# The $\mathbb{C}$ Unit Distance Graph.

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**ABSTRACT**

We examine some results on coloring the unit distance graph in the plane. In particular, we examine Coulson's proof that it cannot be 5-colored by polygons, and Woodall's result that $\mathbb{Q}[i]$ is 2-colorable.

The unit distance graph in the plane is the graph whose vertices are the points of $\mathbb{C}$, with edges connecting any two points a unit distance apart. The Hadwiger-Nelson problem asks the chromatic number of this graph. By the theorem of Erdos and de Bruijn, this is the same as the supremum of the chromatic numbers attained by finite subgraphs. It is known that $4 \leq \chi(\mathbb{C}) \leq 7$. We obtain the lower bound by noting that in any hypothetical three-coloring of $\mathbb{C}$, two points at a distance of $\sqrt{3}$ would have to be the same color. Thus all points of $\{|z| = 3\}$ are the same color as 0, but since this set contains a unit distance we need a fourth color. (See diagram 0a.) This construction is due to Moser and Moser. The 7 coloring can be obtained by using regular hexagons of diameter slightly less than 1 as the coloring base (so the set of points of a given color consists of a union of regular hexagons.) See diagram 0b.

There are many interesting directions from which to approach the theory, two of which shall be highlighted: first, Coulson's proof that we cannot 5-color the plain using polygons as a coloring base, and then Woodall's proof that $\mathbb{Q}[i]$ is 2-colorable, which is presented along with a similar proof that $\mathbb{Q}[\omega]$ is 3 colorable, where $\omega$ is a third root of unity. The natural conjecture is that $\mathbb{Q}[\alpha]$ is 3-colorable for all $\alpha$ quadratic over $\mathbb{Q}$. "Moser's extension" for instance (i.e., the extension containing $\omega$ and $\alpha$ where $|\alpha - \sqrt{3}i| = 1$ and $|\alpha| = \sqrt{3}$ ) is the "simplest" non 3-colorable number field; its degree is 4.

**Theorem 1** *(Coulson[2]): at least 6 colors are required to color $\mathbb{C}$ if we use polygons (with convex interior and measure bounded below) as the coloring base. (I.e., if the set of points of each color is a union of polygons with the desired properties.)*

**Proof.** There are several lemmas:

**Lemma 2** *Any path-connected set $S$ of diameter more than 1 contains a unit distance.*

**Proof.** Take two points x,y in S such that d(x,y) ¿ 1. Let $\alpha : [0,1] \longrightarrow S$ be a path connecting x and y (i.e., $\alpha$ continuous, $\alpha(0) = x$ and $\alpha(1) = y$.)

Then $d(x, \alpha(t))$ achieves the value 1 for some $t \in [0,1]$ by the intermediate value theorem. ■

**Lemma 3** *Any coloring by polygons contains a point where (at least) three colors meet (see diagram 1.)*

**Proof.** Assume otherwise. A region is defined to be a "maximal" connected set whose points are all the same color (again, see diagram 1.) By lemma 0, no region has diameter greater than 1. By assumption, each region is surrounded by a region of a different color. By our lower bound on the measure of polygons, we cannot have a region surrounded by a region surrounded by a region ad infinitum (since eventually one of these regions will have diameter greater than 1, as it will surround a set of measure greater than $\pi$.) Thus we have a contradiction (see diagram 2a.)

A tricolor point is one where (at least) three colors meet, i.e., it is a limit point of regions of three different colors. ■

**Lemma 4** *The circle C of radius 1 around a tricolor point X must pass through at least two colors (which are neccessarily different from those which meet at X.) This establishes that a coloring by polygons uses at least 5-colors.*

**Proof.** (see diagram 2b) Say colors a, b, and c meet at X. Let $R_Y$ denote the region a point Y is in. If Y is a point on C, then d(X,Y) = 1. For all but finitely many such Y, we know $\exists \varepsilon > 0$ s.t. $d(Y, y) < \varepsilon \Longrightarrow y \in R_Y$ (in other words, Y is an interior point of $R_Y$ for all but finitely many Y on C, since the $R_Y$ are finite unions of polygons with measure bounded below, and C is a circle.) But the ball of radius $\varepsilon$ around X contains points of colors a, b, and c, so $R_Y$ requires a fourth color, d, whenever Y an interior point of $R_Y$ (remember, all points of $R_Y$ are the same color.) For such a Y, consider $Y' \in R_Y \cap C$. For all but finitely many such $Y'$, the points on C which are at a distance of 1 from Y' are interior points of their regions, and thus require a fifth color by the same argument. ■

**Lemma 5** *Say we have a 5-coloring by convex polygons, and say the regions which cover C are (moving clockwise, in order) $S_1, S_2, ..., S_n$. Then setting $Y_i$ to be the place where $S_i$ and $S_{i+1}$ meet, we have that for $Y$ on C, $d(Y, Y_i) = 1 \Longrightarrow Y = Y_j$ for some j. (See diagram 3.)*

**Proof.** For $Y$ on C with $d(Y, Y_i) = 1$, we have $Y = Y_j$ for some j or Y an interior point of $R_Y$. In the latter case, a contradiction arises because there is an $\varepsilon$ ball around Y contained entirely within $R_Y$, but any $\varepsilon$ ball around $Y_i$ contains both points of color d and points of color e. This proves the lemma. ■

Note that if $d(Y_i, Y_j) = 1$, and $Y_j$ is $\pi/3$ radians clockwise of $Y_i$ , then the region clockwise of $Y_i$ is the same color as the region clockwise of $Y_j$ (by essentially the same argument.)

With notation as in lemma 3, inscribe a unit hexagon D in C, with $Y_1$ a vertex of D, then the other vertices are $Y_j$ as well. Let us label them $Z_1$ through $Z_6$ going clockwise, with $Y_1 = Z_1$. (See diagram 4.)

**Lemma 6** *From point $Z_i$ , draw a ray R perpendicular to the segment $Z_iZ_{i+1}$. The region lying clockwise of $Z_i$ lies entirely to the right of R.*

**Proof.** (See diagrams 5 and 6.) If we draw a circle $C^{'}$ of radius 1 around $Z_{i+1}$ , it bounds the region lying clockwise of $Z_i$. But since this region's boundary is made up of line segments with length bounded below, the segment which passes through $Z_i$ going outward must lie entirely inside $C^{'}$ and thus entirely to the right of R. ■

Clearly, we also have that the region lying counterclockwise of $Z_i$ is bounded by the ray $R^{'}$ which starts at $Z_i$ and runs perpendicular to $Z_{i-1}Z_i$.

Thus, we have that the region between R and $R^{'}$ near $Z_i$ is colored with a, b, or c. We call this region the wedge region at $Z_i$. (Of course, there might be several such regions, the point is that at least 1 exists, and none is colored with d or e.)

We are now ready to complete the proof. Consider a ray L out of X which separates colors a and b at X. (Such a ray exists WLOG because we can permute the labels a, b, and c, and X is a tricolor point.) This ray passes through D between two vertices (or at a vertex, the argument is similar,) say $Z_i$ and $Z_{i+1}$. A point close to $Z_i$ in the wedge region cannot be colored a or b, so it must be c, but the same is true of the wedge region at $Z_{i+1}$. Since these regions contain points at a unit distance from each other, we have obtained the desired contradiction. ■

Now we examine the other side of the theory (succesful colorings) by coloring numberfields.

**Theorem 7** *(Woodall[1]): $\mathbb{Q}[i]$ (as a subset of $\mathbb{C}$) is 2-colorable.*

**Proof.** There are three steps:

Step 0: show that $|z| = 1$ and $z \in \mathbb{Q}[i]$ imply $z \in O[i]$, where $O$ is the set of rationals with odd denominator in lowest terms.

Step 1: show that $O[i]$ is 2-colorable.

Step 2: choose a system $S$ of coset representatives of $\mathbb{Q}[i]/O[i]$, then color $\mathbb{Q}[i]$ by coloring $s + z$ the same color as $z$, where $s \in S$ and $z \in O[i]$.

Say that $|z| = 1$, where $z \in \mathbb{Q}[i]$. Write $z = \frac{1}{c}(a + bi)$ where $a, b, c \in \mathbb{Z}$ and $c$ positive and minimal, then $a^2 + b^2 = c^2$. If $(a, b) \neq 1$, we have that $\frac{a}{(a,b)}$, $\frac{b}{(a,b)}$, $\frac{c}{(a,b)} \in \mathbb{Z}$ and $(\frac{c}{(a,b)})^{-1}(\frac{a}{(a,b)} + \frac{b}{(a,b)}i) = z$, contradicting our assumption that $c$ minimal. Thus, at least one of $a, b$ must be odd, but both cannot be odd because then $a^2 + b^2 \equiv 2(mod 4)$ and $c^2 \equiv 2(mod 4)$ has no solutions. So exactly one of $a, b$ is odd, thus $c$ is odd, so $\frac{a}{c}$ and $\frac{b}{c}$have odd denominator when written in lowest terms.

We color $O[i]$ as follows: color $2O[i]$ and $1 + i + 2O[i]$ with color 0. Color $1 + 2O[i]$ and $i + 2O[i]$ with color 1. We showed above that if $z = \frac{1}{c}(a + bi)$ has norm 1, then exactly one of $a, b$ is odd and $c$ is odd, so we can see that adding $z$ to something colored with 0 results in something colored with 1 and vice-versa.

Finally, choose a system $S$ of coset representatives of $\mathbb{Q}[i]/O[i]$, and color $\mathbb{Q}[i]$ by making $s + z$ the same color as $z$, where $s \in S$ and $z \in O[i]$. This is well defined (expression as $s + z$ is unique.) If $z_1, z_2 \in \mathbb{Q}[i]$ and $|z_1 - z_2| = 1$, then $z_1 - z_2 \in O[i]$, so $z_1$ and $z_2$ are in the same coset, therefore there exists $s \in S$ and $z_1', z_2' \in O[i]$ with $z_1 = s + z_1'$ and $z_2 = s + z_2'$. But then $|z_1' - z_2'| = 1$, so $z_1'$ and $z_2'$ have different colors, so $z_1$ and $z_2$ have different colors. We have thus successfully colored $\mathbb{Q}[i]$. ■

**Theorem 8** $\mathbb{Q}[\omega]$ *(as a subset of* $\mathbb{C}$*) is 2-colorable, where* $\omega$ *is a third root of unity.*

**Proof.** There are three steps:

Step 0: show that $|z| = 1$ and $z \in \mathbb{Q}[\omega]$ imply $z \in O[\omega]$, where $O$ is the set of rationals with denominator not divisible by three in lowest terms.

Step 1: show that $O[\omega]$ is 2-colorable.

Step 2: choose a system $S$ of coset representatives of $\mathbb{Q}[\omega]/O[\omega]$, then color $\mathbb{Q}[\omega]$ by coloring $s + z$ the same color as $z$, where $s \in S$ and $z \in O[\omega]$.

Say that $|z| = 1$, where $z \in \mathbb{Q}[\omega]$. Write $z = \frac{1}{c}(a + b\omega)$ where $a, b, c \in \mathbb{Z}$ and $c$ positive and minimal, then $a^2 + b^2 - ab = c^2$. If $(a, b) \neq 1$, we have that $\frac{a}{(a,b)}$, $\frac{b}{(a,b)}$, $\frac{c}{(a,b)} \in \mathbb{Z}$ and $(\frac{c}{(a,b)})^{-1}(\frac{a}{(a,b)} + \frac{b}{(a,b)}\omega) = z$, contradicting our assumption that $c$ minimal. Thus, at least one of $a, b$ must not be divisible by 3. Consider the following table, whose entries are possible values of $a^2 + b^2 - ab$ mod 3:

| a,b | 0 | 1 | 2 |
|---|---|---|---|
| 0 | - | 1 | 2 |
| 1 | 1 | 1 | 0 |
| 2 | 2 | 0 | 1 |

We see that $a^2 + b^2 - ab$ can only be divisible by 3 if $a \equiv 1$ and $b \equiv 2$ (or vice-versa.) We compute:

$(3j+1)^2 + (3k-1)^2 - (3j+1)(3k-1) = 9j^2 + 6j + 1 + 9k^2 - 6k + 1 - 9jk + 3j - 3k + 1$

$= 9j^2 + 9j + 9k^2 - 9k - 9jk + 3 \equiv 3(mod 9)$ which is impossible, since $c^2 \equiv 3(mod 9)$ has no solutions. Thus c is not divisible by 3, as desired. Furthermore, we can see that we cannot have $a \equiv 1$ and $b \equiv 2$ or $a \equiv 2$ and $b \equiv 1$ (mod 3).

For $x, y = 0, 1, 2$ we color $x + y\omega + 3O[\omega]$ as follows:

| $x, y$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

This coloring works, since if $z_1$ and $z_2$ have the same color, then they differ by some $\frac{1}{c}(a + b\omega)$ where $a \equiv 1$ and $b \equiv 2$ or $a \equiv 2$ and $b \equiv 1(mod 3)$.

The final part of the proof is exactly the same as in the $\mathbb{Q}[i]$ case. ■

References:

[1]Woodall, D.R. (1973). Distances Realized by Sets Covering The Plane. *Journal of Combinatorial Theory Series A* **14**: 187-200.

[2]Coulson, D. (2004). On the chromatic number of plane tilings. *Journal of the Australian Mathematical Society* **77**: 191–196.

[3]Jensen, T.R. (1995). *Graph Coloring Problems.* 150-152.