The Determinant: a Means to Calculate Volume

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Abstract

This paper gives a definition of the determinant and lists many of its well-known properties. Volumes of parallelepipeds are introduced, and are shown to be related to the determinant by a simple formula. The reader is assumed to have knowledge of Gaussian elimination and the Gram-Schmidt orthogonalization process.

Determinant Preliminaries

We will define determinants inductively using "minors." Given an $n \times n$ matrix A, the (r, s) *minor* is the determinant of the submatrix A_{rs} of A obtained by crossing out row r and column s of A. The *determinant* of an $n \times n$ matrix A, written $\det(A)$, or sometimes as $|A|$, is defined to be the number

$$\sum_{r=1}^{n} (-1)^{r+1} a_{r1} M_{r1}$$

where M_{k1} is the $(k, 1)$ minor of A. This expression is commonly referred to as "expansion along the first column." Of course, for this definition to make sense, we need to give the base case:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc.$$

We now go over some easy properties of the determinant function.

Theorem 1. *If $\tilde{A}$ is obtained from A by interchanging two rows, then* $\det(A) = -\det(\tilde{A})$.

Proof. If we show the result for any two adjacent rows, the general result follows, since the swapping of any two rows may be written as the composition of an odd number of adjacent row swaps. We proceed by induction. For the 2×2 base case, let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Recall that $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. Hence, interchanging two rows gives

$$\tilde{A} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

So $\det(\tilde{A}) = a_{21}a_{12} - a_{22}a_{11} = -(a_{11}a_{22} - a_{12}a_{21}) = -\det(A)$.
For the inductive step, suppose the statement is true for any $(n-1) \times (n-1)$ matrix. Let A be any $n \times n$ matrix, and let $\tilde{A}$ be obtained from A by exchanging rows r and $r + 1$. We have, by the definition of the determinant:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} M_{i1}$$

From the inductive hypothesis, $M_{k1} = -N_{k1}$, where N_{k1} is the $(k,1)$ minor of $\tilde{A}$ and $k \neq r, r+1$. Therefore, setting $\tilde{A} = \{a'_{ij}\}$,

$$\begin{aligned}
\det \tilde{A} &= \sum_{i=1}^{n} (-1)^n a'_{i1} N_{i1} \\
&= -\sum_{i=1}^{r-1} (-1)^{i+1} a_{i1} M_{i1} + (-1)^{r+1} a'_{r1} M_{(r+1)1} + (-1)^{r+2} a'_{(r+1)1} M_{r1} - \sum_{i=s+1}^{n} (-1)^{i+1} a_{i1} M_{i1} \\
&= -\sum_{i=1}^{r-1} (-1)^{i+1} a_{i1} M_{i1} - (-1)^{r+1} a_{r1} M_{r1} - (-1)^{r+2} a_{(r+1)1} M_{(r+1)1} - \sum_{i=s+1}^{n} (-1)^{i+1} a_{i1} M_{i1} \\
&= -\det A,
\end{aligned}$$

as desired. $\square$

To set up the next theorem, we first define what it means for the determinant function to be "multi-linear". The determinant function is *multi-linear* if the following two properties hold. First, for any scalar value q and any row s of an $n \times n$ matrix A,

$$\det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ qa_{s1} & \dots & qa_{sn} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = q \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{s1} & \dots & a_{sn} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Second, given two $n \times n$ matrices A and B of the form

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ b_{s1} & \dots & b_{sn} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

where all corresponding rows between A and B are equal except for a row s, the sum $\det A + \det B$ is equal to the determinant of C, where

$$C = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{s1} + b_{s1} & \dots & a_{sn} + b_{sn} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Theorem 2. *The determinant function is multi-linear.*

Proof. Given an $n \times n$ matrix A, we prove the first condition of multi-linearity by induction on n. First, for the 2×2 case,

$$q \det \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = q(ad - bc) = (qa)d - (qb)c = \det \begin{pmatrix} qa & qb \\ c & d \end{pmatrix} = a(qd) - b(qc) = \det \begin{pmatrix} a & b \\ qc & qd \end{pmatrix}.$$

Now it remains to show that the statement is true for the $n \times n$ case, given the $(n-1) \times (n-1)$ case. Given an $n \times n$ matrix A let $\tilde{A} = \{a'_{ij}\}$ be obtained by multiplying each entry of an arbitrary row of A, say row s, by a scalar q. Expand $\det \tilde{A}$ along the first column as per the definition:

$$\det(\tilde{A}) = \sum_{r=1}^{n} (-1)^{r+1} a'_{r1} N_{r1},$$

where N_{r1} is the $(r, 1)$ minor of $\tilde{A}$. Let M_{r1} be the $(r, 1)$ minor of A. When $r \neq s$, the determinant N_{r1} will also have a row where each term is multiplied by a scalar q. Hence, by inductive hypothesis, $N_{r1} = qM_{r1}$. When $r = s$, the (r, 1) minor N_{r1} is not multiplied by q; hence the $(n-1) \times (n-1)$ determinant N_{r1} is equal to the determinant M_{r1}. Instead, the $(r, 1)$ entry of A is multiplied by q. Thus,

$$\det \tilde{A} = \sum_{i=1}^{n} (-1)^{r+1} q a_{r1} M_{r1} = q \sum_{i=1}^{n} (-1)^{r+1} a_{r1} M_{r1} = q \det A.$$

The second condition of multi-linearity follows from a very similar inductive argument, so we leave it out.

□

Theorem 3. *The determinant of an upper triangular matrix is the product of the diagonal entries.*

Proof. By induction. The base case follows from an easy calculation. Now suppose the result is true for any $(n-1) \times (n-1)$ matrix.

Given the $n \times n$ upper triangular matrix:

$$A = \begin{pmatrix} a_{11} & a_{21} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{22} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ 0 & \dots & 0 & 0 & a_{nn} \end{pmatrix}$$

Simply expand along the first column, as per our definition of the determinant. Then we obtain:

$$\det(A) = \sum_{i=1}^{n} a_{i1} M_{i1} = a_{11} M_{11} + a_{21} M_{21} + \ldots + a_{n1} M_{n1}.$$

Since $a_{i1} = 0$ for all $i = 2, 3, \ldots, n$, all that remains is the term $a_{11}M_{11}$. But M_{11} is an $(n-1) \times (n-1)$ upper triangular matrix comprising the entries in the lower right of the matrix A. By the inductive hypothesis, this (1,1) minor is the product of its diagonal entries: $M_{11} = a_{22} \ldots a_{nn}$ Thus, $\det(A) = a_{11}M_{11} = a_{11}a_{22} \ldots a_{nn}$ which is the product of the diagonal entries of A. □

The next two theorems will be important in the proof relating volumes and determinants.

Theorem 4. *For any matrix A, we have* $\det(A) = \det(A^T)$.

Proof. In order to prove this, we will need a closed form equation for the determinant of a matrix in terms of its entries that follows easily from observation: Let $A = \{a_i\}_{i=1}^{n}$, then

$$\det A = \sum_{\sigma} sgn(\sigma) a_{\sigma_1} a_{\sigma_2} \cdots a_{\sigma_n},$$

where the sum is taken over all possible permutations σ of $(1, \ldots, n)$. The sign of the permutation $sgn(\sigma)$ is $+1$ if σ is an even permutation and -1 otherwise. It is obvious from this formula that $\det A = \det A^T$, as needed. □

Theorem 4 gives us quite a bit. All the theorems having to do with columns are now true for rows as well. Further, in the definition of the determinant, we now see how we can "expand" along other rows or columns and still get the same answer up to a minus sign.

Recall that an elementary matrix is one that adds a multiple of one row to another. Further, Gaussian elimination without pivoting is the procedure used to find elementary matrices $E_1, \ldots, E_n$ such that $E_1 \cdots E_n A$ is an upper-triangular matrix. We will simply refer to this as Gaussian elimination.

Theorem 5. *Let E be an elementary $n \times n$ matrix and A an arbitrary $n \times n$ matrix. Then* $\det(A) = \det(EA) = \det(AE)$.

Proof. Multiplication by an elementary matrix adds one row to another. We will use Theorem 2. Suppose $\tilde{A}$ is obtained from A by adding row i to row j. Let B be the matrix obtained from A by replacing row i with the elements in row j. By Theorem 1, $\det(B) = -\det(B)$, so $\det(B) = 0$. Thus, by multi-linearity of the determinant, $\det(A) = \det(\tilde{A}) + \det(B) = \det(\tilde{A})$.

Multiplying A on the right by an elementary matrix operates on A by adding a multiple of a one column to another. Taking transposes and using Theorem 4 finishes the proof: $\det(AE) = \det((AE)^T) = \det(E^T A^T) = \det(A^T) = \det(A)$. □

Theorem 6. *For any $n \times n$ matrices A and B, we have* $\det(A)\det(B) = \det(AB)$.

Proof. If A and B are upper triangular matrices, Theorem 3 gives that $\det(A)\det(B) = \det(AB)$. We will use this to prove the general case.

By Gaussian elimination we may write $A = E_1 \cdots E_n U$ and $B = LF_1 \cdots F_n$ where E_i and F_i^T are elementary matrices and U and V are upper triangular matrices. Then by successively applying Theorem 5, $\det(A) = \det(U)$ and $\det(B) = \det(V)$, and so $\det(AB) = \det(UV) = \det(U)\det(V) = \det(A)\det(B)$, as needed. □

Remark: Any elementary matrix E has $\det(E) = 1$. Indeed, $\det(EA) = \det(A) = \det(E)\det(A)$.

Volumes of Parallelepipeds

We will define the n-dimensional parallelepiped P in vector space $\mathbb{R}^n$, from which we take any vectors $x_1, \ldots, x_k$. Take the span of these vectors with the coefficients t_i, as follows: Let $P = \{t_1 x_1 + \cdots + t_k x_k | 0 \le t_i \le 1, i = 1, ..., k\}$. Then P is a k-dimensional parallelepiped in vector space $\mathbb{R}^n$, where the vectors $x_1, ..., x_k$ are edges of P. In order to define volume, we need the following

Lemma 1. *For any vectors $v, w_1, \ldots, w_m$ in $\mathbb{R}^k$, we may find vectors B and C so that $v = B+C$, where B is perpendicular to all w_i, $i = 1, \ldots, m$ and C is in the span of w_i, $i = 1, \ldots, m$.*

Proof. Apply the Gram-Schmidt process to $w_1, \ldots, w_m$ to retrieve vectors $a_1, \ldots, a_{m'}$ that are orthonormal and span the same space as w_i, $i = 1, \ldots, m$. Let $B = v - \sum_{i=1}^{m'} v \cdot a_i a_i$. Then B is perpendicular to all the a_i and hence to the w_i, and similarly $v - B$ is in the span of the w_i. □

We can now define the volume of P by induction on k. The volume is the product of a certain "base" and "altitude" of P. The *base* of P is the area of the $(k-1)$-dimensional parallelepiped with edges $x_2, ..., x_k$. The Lemma gives $x_1 = B + C$ so that B is orthogonal to all of the x_i, $i \ge 2$ and C is in the span of the $x_i, i \ge 2$. The *altitude* is the length of B. Notice that we have made specific choices for the base and height of a given parallelepiped that depend on the ordering of the vertices. As intuition suggests, this ordering does not matter, and we will see this as a result of the following theorem that relates volume to the determinant function.

Theorem 7. *Given an m-dimensional parallelepiped P in n-dimensional space, the square of the volume of P is the determinant of the matrix obtained from multiplying A by its transpose A^T, where*

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

and the rows of A are the edges of P. More precisely,

$$vol(P)^2 = \det(AA^T).$$

Proof. First note that A may be a rectangular matrix (nowhere is it specified that m must equal n). Since we may only take determinants of square matrices, we need to make sure that AA^T is a square matrix. Indeed, as P is an m-dimensional parallelepiped in n-dimensional space, A is an $m \times n$ matrix, and hence, A^T is an $n \times m$ matrix so (AA^T) is an $m \times m$ square matrix.

We will prove the theorem by induction on m. The base case $m = 1$ has AA^T equal to the square of the length of the vector A, and hence is trivially $vol(P)^2$.

Suppose, for the inductive step, that $\det(DD^T) = vol(P)^2$ for any $(m-1) \times n$ matrix D. For the $m \times n$ case, let

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

By Lemma 1, we may write $\alpha_1 = B + C$ such that B is orthogonal to all of the vectors $\alpha_2, \ldots, \alpha_m$ and C is in the span of $\alpha_2, \ldots, \alpha_m$.

Let

$$\tilde{A} = \begin{pmatrix} B \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

As C is in the span of $\alpha_2, \ldots, \alpha_m$ and $\alpha_1 = B + C$ we see that there exist elementary matrices E_i, $i = 1, \ldots, m-1$ so that $A = E_1 \cdots E_{m-1}\tilde{A}$. Thus, by the product formula,

$$\begin{aligned} \det(A^T A) &= \det(E_1 \cdots E_{m-1}\tilde{A}\tilde{A}^T E_{m-1}^T \cdots E_1^T) \\ &= \det(\tilde{A}\tilde{A}^T). \end{aligned}$$

To use the inductive hypothesis, let $D = \begin{pmatrix} \alpha_2^T & \cdots & \alpha_m^T \end{pmatrix}^T$, the matrix obtained by removing the first row from A. By the definition of matrix multiplication,

$$\tilde{A}\tilde{A}^T = \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^T D^T \end{pmatrix} = \begin{pmatrix} BB^T & BD^T \\ DB^T & DD^T \end{pmatrix}$$

Since B is orthogonal to the rows of D, we have BD^T and DB^T are both filled by zeros. Hence expanding along the first row of $\tilde{A}\tilde{A}^T$ gives $\det(\tilde{A}\tilde{A}^T) = BB^T \det(DD^T)$. By inductive hypothesis, $\det(DD^T)$ is the square of the volume of the base of P. And by choice of B, BB^T is the square of the length of the altitude of P. Hence $\det(\tilde{A}\tilde{A}^T) = (vol(P))^2$, as desired. $\square$

Remarks. It is not necessarily true that $vol(P)^2 = \det(A^T A)$ if A is not square. In fact, in the case when A is not square, one of $\det(A^T A)$ and $\det(AA^T)$ is always zero! Furthermore, since $vol^2(P) \geq 0$, Theorem 7 shows that the determinant of any matrix multiplied by its transpose is nonnegative. Lastly, as swapping rows only changes the sign of the determinant, we see that in the definition of the volume of a parallelepiped, a different choice of base would result in the same value for volume.

Corollary 1. *Given an m-dimensional parallelepiped P in m-dimensional space, we have $vol(P) = |\det(A)|$.*

Proof. If A is a square matrix, applying the product formula gives $\det(AA^T) = \det(A)^2$. $\square$

As an application of this relation between volume and the determinant we have the following:

Corollary 2. *Let T be a triangle with vertices (x_1, y_1), (x_1, y_2) and (x_3, y_3). Then*

$$Area(T) = \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|$$

Proof. Subtracting the last row from each of the others gives

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \det\begin{pmatrix} x_1 - x_3 & y_1 - x_3 & 0 \\ x_2 - x_3 & y_2 - x_3 & 0 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Now expanding the determinant along the last row gives

$$\det\begin{pmatrix} x_1 - x_3 & y_1 - x_3 & 0 \\ x_2 - x_3 & y_2 - x_3 & 0 \\ x_3 & y_3 & 1 \end{pmatrix} = \det\begin{pmatrix} x_1 - x_3 & y_1 - x_3 \\ x_2 - x_3 & y_2 - x_3 \end{pmatrix} = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

The absolute value of this last quantity is, by Corollary 1, exactly $2 \times Area(T)$. $\square$

References

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