HELLY'S THEOREM

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Theorem 1. *(Eduard Helly) For a finite collection of convex subsets $X_1, X_2, \ldots, X_n \subset \mathbb{R}^d$, where $n > d$, if the intersection of every $d+1$ of these sets is nonempty, then*

$$\bigcap_{j=1}^{n} X_j \neq \emptyset.$$

Pf:

By mathematical induction. Let proposition P_n be Helly's Theorem in the case of n subsets in $\mathbb{R}^d$. Since $n > d$, we can use P_{d+1} as our base case. P_{d+1} is clearly true, because if the intersection of $d+1$ of them are non-empty, then the intersection of all of them are non-empty.

Lemma 1. *(Johann Radon) Any set with $d+2$ points in $\mathbb{R}^d$ can be partitioned into 2 disjoint, non-empty sets such that the convex hulls of these sets have a non-empty intersection.*

Pf:
Let $X = \{p_1, p_2, \ldots, p_{d+2}\} \subset \mathbb{R}^d$, let $X' = \{p'_1, p'_2, \ldots, p'_{d+2}\} \subset \mathbb{R}^{d+1}$, let $\vec{p}_i$ be the position vector of the point p_i and let the vector $\vec{p}'_i$ be such that

$$\text{if } \vec{p}_i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \text{ then } \vec{p}'_i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \\ 1 \end{bmatrix}.$$

Since $p'_i \in \mathbb{R}^{d+1}$, the position vectors of the points in X' cannot be mutually linearly independent because $|X'| = d+2$. This means that there exists some non-trivial solution for the equation

$$\sum_{i=1}^{d+2} \alpha_i \vec{p}'_i = 0,$$

which implies that there exists some non-trivial solution to $\alpha_1, \alpha_2, \ldots, \alpha_n$ s.t. the two equations

$$\sum_{i=1}^{d+2} \alpha_i \vec{p}_i = 0 \quad \text{and} \quad \sum_{i=1}^{d+2} \alpha_i = 0$$

are satisfied.

Let $I = \{1 \leq i \leq d+2 \mid \alpha_i > 0\}$, let $J = \{1 \leq j \leq d+2 \mid \alpha_j \leq 0\}$, let $K = \{p_i \mid i \in I\}$ and let $L = \{p_i \mid i \in J\}$.

It is easy to see that $K, L \neq \emptyset$ because if either of them is empty, then either $\sum_{i=1}^{d+2} \alpha_i = 0$ is not satisfied or the solution to $\sum_{i=1}^{d+2} \alpha_i \vec{p}_i = 0$ is trivial. This means that the point p with position vector $\vec{p} = \frac{\sum_{i \in I} \alpha_i \vec{p}_i}{\sum_{i \in I} \alpha_i}$ exists, and is in the convex hull of K because it is a convex combination of the position vectors of the points in K, i.e.

$$\forall i \in I, \frac{\alpha_i}{\sum_{i \in I} \alpha_i} \geq 0 \quad \text{and} \quad \sum_{i \in I} \left(\frac{\alpha_i}{\sum_{i \in I} \alpha_i}\right) = 1.$$

Also,

$$\begin{aligned} \vec{p} &= \frac{\sum_{i \in I} \alpha_i \vec{p}_i}{\sum_{i \in I} \alpha_i} \\ &= \frac{\sum_{i=1}^{d+2} \alpha_i \vec{p}_i - \sum_{i \in J} \alpha_i \vec{p}_i}{\sum_{i=1}^{d+2} \alpha_i - \sum_{i \in J} \alpha_i} \\ &= \frac{0 - \sum_{i \in J} \alpha_i \vec{p}_i}{0 - \sum_{i \in J} \alpha_i} \\ &= \frac{\sum_{i \in J} \alpha_i \vec{p}_i}{\sum_{i \in J} \alpha_i} \end{aligned}$$

This shows that p is also in the convex hull of L because the position vector of p is a convex combination of the position vectors of the points in L. Hence, $p \in K \cap L$, so the convex hulls of K and L have a non-empty intersection. □

Lemma 2. *The intersection of any 2 convex sets is a convex set.*

Pf:

By contradiction. Let A, B be convex sets, and assume that $A \cap B = C$ is not convex. This implies that there exists two points $a, b \in C$, with position vectors $\vec{a}, \vec{b}$ respectively, such that for some $\alpha \in [0,1]$, the point p with position vector $\vec{p} = \alpha\vec{a} + (1-\alpha)\vec{b}$ is not in C. Since $p \notin C$, we can assume without loss of generality that $p \notin A$, which means that A is not convex because $a, b \in C$ implies $a, b \in A$. Contradiction. □

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Although we already have a base case, we shall now consider P_{d+2}, which will later be used in conjunction with the inductive hypothesis to prove the inductive step.

Choose a common point p_i of all sets X_j, where $j \neq i$. We assume that $p_i \notin X_i$, since otherwise we are done. Let $A = \{p_1, \ldots, p_{d+2}\}$.

By Lemma 1, there exists a nontrivial, disjoint partition A_1, A_2 of A such that the convex hulls of A_1 and A_2 intersect at some point p. Also, observe that $\forall i \in [d+2]$, the only point that is not in X_i but is in A is p_i. Note that since $p_i \in A$ and $A_1 \cup A_2 = A$, we can assume without loss of generality that $p_i \in A_1$. This means that $p_i \notin A_2$, so $A_2 \subset X_i$. Since X_i is convex, it has to contain the convex hull of A_2, and in particular, the point p. Hence, p is common to all the X_i's, and so P_{d+2} is true.

Now, we are ready to prove the inductive step. Assume there exists some $k \in \mathbb{N}$ with $k > d$ such that P_k is true.

Consider P_{k+1}, and let $Y_i = X_i \cap X_{k+1}$.

$$\begin{array}{rcll} \bigcap_{R\subset[k],||R||=d+1} Y_i & = & [\bigcap_R X_i] \cap X_{k+1} & \\ & \neq & \emptyset & \because P_{d+2} \text{ is true .} \end{array}$$

By Lemma 2, $\forall i \in [k+1]$, Y_i is also convex. Since the Y_i's are convex and every $d+1$ of them have a nonempty intersection, by the inductive hypothesis, $\bigcap_{i=1}^{k} Y_i \neq \emptyset$, which implies that $\bigcap_{i=1}^{k+1} X_i \neq \emptyset$. Hence, P_k is true implies that P_{k+1} is true, and this proves the theorem. □

Now that we have proven Helly's theorem for a finite number of convex sets in $\mathbb{R}^d$, we will try to extend this theorem to an infinite number of convex sets. However, we have to add an additional restriction of compactness in place of removing the finiteness restriction on the number of sets. Helly's theorem for an infinite number of convex sets is thus stated as follows:

Theorem 2. *For any infinite collection of convex, compact subsets $X_1, X_2, \cdots \in \mathbb{R}^d$, if the intersection of every $d+1$ of these sets is nonempty, then*

$$\bigcap_{j\to\infty} X_j \neq \emptyset.$$

Before we attempt to prove this theorem, let us demonstrate that the restriction of compactness is necessary. This will be done by creating two counter-examples; in the first, we will show that restricting ourselves only to closed, convex sets is insufficient, while in the second, we will show that restricting ourselves to bounded, convex sets is also insufficient.

False Assertion 1. *For any infinite collection of convex,* ***closed*** *subsets $X_1, X_2, \cdots \in \mathbb{R}^d$, if the intersection of every $d+1$ of these sets is nonempty, then*

$$\bigcap_{j=1}^{\infty} X_j \neq \emptyset.$$

This counter-example will be in the case where $d = 1$. Consider the sets $A_i = \mathbb{R}\backslash(-\infty, i)$, where $i \in \mathbb{N}$. Since the set $(-\infty, i)$ is open, its complement, A_i, is by definition closed. Also, any two A_i's have a non-empty intersection because $A_i \subset A_j$ if $i > j$. However,

$$\bigcap_{i=1}^{\infty} A_i = \emptyset$$

which is contrary to Proposition 1. □

False Assertion 2. *For any infinite collection of convex,* ***bounded*** *subsets $X_1, X_2, \cdots \in \mathbb{R}^d$, if the intersection of every $d+1$ of these sets is nonempty, then*

$$\bigcap_{j=1}^{\infty} X_j \neq \emptyset.$$

This counter-example will again be in the case where $d = 1$. Consider the sets $A_i = (0, \frac{1}{i})$, where $i \in \mathbb{N}$. The A_i's are clearly bounded, and any two A_i's have a non-empty intersection because $A_i \subset A_j$ if $i > j$. Again however,

$$\bigcap_{i=1}^{\infty} A_i = \emptyset$$

which contradicts Proposition 2. □

Now, we will prove Helly's theorem for an infinite number of compact, convex sets.

Pf:

In Theorem 1, we proved that for any n convex sets $A_1, \ldots, A_n$ with every $d+1$ of them having a nonempty intersection, there exists some point x_n s.t.

$$x_n \in \bigcap_{i=1}^{n} A_i.$$

Consider the infinite set $Z_i = \{x_i, x_{i+1}, \ldots\}$. For all $j \in \mathbb{N}$, $j > i$, $x_j \in A_i$ because x_j is a point in the intersection of a set of sets, one of which is A_i. This means that for all $i \in \mathbb{N}$, $Z_i \subset A_i$. Since we restricted that A_i to be compact, A_i is bounded, and so its subset Z_i also has to be bounded. We also know that Z_i is infinite, which implies that Z_i has at least one limit point, because any infinite bounded set has a limit point.

Observe that if $i > j$ with $i, j \in \mathbb{N}$, then $Z_i \subset Z_j$, which means that the set of limit points of Z_i is a subset of the set of limit points of Z_j. Since every Z_i has at least one limit point, this means that for every $i > j$, Z_i shares at least one limit point with Z_j. Also, for all $k < j$ with $k \in \mathbb{N}$, $Z_j \subset Z_k$, and this implies that all the limit points of Z_j are also limit points of Z_k, which means that for every $k < j$, Z_k and Z_j share a limit point. Since this is true for all $j \in \mathbb{N}$, all the Z_i's share a limit point, q.

Now, since $Z_i \subset A_i$, the limit points of Z_i are also limit points of A_i. This means that q is a limit point of all the A_i's because it is a limit point of all the Z_i's. Moreover, since all the A_i's are closed, for every $i \in \mathbb{N}$, $q \in A_i$, so

$$\bigcap_{i=1}^{\infty} A_i \neq \emptyset$$

□