Triads and Topos Theory

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1 Abstract

This paper describes how to use topos theory to analyze pieces of music. Specifically, it aims to do the following things:

- discuss the reasons for studying the topos $Sets^{\mathcal{T}}$, the category of sets equipped with an action by the monoid $\mathcal{T}$ of affine maps on $\mathbb{Z}/12\mathbb{Z}$ preserving the C-major triad,
- define a family of 144 affine $\mathcal{T}$-actions $\mu[m, n]$ in $Sets^{\mathcal{T}}$ that naturally arises from the structure of $\mathcal{T}$.
- describe the subobject classifier for $Sets^{\mathcal{T}}$,
- calculate the Lawvere-Tierney topologies for $Sets^{\mathcal{T}}$,
- and show how we use the above concepts to comprehend pieces of music.

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3 Introduction

3.1 Why Does Music Theory use Mathematics?

Music theory is a field of study which analyzes the style and mechanics of music. A central concern of music theory is locating patterns in pieces of music and providing a system in which listeners can acquire a better understanding for a piece of music, and in which they can describe this method of comprehension to others. Mathematics is a natural tool to apply to this field of music theory, as it gives us a set of analytical tools with which to study both local and global patterns that arise in a piece of music. Examples of fields of music theory which use mathematics are set theory, which breaks pieces of music into small cells of notes that "reoccur" throughout the piece in different guises, and transformation theory, which seeks to study which transformations – e.g. inversion or transposition by 5 – are central to a piece of music.

In this paper, we will explore the **topos of triads** $Sets^{\mathcal{T}}$ of triads and show how this sophisticated object can lead to some deep music-theoretical insights, by recreating and expanding on Thomas Noll's study of Scriabin's Etude Op. 65, No. 3.

3.2 Why $Sets^{\mathcal{T}}$?

In the first half of our paper, we aim to prove that the monoid $\mathcal{T}$ of affine maps preserving the C-major triad is of definite mathematical interest, and that it has mathematical properties unique to it which make it an object that merits further study (aside from its musical relevance.) Once we have such a collection of functions on sets, it then seems natural to study its possible actions on sets; from there, we will examine the category of all such $\mathcal{T}$-sets, with objects all $\mathcal{T}$-actions and morphisms $\mathcal{T}$-equivariant functions, which turns out to be a topos.

3.3 Why Topos Theory?

The topos structure of $Sets^{\mathcal{T}}$ requires that $Sets^{\mathcal{T}}$ has all finite limits, is cartesian closed, has a subobject classifier. In this paper, we focus mainly on the third property, the subobject classifier – a tool which gives us a notion of a "characteristic morphism" in the category of $Sets^{\mathcal{T}}$, and thus a new way of studying pieces of music by using characteristic functions idiomatic to those pieces to analyze their musical structure.

Before this abstraction, however, we begin with some basic music theory and motivation for why $\mathcal{T}$ is an interesting mathematical object to study.

4 Motivating Our Choice of $\mathcal{T}$

4.1 Music Theory Essentials

Definition 1. A **pitch class** is an equivalence class of pitches, with two pitches regarded as equivalent if they differ by a whole number of octaves. By restricting to the pitch classes which correspond to musical notes (i.e. C, $D_\flat$) we have a set of 12 objects, each corresponding to a different class of pitches. These pitch classes can be thought of as the commutative ring $\mathbb{Z}/12\mathbb{Z}$ in two distinct ways: using either the **semitone encoding**,

Semitone encoding											
0	1	2	3	4	5	6	7	8	9	10	11
C	$C^\sharp$	D	$E_\flat$	E	F	$F^\sharp$	G	$A_\flat$	A	$B_\flat$	B

which orders the pitch classes by the increasing frequency of their representatives within one octave, or the **circle of fifths encoding**,

Circle of Fifths encoding											
0	1	2	3	4	5	6	7	8	9	10	11
C	G	D	A	E	B	$F^\sharp$	$D_\flat$	$A_\flat$	$E_\flat$	$B_\flat$	F

which begins with the pitch class C and continually advances by a **perfect fifth**. In this paper, we follow Noll's work and use the circle of fifths encoding.

Definition 2. An **interval**[1] is an ordered pair (a, b) of pitch classes , $a, b \in \mathbb{Z}/12\mathbb{Z}$.

Definition 3. A **triad** consists of an unordered set $\{a, b, c\}$ of three distinct pitch classes $a, b, c \in \mathbb{Z}/12\mathbb{Z}$.

Definition 4. A interval (a, b) is called **consonant** if the distance from a to b is a unison, octave, perfect fifth, a major or minor third, or a major or minor sixth: under the semitone encoding, this means $b - a$ lies in the set $\{0, 7, 3, 4, 8, 9\}$. An interval that is not consonant is termed **dissonant**: the intervals that fall in this category are major and minor seconds, major and minor sevenths, perfect fourths, and the tritone, i.e. those (a, b) with $b - a \in \{1, 2, 5, 6, 10, 11\}$ under the semitone encoding.

It bears noting that this is not a completely universal definition: for example, the perfect fourth is often regarded as a consonant interval in practice in certain contexts, i.e. when it is being supported by a perfect fifth. But as an isolated interval, it is generally regarded as dissonant, and we will do so for the entirety of this paper.

4.2 Tone Perspectives and Prime Form

Under the encoding above, we have a method of looking at musical notes in a mathematical context as elements of $\mathbb{Z}/12\mathbb{Z}$. This allows us to define the concept of an affine map on musical notes, and the notion of affine maps which "preserve" a given musical substructure; mathematical objects that we will use (after some abstraction) to study recurrent patterns in Scriabin's Op. 65 No. 3.

[1]Though this is not the typical notion of interval, it is the one that Noll uses in his paper, and which we will use here.

Definition 5. A **tone perspective** is an affine map $f : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$, i.e. f is of the form $f(x) = t + s \cdot x$. We denote such maps by ${}^{t}s$, and call the collection of all such maps $\mathbb{A}$.

It bears noting that this collection forms a monoid[2] by defining the product of two tone perspectives as the composition of their maps ${}^{t}s \circ {}^{t'}s' \to {}^{t+s\cdot t'}s \cdot s'$.

Proposition 1. *A tone perspective is determined by where it sends the elements 0 and 1; in fact, there is a bijection between* $(\mathbb{Z}/12\mathbb{Z})^2$ *and* $\mathbb{A}$*, given by the map* ${}^{t}s \mapsto ({}^{t}s(0), {}^{t}s(1)) = (t, s + t)$.

Proof. For a given tone perspective ${}^{t}s$, note that ${}^{t}s(0) = t$ and ${}^{t}s(1) = t + s$, so from the values ${}^{t}s$ takes on 0 and 1, we can determine the coefficients of ${}^{t}s$ by setting

$$t = {}^{t}s(0) \quad \text{and} \quad s = {}^{t}s(1) - {}^{t}s(0)$$

□

Corollary 1. *There are 144 tone perspectives.*

Proposition 2. *A tone perspective* ${}^{t}s$ *is invertible if and only if* $s \in \{1, 5, 7, 11\}$.

Proof. For ${}^{t}s$ a tone perspective: if $s \in \{1, 5, 7, 11\}$, then note that $\exists s^{-1} \in \mathbb{Z}/12\mathbb{Z}$, so

$${}^{-ts^{-1}}s^{-1} \circ {}^{t}s = {}^{-ts^{-1}+ts^{-1}}s^{-1}s = {}^{0}1,$$

and

$${}^{t}s \circ {}^{-ts^{-1}}s^{-1} = {}^{-t+t}s^{-1}s = {}^{0}1.$$

Conversely, if $s \notin \{1, 5, 7, 11\}$, then $\nexists s^{-1} \in \mathbb{Z}/12\mathbb{Z}$, so for any other ${}^{t'}s'$,

$${}^{t'}s' \circ {}^{t}s = {}^{t'+ts'}s's \neq {}^{0}1.$$

So no inverse can exist. □

Corollary 2. *The group of invertible tone perspectives has 48 elements.*

Definition 6. The T/I **group** is the group of tone perspectives of the form ${}^{t}s$, with $t \in \mathbb{Z}/12\mathbb{Z}$ and $s \in \{1, -1\}$.

Definition 7. A triad is said to be in **prime form**[3] if it is of the form $\{0, a, a + b\}$, where $0 \leq a \leq b \leq 12 - a - b$.

Example 1. $\{0, 1, 4\}$ is in prime form, as $0 \leq 1 \leq 3 \leq 8$; conversely, $\{0, 1, 10\}$ is not in prime form, as $0 \leq 1 \leq 9 \nleq 2$.

[2] A monoid is a set equipped with an associative and unital product.

[3] See Richard Cohn's paper on Neo-Riemannian Operations in the bibliography for more.

Proposition 3. *Every triad is isomorphic to a triad in prime form via a map in the T/I-group.*

Proof. For a given triad $T = \{x, y, z\}$, we can get the set of T's differences, $\{y - x, z - y, x - z\}$. Now, in this set, we have one of two possibilities:

- either there is an ascending chain $y - x \leq z - y \leq x - z,\ z - y \leq x - z \leq y - x,\ x - z \leq y - x \leq z - y$,
- or there is a descending chain $y - x \geq z - y \geq x - z,\ z - y \geq x - z \geq y - x,\ x - z \geq y - x \geq z - y$.

If we have an ascending chain, then we simply reorder the elements of T into the new ordered triad T' so that we have the specific ascending chain $y - x \leq z - y \leq x - z$. Translation by $-x$ then yields a triad in prime form, as we get a triad of the form $\{0, y - x, z - x\}$ where

$$0 \leq y - x,$$

$$y - x \leq z - y = (z - x) - (y - x), \text{and}$$
$$(z - x) - (y - x) = z - y \leq x - z = 12 - (z - x).$$

Conversely, if we have a descending chain, we apply the inversion map to T to get the new triad $T' = \{-x, -y, -z\}$; if T had a descending chain, T' has an ascending chain as multiplication by -1 reverses the inequalities. So we are done by the above proof. □

Corollary 3. *The following is a complete list of T/I-classes of triads.*

$\{0,1,2\}$	$\{0,1,5\}$	$\{0,2,5\}$	$\{0,3,6\}$
$\{0,1,3\}$	$\{0,1,6\}$	$\{0,2,6\}$	$\{0,3,7\}$
$\{0,1,4\}$	$\{0,2,4\}$	$\{0,2,7\}$	$\{0,4,8\}$

Definition 8. To any given triad $X = \{a, b, c\}$, we assign the monoid $\mathcal{T}_X$ consisting of all tone perspectives ${}^ts$ such that ${}^ts(X) \subseteq X$; thus, $\mathcal{T}_X$ is the monoid of all tone perspectives preserving X. We call $\mathcal{T}_X$ the **triadic monoid associated to** X; when we discuss the triadic monoid associated to the C-major triad $\{0, 1, 4\}$, we will often omit the subscript and just write $\mathcal{T}$, and refer to $\mathcal{T}$ as **the triadic monoid**.

4.3 Chart of Triadic Monoids

$\mathcal{T}_{\{0,1,2\}} : \{{}^00, {}^01, {}^10, {}^211, {}^20\}$
$\mathcal{T}_{\{0,1,3\}} : \{{}^00, {}^01, {}^10, {}^30\}$
$\mathcal{T}_{\{0,1,4\}} : \{{}^00, {}^01, {}^04, {}^10, {}^13, {}^48, {}^49, {}^40\}$
$\mathcal{T}_{\{0,1,5\}} : \{{}^00, {}^01, {}^05, {}^10, {}^50\}$
$\mathcal{T}_{\{0,1,6\}} : \{{}^00, {}^01, {}^06, {}^10, {}^66, {}^67, {}^60\}$

$\mathcal{T}_{\{0,2,4\}} : \{{}^00, {}^06, {}^01, {}^07, {}^20, {}^26, {}^45, {}^411, {}^40, {}^46\}$
$\mathcal{T}_{\{0,2,6\}} : \{{}^00, {}^06, {}^01, {}^07, {}^03, {}^09, {}^20, {}^26, {}^22, {}^28, {}^63, {}^69, {}^64, {}^610, {}^60, {}^66\}$
$\mathcal{T}_{\{0,3,6\}} : \{{}^00, {}^01, {}^02, {}^04, {}^05, {}^06, {}^08, {}^09, {}^010, {}^30, {}^34, {}^38, {}^60, {}^62, {}^63, {}^64, {}^66, {}^67, {}^68, {}^610, {}^611\}$
$\mathcal{T}_{\{0,4,8\}} : \{{}^0k, {}^4l, {}^8m : k, l, m \in \mathbb{Z}/12\mathbb{Z}\}$

We omit the monoids for $\{0, 2, 5\}$, $\{0, 2, 7\}$, $\{0, 3, 7\}$ here for reasons we will demonstrate in the next subsection.

4.4 Examining Noll's Claim of the Uniqueness of $\{0, 1, 4\}$

Musically, the major and minor chords are decidedly objects of interest: however, the mathematical justification for why we would want to study $Sets^{\mathcal{T}}$ is not necessarily obvious. In his paper, Thomas Noll conjectures that a mathematical property unique to the C-major triad is that it consists of "a concord of its consonant internal intervals in the sense that their perspectival extrapolations constitute its self-perspectives" – i.e. under the bijection in Prop. 1, the elements of the triadic monoid $\mathcal{T}$ correspond to the set of consonant intervals generated by its elements $((0, 0), (0, 1), (0, 4), (1, 0)....$ Noll then uses this characterization to motivate a study of $\mathcal{T}$ and sets equipped with a $\mathcal{T}$-action.

But does this property elegantly generalize from the C-major chord to all of the major chords? Indeed, to any chord that has a triadic monoid isomorphic to $\mathcal{T}$? We categorize all of the triadic monoids below in an effort to answer these questions.

Proposition 4. *If $f : X \to X'$ is an affine isomorphism of triads, then $\mathcal{T}_{X'} = f \circ \mathcal{T}_X \circ f^{-1}$.*

Proof. Let $g \in \mathcal{T}_X$. Then for any $f(x) \in X'$,

$$f \circ g \circ f^{-1}(f(x)) = f(g(x)) \in X' \Rightarrow f \circ g \circ f^{-1} \in \mathcal{T}_{X'}$$
$$\Rightarrow \mathcal{T}_{X'} \supseteq f \circ \mathcal{T}_X \circ f^{-1}$$

Conversely, let $h \in \mathcal{T}_{X'}$; then $\forall x \in X$,

$$h \circ f(x) \in X' \Rightarrow f^{-1} \circ h \circ f(x) \in X$$
$$\Rightarrow f^{-1} \circ h \circ f \in \mathcal{T}_X \Rightarrow \mathcal{T}_{X'} \subseteq f \circ \mathcal{T}_X \circ f^{-1}.$$

□

Given this proposition and the corollary from the earlier subsection on prime form, we only have at most 12 possible distinct triadic monoids; below, we show that there are exactly 8 non-isomorphic triadic monoids.

Proposition 5. *The following is a complete list of those monoids which can occur as a triadic monoid.*

$$\begin{array}{cccc} \mathcal{T}_{\{0,1,2\}} & \mathcal{T}_{\{0,1,4\}} & \mathcal{T}_{\{0,2,4\}} & \mathcal{T}_{\{0,3,6\}} \\ \mathcal{T}_{\{0,1,3\}} & \mathcal{T}_{\{0,1,6\}} & \mathcal{T}_{\{0,2,6\}} & \mathcal{T}_{\{0,4,8\}} \end{array}$$

Proof. By corollary 3, it suffices to check only the monoids

$$\begin{array}{cccc} \mathcal{T}_{\{0,1,2\}} & \mathcal{T}_{\{0,1,5\}} & \mathcal{T}_{\{0,2,5\}} & \mathcal{T}_{\{0,3,6\}} \\ \mathcal{T}_{\{0,1,3\}} & \mathcal{T}_{\{0,1,6\}} & \mathcal{T}_{\{0,2,6\}} & \mathcal{T}_{\{0,3,7\}} \\ \mathcal{T}_{\{0,1,4\}} & \mathcal{T}_{\{0,2,4\}} & \mathcal{T}_{\{0,2,7\}} & \mathcal{T}_{\{0,4,8\}}. \end{array}$$

By composing ${}^{5}0$ with ${}^{1}7$, we can send $\{0,1,4\}$ to $\{0,3,7\}$; similarly, ${}^{1}1 \circ {}^{0}7$ sends $\{0,2,5\}$ to $\{0,1,3\}$ and ${}^{0}7$ sends $\{0,1,2\}$ to $\{0,2,7\}$. Also, by examining the actual monoids

$$\mathcal{T}_{\{0,1,2\}} = \{{}^{0}0, {}^{0}1, {}^{1}0, {}^{2}11, {}^{2}0\}, \mathcal{T}_{\{0,1,5\}} = \{{}^{0}0, {}^{0}1, {}^{0}5, {}^{1}0, 5_0\},$$

we can see that these are also isomorphic (though not via conjugation by an affine transformation of $\mathbb{Z}/12\mathbb{Z}$) via the map ${}^{0}5 \mapsto {}^{2}11, {}^{1}0 \mapsto {}^{0}0, {}^{0}0 \mapsto {}^{1}0$ from the generators of $\mathcal{T}_{\{0,1,2\}}$ to the generators of $\mathcal{T}_{\{0,1,5\}}$.

None of the remaining 8 monoids are isomorphic, as they are all of different cardinalities (see the chart in the earlier subsection for an explicit description of all such monoids.) □

Finally, we make a few brief observations on consonance before studying the possible interpretations of Noll's statement:

Proposition 6. *Let $(a,b) \in \mathbb{Z}/12\mathbb{Z}$. Then, the following properties of (a,b) hold:*

- *The consonance or dissonance of (a,b) is invariant under translation.*
- *If $b-a \in \{0,3,4,8,9\}$, then (a,b) remains consonant under any affine automorphism of $\mathbb{Z}/12\mathbb{Z}$.*
- *If $b-a=1$, then (a,b) becomes dissonant under inversion, multiplication by 5 or by 7.*
- *If $b-a=11$, (a,b) becomes consonant under inversion; under any other affine automorphism, it remains dissonant.*
- *If $b-a=7$, (a,b) becomes consonant under multiplication by 7; under any other affine automorphism, it remains dissonant.*
- *If $b-a=5$, (a,b) becomes consonant under multiplication by 5; under any other affine automorphism, it remains dissonant.*

- *If $b - a \in \{2, 6, 10\}$, then (a, b) remains dissonant under any affine automorphism of $\mathbb{Z}/12\mathbb{Z}$.*

Proof. For any interval (a, b) and a translation $(a + x, b + x)$, $b - a = b + x - a - x$, so (a, b) is consonant if and only if $(a + x, b + x)$ is.

For $b - a = 1$, we have that $a - b = 11$, $5a - 5b = 5$ and $7a - 7b = 7$; so (a, b) in this case becomes dissonant under inversion, multiplication by 5 and multiplication by 7.

The other cases are similar, and are left as an exercise. □

Now we have the mathematical background to properly explore any of the possible implications of Noll's statement. If we read his statement literally, we get that a characterization of the C-major chord is that there is a bijection between $\mathcal{T}$ and the set of consonant intervals generated by the elements of $\{0, 1, 4\}$, $\{(0, 0), (0, 1), (0, 4), (1, 1), (1, 4), (4, 0), (4, 1), (4, 4)\}$, given by ${}^ts \to (t, s + t)$. This characterization is not unique to the C-major chord: there are chords like $\{1, 2, 10\}$, the G-minor chord, and $F^\sharp$-major $\{6, 7, 10\}$, that share this property. This does not surprise us though, as we expect other major and minor chords to have similar properties. However, not all major and minor chords have this property! For example, the $A_\flat$-major chord $\{0, 8, 9\}$ has ^{88} as an element in its associated triadic monoid, but $(8, 8 + 8)$ is not an interval contained in the $A_\flat$ chord. So this property does not suffice to characterize either the major chords or the chords with triadic monoids isomorphic to $\mathcal{T}$.

But what if we weaken this criterion? There are two natural directions in which we could alter our statement: either

- we require only that there is **some** bijection between the set $\mathcal{T}$ and the set of consonant intervals generated by a triad, or
- we require only that the map ${}^ts \to (t, s + t)$ takes each element of $\mathcal{T}$ to **some** consonant interval.

Proposition 7. *The major and minor triads and the suspended second chord – i.e. all chords of the forms $\{0, 1, 4\}, \{0, 3, 4\}$ and $\{0, 1, 2\}$ – are the unique triads X (up to translation) such that there is a bijection between $\mathcal{T}_X$ and the set of consonant intervals generated by that triad X.*

Proof. By proposition 5 and the explicit calculations on the chart in section 8, we have a complete characterization of all of the distinct triadic monoids. As there are at most 9 consonant intervals possible for any given triad, we can thus quickly eliminate half of the remaining triads as they all have too many elements in their triadic monoids, leaving us with

$$\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\}, \{0, 1, 6\}.$$

and their associated chords reached by translation and multiplication by 5 or 7. (We check $\{0,1,5\}$ here because while it is isomorphic to $\{0,1,2\}$, it is not conjugate-isomorphic to it, and some of our techniques will rely on this property.) It thus suffices to check by hand these five cases.

So. From the chart earlier, $|\mathcal{T}_{\{0,1,3\}}| = 4$, but by the consonance observations above, under any automorphism the number of consonant intervals will be at least 5, as $(0,0),(1,1),(3,3)(0,3),(3,0)$ are all invariant under any automorphism of $\mathbb{Z}/12\mathbb{Z}$. So there cannot be any such association between $\{0,1,3\}$'s intervals {or any associated triad} and its associated triadic monoid. Similarly, $|\mathcal{T}_{\{0,1,6\}}| = 7$, but by the consonance observations above, under any automorphism the number of consonant intervals will be at most 6, as $(0,6)$ and $(6,0)$ are dissonant under any automorphism of $\mathbb{Z}/12\mathbb{Z}$. So there also cannot be any such association between $\{0,1,6\}$'s intervals and its associated triadic monoid. As well, for $\{0,1,5\}$, $|\mathcal{T}_{\{0,1,5\}}| = 5$; but the intervals $(0,0),(1,1),(5,5),(1,5),(5,1)$ will all remain consonant under any affine isomorphism, while $(0,1)$ is consonant under inversion, $(0,5)$ becomes consonant under multiplication by 5, and $(5,0)$ becomes consonant under multiplication by 7. So under any transformation there are at least 6 consonant intervals in any triad with triadic monoid $\cong \mathcal{T}_{\{0,1,5\}}$, and thus those triads also fail to satisfy our condition.

For $\{0,1,2\}$, there are precisely 5 consonant intervals $\{(0,0),(0,1),(1,1),(1,2),(2,2)\}$ and 5 tone perspectives fixing the triad, $\{{}^{0}0,{}^{0}1,{}^{1}0,{}^{2}11,{}^{2}0\}$, so such a correspondence exists for this specific triad: however, multiplication by 5,7 or 11 will decrease the number of such consonant intervals, as a perfect fifth becomes a dissonant interval under any non-translation affine isomorphism.

$\{0,1,4\}$ has a similar correspondence between its consonant intervals $\{(0,0),(0,1),(0,4),(1,1),(1,4),(4,0),(4,1),(4,4)\}$ and its tone perspectives $\{{}^{0}0,{}^{0}1,{}^{0}4,{}^{1}0,{}^{1}3,{}^{4}8,{}^{4}9,{}^{4}0\}$: also, the number of consonant intervals for this triad is constant under inversion, as the perfect fifth $(0,1)$ which becomes dissonant is balanced out by the perfect fourth $(1,0)$ which becomes consonant. Under multiplication by 5 or 7, however, this doesn't occur, so translations of $(0,1,4)$ and the inversion of $(0,1,4)$, the minor chord $(0,3,4)$ are the only triads associated to $\mathcal{T}_{\{0,1,4\}}$ that have a 1-1 correspondence between their consonant intervals and this triadic monoid. □

Proposition 8. *The major triad* $\{0,1,4\}$ *and the triad* $\{0,1,3\}$ *are the only two triads up to affine isomorphism such that the interval associated to any tone perspective in their triadic monoid via the map*

$$^{t}s \to (t, t+s)$$

is ***consonant****.*

Proof. From proposition 4, we have that the image of a tone perspective ${}^{t}s$ under conjugation by a translation map is ${}^{(t-(s-1)\cdot x)}s$, under conjugation by multiplication by 5 is ${}^{5t}s$, and under conjugation by multiplication by 7 is ${}^{7t}s$; so in every case we have that, as

$$s + t - t = (t - (s-2)\cdot x) - (t - (s-1)\cdot x) = 5t + s - 5t = 7t + s - 7t = s,$$

if every tone perspective in a given triadic monoid $\mathcal{T}$ has a consonant interval associated to it by the map ${}^{t}s \to (t, s+t)$, then all of the triadic monoids $\cong \mathcal{T}$ also have this property. So it suffices to check our nine cases: examining the earlier chart shows that exactly 2 triadic monoids satisfy this property, namely those associated to the triads $\{0,1,4\}$ and $\{0,1,3\}$. □

So neither condition suffices to characterize the major and minor triads; and, while applying both criterion **will** suffice to uniquely characterize the major and minor triads, the property of having one bijection between the triadic monoid $\mathcal{T}$ and the set of consonant intervals generated by the triad and a second bijection between $\mathcal{T}$ and consonant intervals generated by $\mathcal{T}$'s tone perspectives seems rather stretched and less natural than Noll's initial description of the relation. A stronger mathematical reason is desired; luckily, one lies unstated but strongly implied in Noll's later work, which will even serve to motivate our next direction of study.

5 $\mathcal{T}$-actions on $\mathbb{Z}/12\mathbb{Z}$: Why and How

5.1 Chart of Generators of Triadic Monoids

$\mathcal{T}_{\{0,1,2\}} : \{{}^{0}0, {}^{2}11, {}^{1}0\}$
$\mathcal{T}_{\{0,1,3\}} : \{{}^{0}0, {}^{1}0, {}^{3}0\}$
$\mathcal{T}_{\{0,1,4\}} : \{{}^{1}3, {}^{4}8\}$
$\mathcal{T}_{\{0,1,5\}} : \{{}^{0}0, {}^{0}5, {}^{1}0\}$
$\mathcal{T}_{\{0,1,6\}} : \{{}^{1}0, {}^{6}6, {}^{6}7\}$
$\mathcal{T}_{\{0,2,4\}} : \{{}^{0}0, {}^{2}6, {}^{4}5, {}^{4}11, {}^{4}6\}$
$\mathcal{T}_{\{0,2,6\}} : \{{}^{0}7, {}^{0}3, {}^{2}2, {}^{6}3, {}^{6}9\}$
$\mathcal{T}_{\{0,3,6\}} : \{{}^{0}2, {}^{0}5, {}^{0}9, {}^{3}8, {}^{6}7\}$
$\mathcal{T}_{\{0,4,8\}} : \{{}^{0}2, {}^{0}5, {}^{0}7, {}^{0}3, {}^{4}1, {}^{8}1\}$

5.2 The Minimality of $\mathcal{T}_{\{0,1,4\}}$

Without deciding what triadic monoid $\mathcal{T}$ we wish to study $Sets^{\mathcal{T}}$ in relation to, we can still ask ourselves "Which specific sets equipped with $\mathcal{T}$-actions are going to be objects of interest?" Intuitively, we would want these $\mathcal{T}$-sets to somehow mirror the musical structure of the triadic monoid $\mathcal{T}$ – i.e. that these sets would consist of subsets of $\mathbb{Z}/12\mathbb{Z}$ which $\mathcal{T}$ would act by affine maps on. To define any such action for a given triadic monoid $\mathcal{T}$, it suffices to determine which maps the generators of $\mathcal{T}$ get sent to, and thus to make choices for the following elements depending on our specific triadic monoid. So how many such maps exist?

If we choose to look at the specific $\mathcal{T}$-actions defined on the generators of $\mathcal{T}$ by ${}^{t}s \cdot x = {}^{t'}s(x)$, for any t', we then have 12^3 possible actions of interest for every monoid **not** $\mathcal{T}_{\{0,1,4\}}$, and in the case of at least half of the monoids, quite a few more, as we can send each generating element to 12 other tone perspectives. As $\mathcal{T}_{\{0,1,4\}}$ is generated by just 2 elements, however, it has a comparatively small and easily understood family of 144 such affine actions. This "minimality" is a characteristic that uniquely characterizes the family of triads affine isomorphic to the C-major chord, as they are the unique triads with associated triadic monoids generated by ≤ 2 elements: furthermore, this mathematical property serves to motivate our next area of study, the collection of all 144 such $\mathcal{T}$-actions.

Definition 9. We define $\mu[m,n] : \mathcal{T} \times \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ as the affine $\mathcal{T}$-action such that $\mu[m,n]({}^{1}3, x) = {}^{m}3(x)$ and $\mu[m,n]({}^{4}8, x) = {}^{m}8(x)$. (We do not define this action on the other elements of $\mathcal{T}$, as we have already defined it on $\mathcal{T}$'s generators.)

Example 2. The natural action of $\mathcal{T}$ on $\mathbb{Z}/12\mathbb{Z}$, ${}^{t}s \cdot x = t + sx$, is just $\mu[1,4]$ under this notation.

Definition 10. For an $\mathcal{T}$-action μ, $|\mu|$ denotes the underlying set of μ.

Definition 11. For a given action μ, a **subaction** ν of μ is a subset $|\nu| \subseteq |\mu|$ equipped with the same action as μ, closed under this action.

Given this notion, a natural question arises: what triads can be expressed as subactions of some $\mu[m,n]$, and which triads can be expressed as minimal subactions (under the ordering of inclusion)?

Proposition 9. *There are precisely 2 triads (up to affine transformation) that are minimal (under inclusion) subactions of some $\mu[m,n]$: $\{0,1,4\}$ and $\{0,2,6\}$, the C-major triad $\{C,E,G\}$ for $\mu[1,4]$ and the "stretched" triad $\{D,F^{\sharp},C\}$ for $\mu[6,2]$.*

Proof. Note first that any triads affine isomorphic to $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,5\}$, $\{0,1,6\}$, $\{0,2,4\}$ are not preserved by any tone perspectives ${}^{t}s$ for $s \in \{3,8\}$, and so no $\mu[m,n]$ action can preserve these triads.

Conversely, to tell if a given triad can be fixed by some $\mu[m,n]$, it suffices to check all of the $\mu[m,n]$ such that ${}^{m}3$ and ${}^{n}8$ are in its triadic monoid: so,

1. to see if $\{0,1,4\}$ can be described as a minimal subaction of some $\mu[m,n]$, it suffices to check if it is a minimal subaction of $\mu[1,4]$,

2. to see if $\{0,2,6\}$ is a minimal subaction, it suffices to check $\mu[6,2]$ or $\mu[0,2]$,

3. to see if $\{0,3,6\}$ is a minimal subaction, it suffices to check $\mu[6,6]$ or $\mu[6,3]$, and finally

4. to see if $\{0,4,8\}$ is a minimal subaction, it suffices to check $\mu[a,b]$, for $a,b \in \{0,4,8\}$.

Checking these results by simple calculations {omitted here} gives us that the triads $\{0,1,4\}$ and $\{0,2,6\}$ are the unique triads that can be expressed as minimal subactions of some $\mu[m,n]$, as all of the possible $\mu[m,n]$ for $\{0,3,6\}$ and $\{0,4,8\}$ have minimal subactions of only 1 or 2 elements. $\square$

6 Sets$^{\mathcal{T}}$ as a Topos

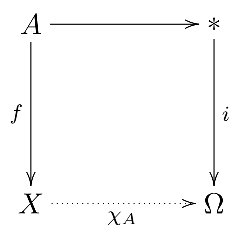
6.1 Definition of a Topos

Definition 12. An **elementary topos**, or simply a **topos**, is a category $\mathcal{E}$ with the following three properties:

- $\mathcal{E}$ has all finite limits.
- $\mathcal{E}$ has a subobject classifier.
- $\mathcal{E}$ is cartesian closed.

As stated before, we are going to focus on the subobject classifier in $Sets^{\mathcal{T}}$, which we define below.

Definition 13. A **subobject classifier** is an object Ω and a morphism $i : * \to \Omega$, $*$ a terminal object, such that for any objects A, X and a monomorphism $f : A \to X$, there is a unique arrow χ_A such the square

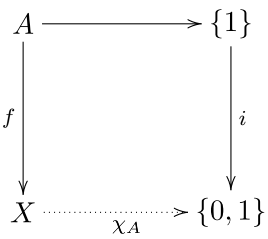


is a pullback square.

To illuminate what's going on, we use the example topos *Sets*, where *Sets* is the category with objects all sets and arrows all functions between sets.

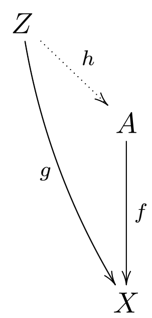
Proposition 10. *The subobject classifier of Sets is the function* $i : \{1\} \to \{0,1\}, i(1) = 1$.

Proof. This means that for any sets A, X with an injection of A into X, there must be a unique function χ_A such that the diagram



is a pullback square.

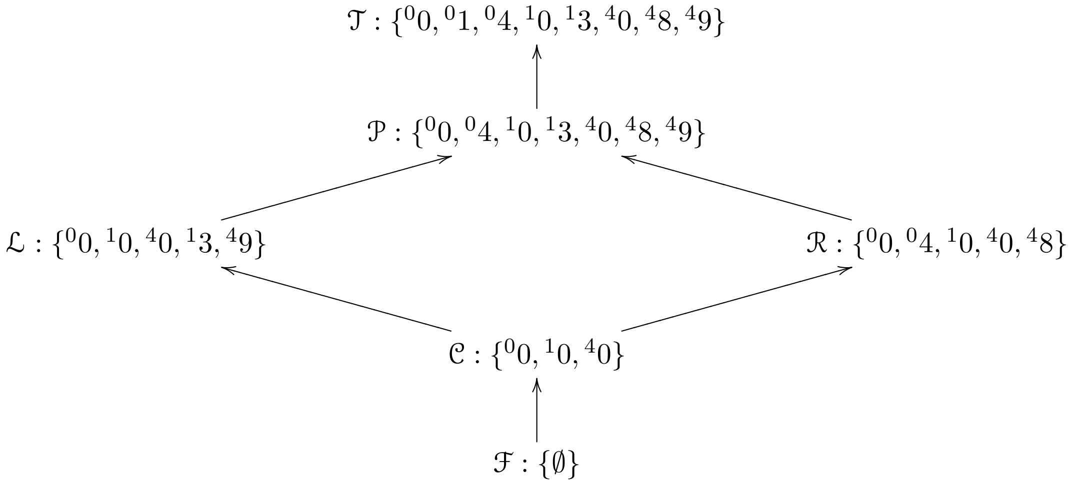
In the case of *Sets*, this unique function is the characteristic function of A, as we show here: because $\chi_A(A) = \{1\}$, we have that the square commutes. Also, for any other function α such that $\alpha(A) = \{1\}$, if there exists a $x \in X$ such that $\chi_A(x) = 0$ and $\alpha(x) = 1$, then the set $\alpha^{-1}(1)$ with inclusion map $g : \alpha^{-1}(1) \to X$ has no map h into A such that the triangle



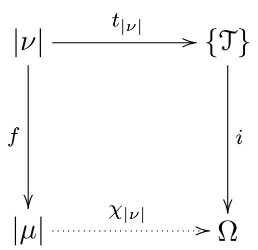
commutes.

So the function such that the earlier square is a pullback square is unique: therefore, $i : \{1\} \to \{0, 1\}, i(1) = 1$ is the subobject classifier of *Sets*. $\square$

Proposition 11. *$Sets^{\mathcal{T}}$ has a subobject classifier. Namely, it has the subobject classifier $i : \{\mathcal{T}\} \to \Omega$, with $\Omega = \{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{F}\}$ consisting of all of the left ideals of $\mathcal{T}$, with the $\mathcal{T}$-action $\omega : \mathcal{T} \times \Omega \to \Omega$, $\omega(m, \mathcal{B}) = \{t \in \mathcal{T} : t \circ m \in \mathcal{B}\}$*



Proof. To show that our category $\mathbf{Sets}^{\mathcal{T}}$ has $i : \{\mathcal{T}\} \to \Omega$ as a subobject classifier, we must show that for any inclusion $f : |\nu| \to |\mu|$, ν a $\mathcal{T}$-subaction of μ, there is a unique pullback square



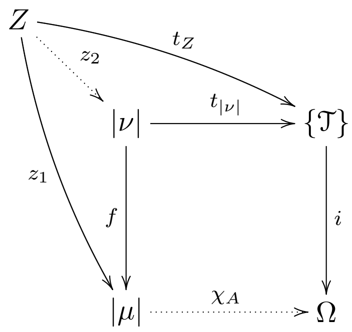
where $t_{|\nu|}$ is the unique map $|\nu| \to \mathcal{T}$.

We know the following things:

1. Given that we want this diagram to commute, we need our arrow $\chi_{|\nu|}$ to be equivariant under the action of $\mathcal{T}$ – i.e. that $(\forall x \in |\mu|)(\forall t \in \mathcal{T}), t \cdot \chi_{|\nu|}(x) = \chi_{|\nu|}(t \cdot x)$.

2. Also, because this diagram commutes, we have that (as $i \circ t_{|\nu|}(|\nu|) = \{\mathcal{T}\}$,) $\chi_{|\nu|} \circ f(|\nu|)$ must be equal to $\{\mathcal{T}\}$.

3. Furthermore, for any Z a $\mathcal{T}$-set with $z_1 : Z \to |\mu|$ such that the square with vertex Z commutes, there must be a unique map $z_2 : Z \to |\nu|$ making the diagram below commute, as $\chi_{|\nu|}$ makes this square a pullback square.



By condition (2), we know that we have to send all of the elements of $|\nu|$ to $\mathcal{T}$. Conversely, if we send any more than these elements to $\mathcal{T}$, we can create a new set $Z = \chi_{|\nu|}^{-1}(\mathcal{T})$ with the inclusion map z_1 into $|\mu|$: then, because this set $Z \notin |\nu|$, there is no map z_2 that factors through $|\nu|$ and is equal to the inclusion map of Z into $|\mu|$. So it must send **exactly** the elements of $|\nu|$ to $\mathcal{T}$, and no others.

This then, with the requirement that $\chi_{|\nu|}$ is equivariant, forces the following definition:

$$\chi_{|\nu|}(x) = \{h \in \mathcal{T} | h \cdot x \in |\nu|\}$$

First, we observe that this is a well-defined function. This is just checking that $\forall x \in |\mu|$, we have that $\chi_{|\nu|}(x)$ is a left ideal; so, taking any $g \in \chi_{|\nu|}(x)$, we observe that

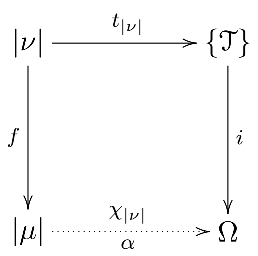
$$\forall h \in \mathcal{T}, (h \cdot g) \cdot x = h \cdot (g \cdot x) \in |\nu|,$$

because $g \cdot x \in |\nu|$ and ν is closed under $\mathcal{T}$-actions: and thus that $\chi_{|\nu|}(x)$ is closed under action by $\mathcal{T}$.

Second, we note that it trivially commutes, as $i \circ t_{|\nu|}(|\nu|) = \chi_{|\nu|} \circ f(|\nu|) = \{\mathcal{T}\})$.

Third, we note that this is a pullback square, as for any Z with $z_1 : Z \to |\mu|$ such that the square with vertex Z commutes, we have that, as $i \circ t_Z = \chi_{|\nu|} \circ z_1$, all of Z is thus mapped to $\mathcal{T}$ and thus $Z \subseteq |\nu|$: so there is a unique map $Z \to |\nu|$, $z \mapsto f^{-1}(z_1(z))$.

Finally, we note that this is the unique arrow such that this forms a pullback square, as for any other map α such that the following diagram commutes,



we have that α and $\chi_{|\nu|}$ agree on $|\nu| = \alpha^{-1}(\mathcal{T}) = \chi_{|\nu|}^{-1}(\mathcal{T})$ because the square commutes.

As well, $\forall x \in |\mu|$, we have that

$$(\forall h \in \mathcal{T}) \text{ s.t. } (h \cdot x \in |\nu|),\ h \cdot \alpha(x) = \alpha(h \cdot x) = \mathcal{T}.$$

For such a h, $h \notin \alpha(x) \Rightarrow h \cdot \alpha(x) \neq \mathcal{T}$ by the multiplication table below; so $\forall h \in \mathcal{T}$ s.t. $h \cdot x \in |\nu|$, $h \in \alpha(x)$. So $\alpha(x) \supseteq \chi_{|\nu|}$.

	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{F}$
${}^0 1$	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{F}$
${}^0 0$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{F}$
${}^1 0$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{F}$
${}^4 0$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{F}$
${}^1 3$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{F}$
${}^4 9$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{F}$
${}^4 8$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{L}$	$\mathcal{T}$	$\mathcal{L}$	$\mathcal{F}$
${}^0 4$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{L}$	$\mathcal{T}$	$\mathcal{L}$	$\mathcal{F}$

To see the other containment, note that if $\exists h \in \alpha(x)$ such that $h \cdot x \notin |\nu|$, then we either have

- $\alpha(x) \in \{\mathcal{L}, \mathcal{R}, \mathcal{P}\}$ and $\chi_{|\nu|}(x) = \mathcal{C}$, in which case action by either ${}^1 3$ or ${}^4 8$ causes α to take on the value $\mathcal{T}$ where $\chi_{|\nu|}$ does not, a contradiction,
- or $\alpha(x) = \mathcal{P}$ and $\chi_{|\nu|}(x) \in \mathcal{L}, \mathcal{R}, \mathcal{C}$, in which case action by one of either ${}^1 3$ or ${}^4 8$ causes α to take on the value $\mathcal{T}$ where $\chi_{|\nu|}$ does not, a contradiction.

So $i : \{\mathcal{T}\} \to \Omega$ is a subobject classifier. □

7 Possible Subobjects and the Lawvere-Tierney Topologies

So what can we do with this subobject classifier? We want to get an understanding of the underlying shape of the musical objects we study; thus, we can look for the Lawvere-Tierney topologies with respect to our subobject classifier, and examine how we can apply these arrows to pieces of music. Intuitively, a Lawvere-Tierney topology is a way to define a sort of "topology" on a topos. For the purposes of studying $Sets^{\mathcal{T}}$, we will use them to determine which of the subobjects above are "characterizing" in some way our topos, and apply these maps to analyze pieces of music.

Definition 14. A **Lawvere-Tierney topology** on a topos $\mathcal{E}$ is a map $j : \Omega \to \Omega$ such that the following three properties hold:

1. $j \circ \mathcal{T} = \mathcal{T}$
2. $j \circ j = j$
3. $j \circ \cap = \cap \circ (j \times j)$,

where $\cap : \Omega \times \Omega \to \Omega$ is the intersection map

To find the Lawvere-Tierney topologies, we must first define how $\mathcal{T}$ acts on Ω; with that done, we simply find all of the possible subobjects of Ω and check which characteristic arrows satisfy the axioms to be Lawvere-Tierney topologies.

Definition 15. $\mathcal{T}$ acts on Ω by left composition: i.e. given a left ideal X of Ω and an element $t \in \mathcal{T}$, we define $t \cdot X$ as $\{t \circ x : x \in X\}$.

Proposition 12. *Under the action of $\mathcal{T}$ on Ω, the subobject classifier of $Sets^{\mathcal{T}}$, there are 22 possible subobjects of Ω.*

Proof. Pick some subobject of Ω, V, and assume first that $\mathcal{F} \notin V$.

If $\mathcal{T}$ is not a member of V, then we know that V must be $\{\emptyset\}$, as ${}^0 0$ sends any value that is not $\mathcal{F}$ to $\mathcal{T}$.

Conversely, if $\mathcal{T}$ is a member of V, then we have either

- $\mathcal{C} \in V$, and thus that $\mathcal{L}, \mathcal{R} \in V$, as ${}^1 3, {}^4 8$ send $\mathcal{C}$ to $\mathcal{L}$ and $\mathcal{R}$: so V is either $\{\mathcal{T}, \mathcal{L}, \mathcal{R}, \mathcal{C}\}$ or $\{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{C}\}$
- $\mathcal{C} \notin V$, and so either
  - $\mathcal{L}, \mathcal{R} \in V$, and so we have either $\{\mathcal{T}, \mathcal{L}, \mathcal{R}\}$ or $\{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}\}$
  - $\mathcal{L} \in V$ and $\mathcal{R} \notin V$, and so we have either $\{\mathcal{T}, \mathcal{L}\}$ or $\{\mathcal{T}, \mathcal{P}, \mathcal{L}\}$
  - $\mathcal{R} \in V$ and $\mathcal{L} \notin V$, and so we have either $\{\mathcal{T}, \mathcal{R}\}$ or $\{\mathcal{T}, \mathcal{P}, \mathcal{R}\}$
  - $\mathcal{R}, \mathcal{L} \notin V$, and so we have either $\{\mathcal{T}\}$ or $\{\mathcal{T}, \mathcal{P}\}$.

This gives us 11 subobjects. Also, for any subobject V not containing $\mathcal{F}$, we can create a new subobject $V' = V \cup \mathcal{F}$, as every map in $\mathcal{T}$ fixes $\mathcal{F}$. This gives us an additional 11 new subobjects, leaving us with 22 subobjects in total.

□

Proposition 13. *Of these 22 possible subobjects, the characteristic arrows of precisely 6 of them satisfy the axioms of the Lawvere-Tierney topologies:*

$$\chi_{\mathcal{T}},\ \chi_{\mathcal{T},\mathcal{P}},\ \chi_{\mathcal{T},\mathcal{P},\mathcal{R}},\ \chi_{\mathcal{T},\mathcal{P},\mathcal{L}},\ \chi_{\mathcal{T},\mathcal{P},\mathcal{L},\mathcal{R},\mathcal{C}},\ \chi_{\mathcal{T},\mathcal{P},\mathcal{L},\mathcal{R},\mathcal{C},\mathcal{F}}$$

Proof. Examining property 3 of the Lawvere-Tierney topologies, we can notice that for the sets $\{\mathcal{T}, \mathcal{R}\}, \{\mathcal{T}, \mathcal{L}\}, \{\mathcal{T}, \mathcal{L}, \mathcal{R}\}, \{\mathcal{T}, \mathcal{L}, \mathcal{R}, \mathcal{C}\}$ and their corresponding characteristic arrows j,

$$j \circ (\mathcal{R} \cap \mathcal{P}) = j \circ \mathcal{R} = \mathcal{T}, \text{and}$$

$$\cap \circ (j(\mathcal{R}) \times j(\mathcal{P})) = \cap \circ (\mathcal{T} \times \mathcal{P}) = \mathcal{P},$$

so the characteristic arrows of all of these subsets fail to be Lawvere-Tierney topologies.

The characteristic arrow of $\{\mathcal{T}, \mathcal{L}, \mathcal{R}, \mathcal{P}\}$ has

$$j \circ (\mathcal{R} \cap \mathcal{L}) = j \circ \mathcal{C} = \mathcal{P}, \text{and}$$

$$\cap \circ (j(\mathcal{R}) \times j(\mathcal{L})) = \cap \circ (\mathcal{T} \times \mathcal{T}) = \mathcal{T},$$

so it also does not satisfy the Lawvere-Tierney topology axioms; finally, we have that the set $\{\emptyset\}$ cannot be a Lawvere-Tierney topology, as $j \circ \mathcal{T}$ is $\mathcal{F}$.

Also, for any of the sets described above, unioning the set $\{\mathcal{F}\}$ does not change the arguments made above; so we have only six sets left whose characteristic arrows could possibly satisfy the Lawvere-Tierney axioms,

$$\{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{C}, \mathcal{F}\}, \{\mathcal{T}, \mathcal{P}, \mathcal{L}, \mathcal{R}, \mathcal{C}\}, \{\mathcal{T}, \mathcal{P}, \mathcal{L}\}, \{\mathcal{T}, \mathcal{P}, \mathcal{R}\}, \{\mathcal{T}, \mathcal{P}\}, \{\mathcal{T}\}.$$

As the characteristic arrows for these sets all send $\mathcal{T}$ to $\mathcal{T}$, have $j \circ j = j$, and commute with the intersection operator, they are all Lawvere-Tierney topologies. We denote these arrows

$$j_{\mathcal{F}}, j_{\mathcal{C}}, j_{\mathcal{L}}, j_{\mathcal{R}}, j_{\mathcal{P}}, j_{\mathcal{T}}$$

with the subscript indicating the smallest (with respect to inclusion) ideal in the given subobject.

□

8 Uses and Applications

8.1 Calculating Some Characteristic Functions

With the family of 144 $\mathcal{T}$-actions from section 5 and a notion of a "characteristic" morphism from section 7, we are now equipped to describe the characteristic functions assigned to certain triads – namely, those which we found could be described as subobjects of some $\mu[m, n]$. Using the methods outlined earlier, we calculate the characteristic functions for the four triads we could express as subactions of $\mu[m, n]$ actions, the C-major triad $\{0, 1, 4\}$, the stretched triad $\{0, 4, 10\}$, and the triads $\{0, 4, 8\}$ and $\{0, 3, 6\}$ below.

	$\chi_{\{0,1,4\}}$ for the action $\mu[1, 4]$	$\chi_{\{0,4,10\}}$ for the action $\mu[10, 4]$
$\mathcal{T}$	if $x \in \{0, 1, 4\}$	$x \in \{0, 4, 10\}$
$\mathcal{P}$	$x \in \{9\}$	$x \in \{6\}$
$\mathcal{L}$	$x \in \{5, 8\}$	$x \in \{2, 8\}$
$\mathcal{R}$	$x \in \{3, 6, 7, 10\}$	$x \in \{1, 3, 7, 9\}$
$\mathcal{C}$	$x \in \{2, 11\}$	$x \in \{5, 11\}$
$\mathcal{F}$	$x \in \{\emptyset\}$	$x \in \{\emptyset\}$

	$\chi_{\{0,3,6\}}$ for the action $\mu[6,3]$	$\chi_{\{0,4,8\}}$ for the action $\mu[0,4]$
$\mathcal{T}$	if $x \in \{0,3,6\}$	$x \in \{0,4,8\}$
$\mathcal{P}$	$x \in \{\emptyset\}$	$x \in \{\emptyset\}$
$\mathcal{L}$	$x \in \{2,4,7,8,10,11\}$	$x \in \{\emptyset\}$
$\mathcal{R}$	$x \in \{9\}$	$x \in \{1,2,3,5,6,7,9,10,11\}$
$\mathcal{C}$	$x \in \{1,5\}$	$x \in \{\emptyset\}$
$\mathcal{F}$	$x \in \{\emptyset\}$	$x \in \{\emptyset\}$

(It is interesting to note that these characteristic functions are in fact independent of the choice of $\mu[m,n]$, for those triads that are subactions of more than one $\mu[m,n]$: i.e. the characteristic arrow of $(0,3,6)$ in $\mu[6,6]$ is the same function as the characteristic arrow of $\{0,3,6\}$ in $\mu[6,3]$.)

So this gives us a set of characteristic arrows with which to study pieces of music; but do they actually tell us anything? We first notice a few interesting properties about how these arrows and our Lawvere-Tierney topologies interact, and then analyze a piece of music below as a demonstration of the efficacy of the system of analysis we have built up.

8.2 Tonal Systems From Lawvere-Tierney Topologies

Before we begin the musical analysis, it is interesting to notice how the Lawvere-Tierney topologies can characterize "interesting" musical sets. E.g.:for a $\mathcal{T}$-action μ and a subaction ν, if we take the Lawvere-Tierney topologies $j_{\mathcal{T}}, j_{\mathcal{P}}, j_{\mathcal{L}}, j_{\mathcal{R}}, j_{\mathcal{C}}, j_{\mathcal{F}}$ from before and apply them to the characteristic function $\chi_{|\nu|}$, we can get a new function $j \circ \chi_{|\nu|} : \mathbb{Z}/12\mathbb{Z} \to \Omega$. We can then ask ourselves what the set of all of the elements that get sent to $\mathcal{T} \in \Omega$ under this map is; in other words, what the set of all elements that are "j-close" to $|\nu|$ is. We call this set $|J(\nu)| = \{x \in |\mu| : j \circ \chi_{|\nu|}(x) = \mathcal{T}\}$.

Recall that from section 5, we characterized the triads $\{0,1,4\}$ and $\{0,4,10\} = \{0+10, 2+10, 6+10\}$ as the unique triads (up to translation) that can be characterized as the minimal subaction of $\mu[1,4]$ and $\mu[10,4]$, respectively. In the following table, we list the various sets we get by applying the process above to these triads (treated as subactions of various $\mu[m,n]$'s):

$\lvert J_{\mathcal{T}}(\{0,1,4\}\rvert = \{0,1,4\}$	$\lvert J_{\mathcal{T}}(\{0,4,10\}\rvert = \{0,4,10\}$
$\lvert J_{\mathcal{P}}(\{0,1,4\}\rvert = \{0,1,4,9\}$	$\lvert J_{\mathcal{P}}(\{0,4,10\}\rvert = \{0,4,6,10\}$
$\lvert J_{\mathcal{L}}(\{0,1,4\}\rvert = \{0,1,4,5,8,9\}$	$\lvert J_{\mathcal{L}}(\{0,4,10\}\rvert = \{0,2,4,6,810\}$
$\lvert J_{\mathcal{R}}(\{0,1,4\}\rvert = \{0,1,3,4,6,7,9,10\}$	$\lvert J_{\mathcal{R}}(\{0,4,10\}\rvert = \{0,1,3,4,6,7,9,10\}$
$\lvert J_{\mathcal{C}}(\{0,1,4\}\rvert = \mathbb{Z}/12\mathbb{Z}$	$\lvert J_{\mathcal{C}}(\{0,4,10\}\rvert = \mathbb{Z}/12\mathbb{Z}$
$\lvert J_{\mathcal{F}}(\{0,1,4\}\rvert = \mathbb{Z}/12\mathbb{Z}$	$\lvert J_{\mathcal{F}}(\{0,4,10\}\rvert = \mathbb{Z}/12\mathbb{Z}$

Ignoring the trivial sets $J_{\mathcal{T}}, J_{\mathcal{C}}, J_{\mathcal{F}}$, we can see a large number of remarkable things: for $\{0,1,4\}$, we get the major/minor mixture, hexatonic and octatonic systems as sets corresponding to $|J_{\mathcal{P}}(\{0,1,4\}|$, $|J_{\mathcal{L}}(\{0,1,4\}|$ and $|J_{\mathcal{R}}(\{0,1,4\}|$, respectively; for $\{0,4,10\}$, we get the french augmented sixth chord, the wholetone and octatonic systems corresponding to $|J_{\mathcal{P}}(\{0,4,10\}|$, $|J_{\mathcal{L}}(\{0,4,10\}|$ and $|J_{\mathcal{R}}(\{0,4,10\}|$. This is fairly unexpected, and gives rise to some hope that the Lawvere-Tierney topologies are indeed capturing some notion of topology similar to our own.

8.3 Analysis of Scriabin's Etude Op. 65 No. 3

Scriabin's Etude Op. 65 No. 3 is a natural piece to test our theories on. Each left hand chord present in the piece is of the form $\{k, 4+k, 10+k\}$, a stretched triad. Consequently, each such triad can be thought of as a subactions of the $\mathcal{T}$-action $\mu[10-2k, 4-7k]$, and so we can calculate their characteristic functions $\chi_{\{k,4+k,10+k\}}$ of these triads.

	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{F}$
$C : \chi_{\{0,4,10\}}$	0, 4, 10	6	2, 8	1, 3, 7, 9	5, 11	$\emptyset$
$G : \chi_{\{1,5,11\}}$	1, 5, 11	7	3, 9	2, 4, 8, 10	6, 0	$\emptyset$
$D : \chi_{\{2,6,0\}}$	2, 6, 0	8	4, 10	3, 5, 9, 11	7, 1	$\emptyset$
$A : \chi_{\{3,7,1\}}$	3, 7, 1	9	5, 11	4, 6, 10, 0	8, 2	$\emptyset$
$E : \chi_{\{4,8,2\}}$	4, 8, 2	10	6, 0	5, 7, 11, 1	9, 3	$\emptyset$
$B : \chi_{\{5,9,3\}}$	5, 9, 3	11	7, 1	6, 8, 0, 2	10, 4	$\emptyset$
$F^{\sharp} : \chi_{\{6,10,4\}}$	6, 10, 4	0	8, 2	7, 9, 1, 3	11, 5	$\emptyset$
$C^{\sharp} : \chi_{\{7,11,5\}}$	7, 11, 5	1	9, 3	8, 10, 2, 4	0, 6	$\emptyset$
$A_{\flat} : \chi_{\{8,0,6\}}$	8, 0, 6	2	10, 4	9, 11, 3, 5	1, 7	$\emptyset$
$E_{\flat} : \chi_{\{9,1,7\}}$	9, 1, 7	3	11, 5	10, 0, 4, 6	2, 8	$\emptyset$
$B_{\flat} : \chi_{\{10,2,8\}}$	10, 2, 8	4	0, 6	11, 1, 5, 7	3, 9	$\emptyset$
$F : \chi_{\{11,3,9\}}$	11, 3, 9	5	1, 7	0, 2, 6, 8	4, 10	$\emptyset$

With these functions, we consider the various repetitions of the stretched triad to be "local consonances," to borrow a phrase from Noll's paper, and then interpret all other notes ("local dissonances") played over these chords with the characteristic functions $\chi_{\{k,4+k,10+k\}}$ corresponding to these local consonances. We do this over the three tables at the end of the paper, which demonstrate a number of striking patterns in Scriabin's work. Namely:

- The lowest possible truth value $\mathcal{C}$ never occurs when the local dissonances are evaluated with respect to their local characteristic functions: the truth values are limited to $\mathcal{P}, \mathcal{L}$ and $\mathcal{R}$ for all such local dissonances.
- Furthermore, the truth values consist almost entirely of the values $\mathcal{L}$ and $\mathcal{R}$. The $\mathcal{P}$ truth value rarely occurs, and when it does the local dissonance corresponding to $\mathcal{P}$ is always the exact same pitch class as the preceding local consonance. (For example: In measure 7.5-8, a

stretched triad with root F is played, and in measure 8-9, the local dissonance F is played, which has truth value $\mathcal{P}$.) This can be partially understood by watching the root of the bass triad. Frequently when the bass chord moves, it moves by a tritone (i.e. 6), a transformation which is idiomatic to Scriabin's piece; and two stretched triads with root notes that differ by a tritone take the same notes to $\mathcal{L}$ and $\mathcal{R}$, as can be seen in the above table. So, in a sense, the preponderance of $\mathcal{L}$ and $\mathcal{R}$ truth values can be interpreted as a consequence of having "consistent" truth values throughout transitions in the piece.

- Looking at the first three distinct root pitches of the bass triads, we can see that these notes $\{G, C^\sharp, A\} = \{1, 7, 3\} = \{3, 7, 1\}$ form a stretched triad as well. If we look at the sets

$$|J_{\mathcal{P}}(\{3,7,1\}| = \{1,3,7,9\}$$
$$|J_{\mathcal{L}}(\{3,7,1\}| = \{0,1,3,4,6,7,9,10\}$$
$$|J_{\mathcal{R}}(\{3,7,1\}| = \{1,3,5,7,9,11\}$$

  from the previous subsection, we can notice that the notes avoided in all of these sets, $G = 2$ and $A_\flat = 8$, are precisely those notes which never occur as root chords of a stretched triad.

- As well, if we take the characteristic function $\chi_{\{3,7,1\}}$ corresponding to these pitches and apply it to the root notes of all of the bass chords, an interesting pattern appears. Namely, the truth values of all of the root notes are also always $\mathcal{P}, \mathcal{L}$ and $\mathcal{R}$, as Scriabin studiously avoids precisely two stretched triads: those corresponding to $D = (2, 6, 0)$ and $A_\flat = (8, 0, 6)$.

  This provides an interesting link between the "local" behavior of Scriabin's piece, and the avoidance of tones practiced over local consonances, and how this is linked to the "global" behavior of the piece, where the same avoidance is practiced over all of the stretched triads themselves.

9 Directions for Further Study

There are many other possible directions in which to expand this study: for instance, exploring the action of other triadic monoids on sets would allow for different formulations of $\mu[m, n]$, that might be able to characterize different triads. This would allow a study of pieces of music that have recurrent phrases centered around other triads than $\{0, 1, 4\}, \{0, 4, 10\}, \{0, 4, 8\}$,, and $\{0, 3, 6\}$ (up to translation.) Also, exploring systems beyond the triad (i.e. 7th chords) would be a good topic of study, as much of modern music makes use of more complex systems than the triadic one we emphasize here.

On a different tack, simply analyzing other pieces of music, or seeing if it is possible to automatically generate “nice”-sounding pieces of music by using a technique like parsimonious voice leading to move the left hand chords and picking tones with high truth values in the treble register, would be an interesting endeavor to attempt.

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10 Analysis of Scriabin's Etude Op. 65 No. 3

Measure #	Stretched Triad	Local Pitches
$1-1.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$1.5-2$	$C^\sharp:\{7,11,5\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$2-2.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$2.5-3$	$C^\sharp:\{7,11,5\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$3-3.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$3.5-4$	$C^\sharp:\{1,5,11\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$4-4.5$	$A:\{3,7,1\}$	$F^\sharp:\mathcal{R}, B:\mathcal{L}$
$4.5-5$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, F:\mathcal{L}$
$5-5.5$	$B:\{5,9,3\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$5.5-6$	$F:\{11,3,9\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$6-6.5$	$B:\{5,9,3\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$6.5-7$	$F:\{11,3,9\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$7-7.5$	$B:\{5,9,3\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$7.5-8$	$F:\{11,3,9\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$8-9$	$B:\{5,9,3\}$	$G^\sharp:\mathcal{R}, C\sharp:\mathcal{L}, F:\mathcal{P}$
$9-9.5$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, F:\mathcal{L}$
$9.5-10$	$A:\{3,7,1\}$	$F^\sharp:\mathcal{R}, B:\mathcal{L}$
$10-10.5$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, F:\mathcal{L}$
$10.5-11$	$A:\{3,7,1\}$	$F^\sharp:\mathcal{R}, B:\mathcal{L}$
$11-11.5$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, F:\mathcal{L}$
$11.5-12$	$A:\{3,7,1\}$	$F^\sharp:\mathcal{R}, B:\mathcal{L}$
$12-12.5$	$F:\{11,3,9\}$	$D:\mathcal{R}, G:\mathcal{L}$
$12.5-13$	$B:\{5,9,3\}$	$G^\sharp:\mathcal{R}, C^\sharp:\mathcal{L}$
$13-13.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$13.5-14$	$C^\sharp:\{7,11,5\}$	$E:\mathcal{R}, A:\mathcal{L}, G:\mathcal{P}$
$14-14.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$14.5-15$	$C^\sharp:\{7,11,5\}$	$E:\mathcal{R}, A:\mathcal{L}$
$15-15.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$15.5-16$	$C^\sharp:\{7,11,5\}$	$E:\mathcal{R}, A:\mathcal{L}$
$16-17$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}, C^\sharp:\mathcal{P}$
$17-18$	$C^\sharp:\{7,11,5\}$	$D:\mathcal{R}, A:\mathcal{L}$
$18-19$	$G:\{1,5,11\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$19-21$	$C^\sharp:\{7,11,5\}$	$E:\mathcal{R}, D:\mathcal{R}, A:\mathcal{L}$
$21-22$	$C^\sharp:\{7,11,5\}$	$D:\mathcal{R}, A:\mathcal{L}$
$22-23$	$G:\{1,5,11\}$	$B_\flat:\mathcal{R}, E:\mathcal{R}, E_\flat:\mathcal{L}$
$23-25$	$C^\sharp:\{7,11,5\}$	$D:\mathcal{R}, A:\mathcal{L}$
$25-26$	$C^\sharp:\{7,11,5\}$	$D:\mathcal{R}, A:\mathcal{L}$
$26-27$	$G:\{1,5,11\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$

Measure #	Stretched Triad	Local Pitches
$27-29$	$C^\sharp : \{7,11,5\}$	$E:\mathcal{R}, D:\mathcal{R}, A:\mathcal{L}$
$29-30$	$C^\sharp : \{7,11,5\}$	$D:\mathcal{R}, A:\mathcal{L}$
$30-31$	$G:\{1,5,11\}$	$B_\flat:\mathcal{R}, E:\mathcal{R}, E_\flat:\mathcal{L}$
$31-33$	$E:\{4,8,2\}$	$F:\mathcal{R}, G:\mathcal{R}, C:\mathcal{L}$
$33-34$	$E:\{4,8,2\}$	$F:\mathcal{R}, G:\mathcal{R}, C:\mathcal{L}$
$34-35$	$B_\flat:\{10,2,8\}$	$F:\mathcal{R}, G:\mathcal{R}, C:\mathcal{L}$
$35-37$	$E:\{4,8,2\}$	$F:\mathcal{R}, G:\mathcal{R}, C:\mathcal{L}$
$37-37.5$	$E:\{4,8,2\}$	$F:\mathcal{R}, G:\mathcal{R}, C:\mathcal{L}$
$37.5-38$	$B_\flat:\{10,2,8\}$	$B:\mathcal{R}, C^\sharp:\mathcal{R}, F^\sharp:\mathcal{L}$
$38-38.5$	$F^\sharp:\{6,10,4\}$	$G:\mathcal{R}, A:\mathcal{R}, D:\mathcal{L}$
$38.5-39$	$C:\{0,4,10\}$	$E_\flat:\mathcal{R}, D_\flat:\mathcal{R}, A_\flat:\mathcal{L}$
$39-40$	$F^\sharp:\{6,10,4\}$	$G:\mathcal{R}, D:\mathcal{L}$
$40-41$	$C:\{0,4,10\}$	$E_\flat:\mathcal{R}, A_\flat:\mathcal{L}$
$41-43$	$F^\sharp:\{6,10,4\}$	$G:\mathcal{R}, A:\mathcal{R}, D:\mathcal{L}$
$43-44$	$F^\sharp:\{6,10,4\}$	$G:\mathcal{R}, D:\mathcal{L}$
$44-45$	$C:\{0,4,10\}$	$E_\flat:\mathcal{R}, A:\mathcal{R}, A_\flat:\mathcal{L}$
$45-47$	$A:\{3,7,1\}$	$C:\mathcal{R}, B_\flat:\mathcal{R}, F:\mathcal{L}$
$47-48$	$A:\{3,7,1\}$	$C:\mathcal{R}, B_\flat:\mathcal{R}, F:\mathcal{L}$
$48-49$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, B_\flat:\mathcal{R}, F:\mathcal{L}$
$49-51$	$A:\{3,7,1\}$	$C:\mathcal{R}, B_\flat:\mathcal{R}, F:\mathcal{L}$
$51-51.5$	$A:\{3,7,1\}$	$C:\mathcal{R}, B_\flat:\mathcal{R}, F:\mathcal{L}$
$51.5-52$	$E_\flat:\{9,1,7\}$	$E:\mathcal{R}, F_\sharp:\mathcal{R}, B:\mathcal{L}$
$52-52.5$	$B:\{5,9,3\}$	$C:\mathcal{R}, D:\mathcal{R}, G:\mathcal{L}$
$52.5-53$	$F:\{11,3,9\}$	$F^\sharp:\mathcal{R}, A_\flat:\mathcal{R}, D_\flat:\mathcal{L}$
$53-55$	$B:\{5,9,3\}$	$C:\mathcal{R}, D:\mathcal{R}, G:\mathcal{L}$
$55-55.5$	$B:\{5,9,3\}$	$C:\mathcal{R}, D:\mathcal{R}, G:\mathcal{L}$
$55.5-56$	$F:\{11,3,9\}$	$D:\mathcal{R}, G:\mathcal{L}$
$56-56.5$	$B:\{5,9,3\}$	$C:\mathcal{R}, D:\mathcal{R}, G:\mathcal{L}$
$56.5-57$	$F:\{11,3,9\}$	$D:\mathcal{R}, G:\mathcal{L}$
$57-59$	$B:\{5,9,3\}$	$C:\mathcal{R}, D:\mathcal{R}, A_\flat:\mathcal{R}, C^\sharp:\mathcal{L}, G:\mathcal{L},$
$59-60$	$G:\{1,5,11\}$	$E:\mathcal{R}, D:\mathcal{R}, A:\mathcal{L}$
$60-61$	$C^\sharp:\{7,11,5\}$	$E:\mathcal{R}, D:\mathcal{R}, B_\flat:\mathcal{R}A:\mathcal{L}, E_\flat:\mathcal{L}$
$61-63$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}, C^\sharp:\mathcal{P}$
$63-63.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$63.5-64$	$C^\sharp:\{7,11,5\}\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$64-64.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$64.5-65$	$C^\sharp:\{7,11,5\}\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$65-65.5$	$G:\{1,5,11\}$	$E:\mathcal{R}, A:\mathcal{L}$
$65.5-66$	$C^\sharp:\{7,11,5\}$	$B_\flat:\mathcal{R}, E_\flat:\mathcal{L}$
$66-66.5$	$A:\{3,7,1\}$	$F^\sharp:\mathcal{R}, B:\mathcal{L}$
$66.5-67$	$E_\flat:\{9,1,7\}$	$C:\mathcal{R}, F:\mathcal{L}$
$67-67.5$	$B:\{5,9,3\}$	$A_\flat:\mathcal{R}, C^\sharp:\mathcal{L}$

Measure #	Stretched Triad	Local Pitches
$67.5 - 68$	$F : \{11, 3, 9\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$68 - 68.5$	$B : \{5, 9, 3\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$68.5 - 69$	$F : \{11, 3, 9\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$69 - 69.5$	$B : \{5, 9, 3\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$69.5 - 70$	$F : \{11, 3, 9\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$70 - 71$	$B : \{5, 9, 3\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}, F : \mathcal{P}$
$71 - 71.5$	$E_\flat : \{9, 1, 7\}$	$C : \mathcal{R}, F : \mathcal{L}$
$71.5 - 72$	$A : \{3, 7, 1\}$	$F^\sharp : \mathcal{R}, B : \mathcal{L}$
$72 - 72.5$	$E_\flat : \{9, 1, 7\}$	$C : \mathcal{R}, F : \mathcal{L}$
$72.5 - 73$	$A : \{3, 7, 1\}$	$F^\sharp : \mathcal{R}, B : \mathcal{L}$
$73 - 73.5$	$E_\flat : \{9, 1, 7\}$	$C : \mathcal{R}, F : \mathcal{L}$
$73.5 - 74$	$A : \{3, 7, 1\}$	$F^\sharp : \mathcal{R}, B : \mathcal{L}$
$74 - 74.5$	$F : \{11, 3, 9\}$	$D : \mathcal{R}, G : \mathcal{L}$
$74.5 - 75$	$B : \{5, 9, 3\}$	$A_\flat : \mathcal{R}, C^\sharp : \mathcal{L}$
$75 - 75.5$	$G : \{1, 5, 11\}$	$E : \mathcal{R}, A : \mathcal{L}$
$75.5 - 76$	$C^\sharp : \{7, 11, 5\}$	$E : \mathcal{R}, A : \mathcal{L}, G : \mathcal{P}$
$76 - 76.5$	$G : \{1, 5, 11\}$	$E : \mathcal{R}, A : \mathcal{L}$
$76.5 - 77$	$C^\sharp : \{7, 11, 5\}$	$E : \mathcal{R}, A : \mathcal{L}$
$77 - 77.5$	$G : \{1, 5, 11\}$	$E : \mathcal{R}, A : \mathcal{L}$
$77.5 - 78$	$C^\sharp : \{7, 11, 5\}$	$E : \mathcal{R}, A : \mathcal{L}$
$78 - 79$	$G : \{1, 5, 11\}$	$E : \mathcal{R}, A : \mathcal{L}, C^\sharp : \mathcal{P}$
$79 - 80$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$
$80 - 81$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, E_\flat : \mathcal{L}$
$81 - 83$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, E : \mathcal{R}, A : \mathcal{L}$
$83 - 84$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$
$84 - 85$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}$
$85 - 87$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$
$87 - 88$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$
$88 - 89$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}$
$89 - 91$	$C^\sharp : \{7, 11, 5\}$	$E : \mathcal{R}, A : \mathcal{L}$
$91 - 92$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$
$92 - 95$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}$
$95 - 95.5$	$C^\sharp : \{7, 11, 5\}$	$B_\flat : \mathcal{R}, D : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}, A : \mathcal{L}$
$95.5 - 96$	$E : \{4, 8, 2\}$	$F : \mathcal{R}, C^\sharp : \mathcal{R}, G : \mathcal{R}, F_\sharp : \mathcal{L}, C : \mathcal{L}$
$96 - 96.5$	$C^\sharp : \{7, 11, 5\}$	$B_\flat : \mathcal{R}, A_\flat : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}, A : \mathcal{L}$
$96.5 - 97$	$B_\flat : \{10, 2, 8\}$	$B : \mathcal{R}, C^\sharp : \mathcal{R}, G : \mathcal{R}, F_\sharp : \mathcal{L}, C : \mathcal{L}$
$97 - 97.5$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, D : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}, A : \mathcal{L}$
$97.5 - 98$	$E : \{4, 8, 2\}$	$F : \mathcal{R}, C^\sharp : \mathcal{R}, G : \mathcal{R}, F_\sharp : \mathcal{L}, C : \mathcal{L}$
$98 - 98.5$	$G : \{1, 5, 11\}$	$B_\flat : \mathcal{R}, A_\flat : \mathcal{R}, E : \mathcal{R}, E_\flat : \mathcal{L}, A : \mathcal{L}$
$98.5 - 99$	$E : \{4, 8, 2\}$	$B : \mathcal{R}, C^\sharp : \mathcal{R}, G : \mathcal{R}, F_\sharp : \mathcal{L}, C : \mathcal{L}$
$99 - 103$	$C^\sharp : \{7, 11, 5\}$	$D : \mathcal{R}, A : \mathcal{L}$