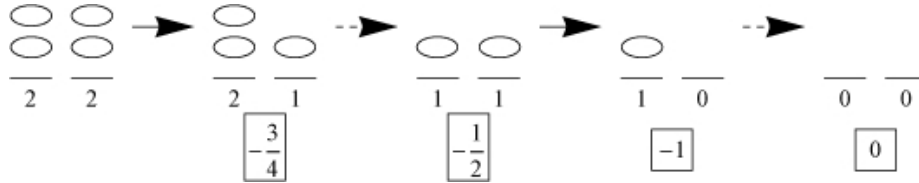
The set of Left's moves, and the set of Right's moves, create $G = \{-1, -\frac{3}{4} \mid -\frac{1}{2}, 0\} = \{-\frac{3}{4} \mid -\frac{2}{4}\} = -\frac{5}{8}$.

Now 3–2.
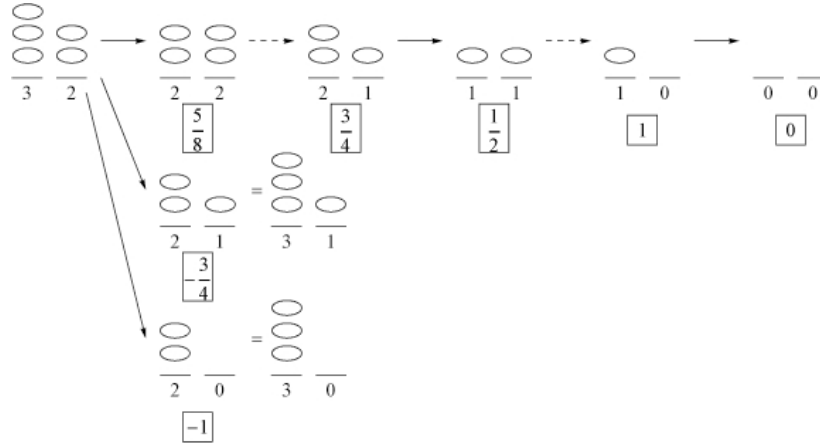
The game 3–2 becomes $\{0, \frac{1}{2}, \frac{5}{8} \mid \frac{3}{4}, 1\} = \frac{11}{16}$.

2–4, 2–5, etc.— Two-pile Nim-games with $x = 2$ and $y \geq 3$ are all thus valued $\frac{11}{16}$.

Following the method described, table 1 is a table of two-pile Nim-game values.

Table 1. Two-Pile Nim-game Values, x–y

	1	2	3	4	5	6	7	8
1	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
2	$\frac{3}{4}$	$-\frac{5}{8}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$
3	$\frac{3}{4}$	$\frac{11}{16}$	$-\frac{21}{32}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$
4	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{43}{64}$	$-\frac{85}{128}$	$\frac{171}{256}$	$\frac{171}{256}$	$\frac{171}{256}$	$\frac{171}{256}$
5	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{171}{256}$	$-\frac{341}{512}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{683}{1024}$
6	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{171}{256}$	$\frac{683}{1024}$	$-\frac{1365}{2048}$	$\frac{2731}{4096}$	$\frac{2731}{4096}$
7	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{171}{256}$	$\frac{683}{1024}$	$\frac{2731}{4096}$	$-\frac{5461}{8192}$	$\frac{10923}{16384}$
8	$\frac{3}{4}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{171}{256}$	$\frac{683}{1024}$	$\frac{2731}{4096}$	$\frac{10923}{16384}$	$-\frac{21845}{32768}$

In the table, the negative values are the losing games, and the values of the winning games are determined by which losing game the player can reach in one move. Therefore all games which can be reduced to 1–1 in one step are valued $\frac{3}{4}$, and all games which can be reduced to 2–2 in one step are valued $\frac{11}{16}$, and so on.

As seen in the table, there is a clear diagonal of losing games. Two-pile Nim-games are losing games when $x = y$. All other two-pile Nim-games are winning games.

7.3. **Three Piles.** In three-pile Nim, we have three piles of beans, with x beans in one pile, y beans in another, and z in the third.

Here are some tables I made for three-pile Nim-games: table 2, and table 3 on page 13. Table 2 supposes $x = 1$, and is fixed. Table 3 supposes $x = 2$.

We can already see a few losing games: 1–2–3, 1–4–5, 1–6–7, and 2–1–3, 2–4–6, 2–5–7. As with two piles, the values of the winning games are determined by which losing game(s) the first player is able to reach.

8. The Game Tree

If we correspond our game values to the number tree, we see figure 2 on page 15. The first number is 0, and with it comes all of the games valued 0—which is just the end game, 0. Then games valued 1 are games in which the player can move to a 0 game in one step. The game 1–1, valued $(-)\frac{1}{2}$ is a losing game, and all of the games that can reach 1–1 in one step are valued $\frac{3}{4}$. Then games valued $(-)\frac{5}{8}$, all of the games that can reach $(-)\frac{5}{8}$ in one step are valued $\frac{11}{16}\ldots$

The number system I assigned follows the path converging to $\frac{2}{3}$. Indeed, that is how the number $\frac{2}{3}$ is created: $\frac{2}{3} = \{0, \frac{1}{2}, \frac{5}{8}, \frac{21}{32} \ldots \mid \ldots \frac{43}{64}, \frac{11}{16}, \frac{3}{4}, 1\}$. The games

TABLE 2. Three-Pile Nim-game Values, 1–y–z

	1	2	3	4	5	6	7	8
1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
2	$\frac{3}{4}$	$\frac{11}{16}$	$-\frac{21}{32}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$
3	$\frac{3}{4}$	$-\frac{21}{32}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$
4	$\frac{3}{4}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{171}{256}$	$-\frac{341}{512}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{683}{1024}$
5	$\frac{3}{4}$	$\frac{43}{64}$	$\frac{43}{64}$	$-\frac{341}{512}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{683}{1024}$
6	$\frac{3}{4}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{2731}{4096}$	$-\frac{5461}{8192}$	$\frac{10923}{16384}$
7	$\frac{3}{4}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$-\frac{5461}{8192}$	$\frac{10923}{16384}$	$\frac{10923}{16384}$
8	$\frac{3}{4}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{10923}{16384}$	$\frac{10923}{16384}$	$\frac{43691}{65536}$

TABLE 3. Three-Pile Nim-game Values, 2–y–z

	1	2	3	4	5	6	7	8
1	$\frac{3}{4}$	$\frac{11}{16}$	$-\frac{21}{32}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$
2	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{11}{16}$
3	$-\frac{21}{32}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$	$\frac{43}{64}$
4	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{171}{256}$	$\frac{683}{1024}$	$-\frac{1365}{2048}$	$\frac{2731}{4096}$	$\frac{2731}{4096}$
5	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{683}{1024}$	$\frac{683}{1024}$	$\frac{2731}{4096}$	$-\frac{5461}{8192}$	$\frac{10923}{16384}$
6	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{43}{64}$	$-\frac{1365}{2048}$	$\frac{2731}{4096}$	$\frac{2731}{4096}$	$\frac{10923}{16384}$	$\frac{2731}{4096}$
7	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{2731}{4096}$	$-\frac{5461}{8192}$	$\frac{10923}{16384}$	$\frac{10923}{16384}$	$\frac{10923}{16384}$
8	$\frac{43}{64}$	$\frac{11}{16}$	$\frac{43}{64}$	$\frac{2731}{4096}$	$\frac{10923}{16384}$	$\frac{2731}{4096}$	$\frac{10923}{16384}$	$\frac{43691}{65536}$

valued less than $\frac{2}{3}$ are losing games, and the games with values greater than $\frac{2}{3}$ are winning games.

Gameplay the way I defined it follows such a pattern—each player makes moves that are games created earlier. If it is a winning game, the first player moves to the highest losing game option, following the path to the losing game preceding it. If it is a losing game, the first player moves to the highest negative game option, which would be following the path to the winning game(s) preceding it. Starting from whichever position, players follow the path upwards until 0 is reached.

Sometimes there were two or more losing games the first player could reach from a winning game, such as with the game 2–2–3, which could move to either 2–2 or 1–2–3. In the way that I assigned the numbers, the player moves to the highest negative value. When actually playing the game, a player could choose other losing games to move to, some would lead him to win in fewer steps. As previously mentioned, moving to the highest value was a rule that helped in assigning numbers to games. However, when following this rule, winning games were valued positive numbers, and losing games negative numbers. This is because the highest valued move for a winning game would be a positive number for the first player, as it would be a losing game, and the highest valued move for a losing game would be the highest negative, but still negative.

9. Patterns

When I was assigning numbers to Nim-games, I noticed patterns in losing games.
Take three-pile Nim, with x, y, and z; all of the losing games fell into certain rules.

If $x = 0$, G is a losing game if $y = z$.
Losing games are 0–0–0, 0–1–1, 0–2–2, 0–3–3, 0–4–4, etc.

If $x = 1$, G is a losing game if $y \equiv 0 \pmod{2}$, $z = y + 1$.
Losing games are 1–0–1, 1–2–3, 1–4–5, 1–6–7, 1–8–9, 1–10–11, etc.

If $x = 2$, G is a losing game if $y \equiv 0, 1 \pmod{4}$, $z = y + 2$.
Losing games are 2–0–2, 2–1–3, 2–4–6, 2–5–7, 2–8–10, 2–9–11, 2–12–14, 2–13–15, etc.

If $x = 3$, G is a losing game when $y \equiv 0, 1 \pmod{4}$:
- If $y \equiv 0$, $z = y + 3$.
- If $y \equiv 1$, $z = y + 1$.

Losing games are 3–0–3, 3–1–2, 3–4–7, 3–5–6, 3–8–11, 3–9–10, etc.

If $x = 4$, G is a losing game if $y \equiv 0, 1, 2, 3 \pmod{8}$, $z = y + 4$.
Losing games are 4–0–4, 4–1–5, 4–2–6, 4–3–7, 4–8–12, 4–9–13, etc.

If $x = 5$, G is a losing game when $y \equiv 0, 1, 2, 3 \pmod{8}$:
- If $y \equiv 0, 2$, $z = y + 5$.
- If $y \equiv 1, 3$, $z = y + 3$.

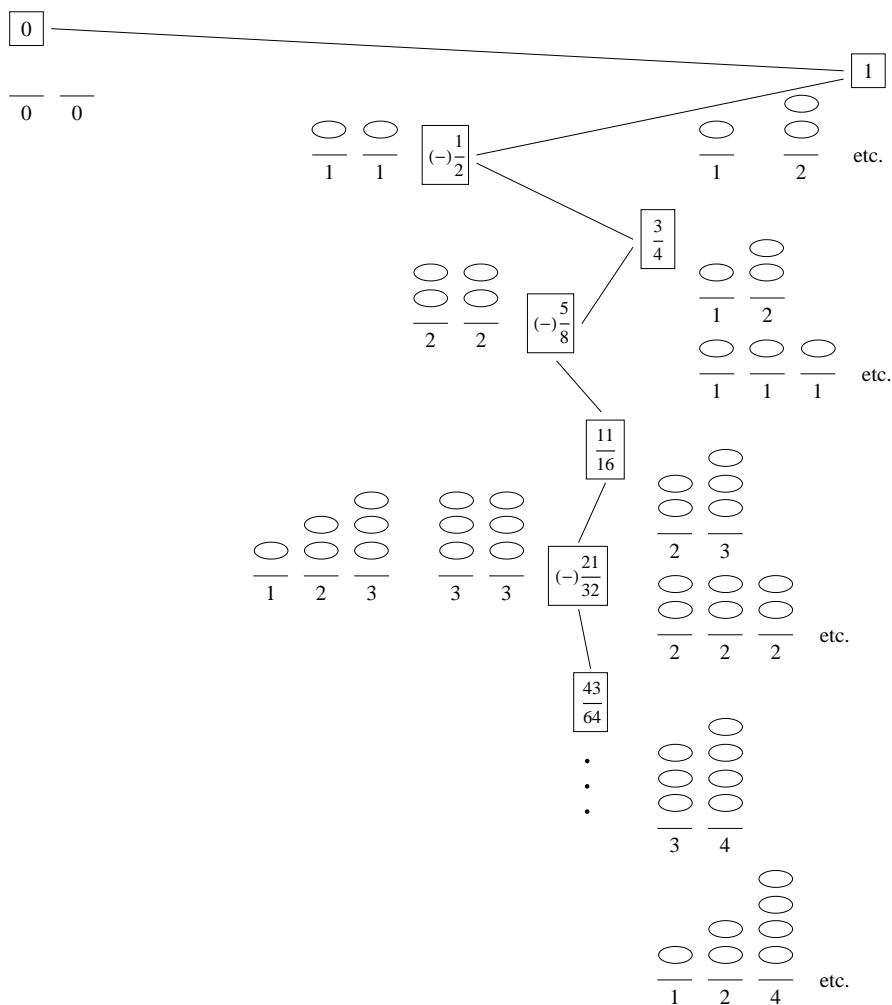Losing games are 5–0–5, 5–1–4, 5–2–7, 5–3–6, 5–8–13, 5–9–12, etc.

FIGURE 2. We correspond our game values to the number tree.

If $x = 6$, G is a losing game when $y \equiv 0, 1, 2, 3 \pmod 8$:

- If $y \equiv 0, 1$, $z = y + 6$.
- If $y \equiv 2, 3$, $z = y + 2$.

Losing games are 6–0–6, 6–1–7, 6–2–4, 6–3–5, 6–8–14, etc.

If $x = 7$, G is a losing game when $y \equiv 0, 1, 2, 3 \pmod 8$:

- If $y \equiv 0$, $z = y + 7$.

- If $y \equiv 1$, $z = y + 5$.
- If $y \equiv 2$, $z = y + 3$.
- If $y \equiv 3$, $z = y + 1$.

Losing games are 7–0–7, 7–1–6, 7–2–5, 7–3–4, 7–8–15, etc.

If $x = 8$, G is a losing game if $y \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{16}$, $z = y + 8$.

Losing games are 8–0–8, 8–1–9, 8–2–10, 8–3–11, 8–4–12, 8–5–13, 8–6–14, 8–7–15, etc.

All other three-pile Nim-games with $x \leq 8$ are winning games.

10. Binary Numbers

I wrote the games in binary numbers, and discovered that losing games and winning games were distinctive when written in binary numbers as well.

Take an arbitrary losing game, 4–8–12. Written in binary numbers, it is 00100–01000–01100. Stacking the numbers on top of each other becomes

0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
0	2	2	0	0

Then take a winning game, 1–3–4. Written in binary numbers, it becomes

0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	2

Here is another losing game, this time with more than three piles: 1–1–1–1.

0	0	0	0	1
0	0	0	0	1
0	0	0	0	1
0	0	0	0	1
0	0	0	0	4

As it turns out, when adding the columns without carrying, all losing games had only even column sums, and all winning games had at least one odd column sum.

Let us introduce the idea of a nim-sum.

Consider two natural numbers a and b. Written in binary, $a = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_m \cdot 2^m$, and $b = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_m \cdot 2^m$. We choose m large enough so that every higher power of 2 is larger than both a and b.

Let $c_i = a_i + b_i \pmod 2$, so $c_i = 0$ or 1.

Definition 10.1. The *nim-sum* of natural numbers a and b,

$$a \oplus b = c_0 + c_1 \cdot 2 + \cdots + c_m \cdot 2^m$$

Notice that $a \oplus b = 0$ if and only if a and b have the same binary representation, i.e. $a = b$. Also the nim-sum operation follows the associative and commutative laws of addition.

Let us apply this concept to a game, which may be represented as $(x_1, \cdots, x_n)$, where x_k is the number of beans in the k-th pile. Let $x = x_1 \oplus \cdots \oplus x_n$.

The claim is that $(x_1, \cdots, x_n)$ is a losing game if and only if $x = 0$. On the other hand, $(x_1, \cdots, x_n)$ is a winning game if and only if $x \neq 0$.

Consider the game $(y_1, \cdots, y_n)$ that results after the first player takes away some number of beans from pile k. Since $x_i = y_i$ for all $i \neq k$, the new nim-sum

$$ y = 0 \oplus y = x \oplus x \oplus y = x \oplus (x_1 \oplus y_1) \oplus \cdots \oplus (x_n \oplus y_n) = x \oplus (x_k \oplus y_k) $$

Theorem 10.2. $(x_1, \cdots, x_n)$ *is a losing game if and only if* $x = 0$.

Proof. Suppose $x = 0$. We must show that $(x_1, \cdots, x_n)$ is a losing game, i.e. any move results in a game with a nim-sum $y \neq 0$. By the above calculation, $y = x_k \oplus y_k$. Since $x_k \neq y_k$, we have $y \neq 0$, so $(y_1, \cdots, y_n)$ is a winning game, implying that $(x_1, \cdots, x_n)$ is a losing game. $\square$

Theorem 10.3. $(x_1, \cdots, x_n)$ *is a winning game if and only if* $x \neq 0$.

Proof. Suppose $x \neq 0$. We must show that there exists a move to a losing game $(y_1, \cdots, y_n)$, i.e. a game such that $y = y_1 \oplus \cdots \oplus y_n = 0$. Let d be the largest non-zero digit in the binary representation of x. Find a pile x_k such that the binary representation of x_k also has a 1 in the d-th digit. Such a x_k must exist, or else the d-th digit of x would be 0. Define $y_k = x \oplus x_k$. The move we want is to take away $(x_k - y_k)$ beans from pile k, but we must first show that $x_k > y_k$. Notice that y_k has a 0 in the d-th binary digit, as x and x_k both have 1. The remaining changes can only amount to adding 2^{k-1}, so we have $x_k - y_k > (2^k - 2^{k-1}) > 0$. Finally, by the earlier calculation, $y = x \oplus (x_k \oplus y_k) = y_k \oplus y_k = 0$, so $(y_1, \cdots, y_n)$ is a losing game. $\square$

11. Conclusion, or How to Win a Game of Nim

Suppose a player is playing a game of Nim. First, he must figure out if the game is a winning game or a losing game. He may convert the piles into binary numbers for such a purpose, or perhaps he has already memorized an extensive list of losing games.

If it is a losing game, he is out of luck. His best bet would be to graciously allow his opponent to start first, or make a move which results in as complicated a winning game as possible, hoping that his opponent does not know Nim strategy, and will therefore play blindly.

If it is a winning game, he can simply make a move that results in a losing game for his opponent. One way to do this would be to write out the game in binary numbers, and make a move that will cause all of the column sums to be even. Another way would be by following the patterns of losing games. Whichever way the player finds a losing game, he can act assuredly knowing that he will win.

References

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